OPTIMUM PROBLEMS WITH MEASURABLE SET-VALUED CONSTRAINTS

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1. Motivation and History

Consider a general Optimal Control problem

(C) Minimize $f_0(x(0), x(1))$ subject to

$$\dot{x}(t) = f(t, x(t), \alpha(t)) \text{ a.e.}, \ \phi(x(0), x(1)) = 0,$$

$$g(t, x(t), \alpha(t)) \in U(t) \text{ a.e.}, \ k(t, x(t)) \in X(t) \ \forall t,$$

- $\alpha(\cdot) \in L^\infty_r[0,1], \quad x(\cdot) \in AC_n[0,1]$.
- $U(\cdot)$ meas, $U(t)$ closed, convex, $\operatorname{int} U(t) \neq \phi$.
- $X(\cdot)$ lsc, $X(t)$ closed convex with $\operatorname{int} X(t) \neq \phi$.

Set $\mathbb{U} := \sigma_\infty(U) := \{u(\cdot) \in L^\infty_m \mid u(t) \in U(t) \text{ a.e.}\}$.

$$\mathbb{X} := \sigma_C(X) = \{s(\cdot) \in C_n \mid s(t) \in X(t) \ \forall t\}$$

($\mathbb{U}$ and $\mathbb{X}$ are closed and convex),

$$Z := C_n[0,1] \times L^\infty_r[0,1], \text{ and, for } z := (x, \alpha) \in Z,$$

$$
\begin{aligned}
F_0(z) &:= f_0(x(0), x(1)) \\
E_0(z)(t) &:= x(t) - x_0 - \int_0^t f(\tau, x(\tau), \alpha(\tau))d\tau \quad (E : Z \to C_n) \\
E_1(z) &:= \phi(x(0), x(1)) \qquad (E_1 : Z \to \mathbb{R}^l) \\
G(z)(t) &:= g(t, x(t), \alpha(t)) \qquad (G : Z \to L^\infty_m) \\
K(z)(t) &:= k(t, x(t)) \qquad (K : Z \to C_n) \\
E(z) &:= (E_0, E_1)
\end{aligned}
$$

The problem (C) is in the framework of:

(P_C) minimize $F_0(z)$ over $G(z) \in \mathbb{U}, K(z) \in \mathbb{X}, E(z) = 0$.

Optimality conditions for (C) can be obtained by deriving optimality conditions for (P_C) then translating back in terms of (C).

The problem (P_C) take the form of:

(P) minimize $F_0(z)$ over $F(z) \in \mathbb{Q}$, $E(z) = 0$,

$F_0 : D \to \mathbb{R}$, $F : D \to X$, $E : D \to Y$, where X, Y, Z are Banach spaces, $D \subset Z$ open, $\mathbb{Q} \subset X$ is closed and convex.

What are the second-order necessary conditions for optimality in (P)?

Some History

- Ben-Tal/Zowe (1982): $\mathbb{Q}$ is a cone.

  – 2nd-order conditions having no extra term. The Critical Directions satisfy $F'(\bar{z})d \in \text{Cone}\,(\mathbb{Q} - F(\bar{z}))$.

- Kawasaki (1988): $\mathbb{Q}$ is a cone.

  – Data is C^2 & Mangasarian-Fromovitz cdn. needed.

  – 2nd-order conditions having an extra term (Envelope-Like Effect). The Critical Directions satisfy $F'(\bar{z})d \in T(F(\bar{z}) \mid \mathbb{Q}) := \overline{\text{cone}}\,(\mathbb{Q} - F(\bar{z}))$. The extra term is $\delta^*(x^* \mid \mathbb{T}^2)$, where

$$\mathbb{T}^2 := T^2(F(\bar{z}), F'(\bar{z})d \mid \mathbb{Q}) := \liminf_{t \to 0^+} \frac{\mathbb{Q} - F(\bar{z}) - tF'(\bar{z})d}{t^2}.$$

- Cominetti (1990): Generalized Kawasaki's work to $\mathbb{Q}$ any closed, convex set with $\mathbb{Q} \neq \phi$.

  – Data is C^2 and M-F condition still needed.

• Ioffe (1989), Penot (1994): Have obtained an extra term of the form $\delta^*(x^* \mid \mathbb{Q}'')$, where

$$\mathbb{Q}'' := Q(F(\bar{z}), F'(\bar{z})d) := \limsup_{t \to 0^+} \frac{\mathbb{Q} - F(\bar{z}) - tF'(\bar{z})d}{t^2}.$$

– Kawasaki and Penot independently realized that $\mathbb{Q}''$ is more complicated to compute than $\mathbb{T}^2$.

• Páles/Zeidan (1994): Have obtained an extra term with no Mangasarian-Fromovitz condition and data not C^2. The extra term is $\delta^*(x^* \mid \mathbb{V})$, where

$$\begin{aligned} \mathbb{V} &:= V(F(\bar{z}), F'(\bar{z})d, \mathbb{Q}) \\ &:= \text{The Dubovitskii/Milyutin-set of} \\ &\quad\ \text{2nd-order admissible variations,} \end{aligned}$$

is defined by $v \in \mathbb{V}$ if and only if $v \in X$ and

$$\exists\ \bar{\epsilon} > 0 :\ \forall\ 0 < \epsilon < \bar{\epsilon}, \quad \forall\ \|w\| < \bar{\epsilon},$$
$$F(\bar{z}) + \epsilon F'(\bar{z})d + \epsilon^2(v + w) \in \mathbb{Q}.$$

$\mathbb{V}$ is open and convex, and $\bar{\mathbb{V}} = \mathbb{T}^2$.

2. 2nd Order Necessary Conditions for (P)

– The Focus is on the set-valued constraint. For simplicity use twice Fréchet differentiable data.

Let $\bar{z}$ be admissible with $E'(\bar{z})$ has a closed Range. A direction $d \in Z$ is critical at $\bar{z}$ for (P) if:

$\bullet F_0'(\bar{z})d \leq 0 \quad \bullet E'(\bar{z})d = 0 \quad \bullet F'(\bar{z})d \in T(F(\bar{z}) \mid \mathbb{Q}) \longleftarrow$

Theorem 1 (P/Z) *If $\bar{z}$ is a min and* int $\mathbb{Q} \neq \phi, \longleftarrow$ *then, for every Critical Direction d satisfying*

$$\mathbb{V} := V(F(\bar{z}), F'(\bar{z})d, \mathbb{Q}) \neq \phi, \qquad \longleftarrow$$

there exist $\lambda \geq 0$, $x^* \in X^*$, $y^* \in Y^*$ not all zero:

$\bullet x^* \in N(F(\bar{z}) \mid \mathbb{Q}) \qquad \longleftarrow$

$\bullet \langle \lambda F_0'(\bar{z}), z\rangle + \langle x^*, F'(\bar{z})z\rangle + \langle y^*, E'(\bar{z})z\rangle = 0 \quad \forall\ z \in Z$

$$\bullet \langle \lambda F_0''(\bar{z})d, d\rangle + \langle x^*, F''(\bar{z}; d)\rangle + \langle y^*, E''(\bar{z}; d)\rangle$$

$$\geq 2\delta^*(x^* \mid \mathbb{V}). \qquad \longleftarrow$$

Remark: – Th. 1 is true with $\mathbb{V}$ replaced by $\mathbb{T}^2$.

– If $F'(\bar{z})d \in$ cone $(\mathbb{Q} - F(\bar{z}))$ then $\mathbb{V}$ is a nonempty cone and thus $\delta^*(x^* \mid \mathbb{V}) = 0$, i.e., no extra term.

- Does $\delta^*(x^* \mid \mathbb{V})$ appear only in pathological cases?

Example: For $a \in \mathbb{R}$, consider

(P) minimize $F_0(z) := z_2$ over

$$F(z)(t) := z_2 + (z_1 - t)^2 + (a-1)z_1^2 \geq 0 \qquad \forall t \in [-1, 1].$$

Then $\bar{z} := (0,0)$ is admissible for all $a \in \mathbb{R}$ and we have $z_2 \geq (1-a)z_1^2$ for z admissible and near $\bar{z}$.

- If $a \leq 1$ then $z_2 \geq 0 \Rightarrow \bar{z} = (0,0)$ is a local min.

- If $a > 1$ then $z_2 := -(1-a)z_1^2 < 0 \Rightarrow \bar{z} = (0,0)$ is not a local minimum.

- Here $Z = \mathbb{R}^2$, $X := C[-1,1]$, $\mathbb{Q} := C_+[-1,1]$.

Define x^* by $\langle x^*, x \rangle := -x(0)$. Then x^* is the only function satisfying the 1st-order conditions.

- $F'(0)d \in$ cone $(C_+[-1,1] - F(0)) \Leftrightarrow d = (0,0)$.

Let $a > 1$. Then the 2nd-order condition trivially holds at $d = (0,0)$ and thus cannot eliminate the non-optimal point $\bar{z}$.

$\bullet F'(0)d \in \overline{\text{cone}}(C_+[-1,1] - F(0)) \Leftrightarrow d_2 \geq 0$. Then $d = \binom{d_1}{0}$ is critical and

$$\mathbb{T}^2 := T^2(F(0), F'(0)\binom{d_1}{0}, \mathbb{Q}) = \{v(\cdot) \in C : v(0) \geq d_1^2\}.$$

The second-order condition reduces to

$$-2ad_1^2 \geq 2\sup\{-v(0) : v \in \mathbb{T}^2\} = -2d_1^2, \quad \forall d_1$$

$\therefore$ when $a > 1$, the 2nd order condition is violated and hence eliminates the optimality of $\bar{z}$.

$\bullet$ Take $a = 1$. Then $\bar{z}$ is optimal for (P). For the Critical Direction $d = \binom{d_1}{0}, d_1 \neq 0$, the right hand side of the 2nd-order condition is $-2d_1^2$ which is negative.

$\therefore$ The extra term $\delta^*(x^* \mid \mathbb{V})$ is crucial in the second-order necessary conditions and cannot be eliminated

3. The Case of Control Constraints

The case when $\mathbb{Q} = \sigma_C(X)$ was studied by Páles-Zeidan in SICOPT 1998. There, all conditions of Theorem 1 were translated in terms of the original state constraints $k(t, x(t)) \in X(t) \quad \forall t$.

Let $\mathbb{Q} := \mathbb{U} := \sigma_\infty(U)$ where $U(\cdot)$ is measurable, $U(t)$ closed and int $U(t) \neq \phi$, a.e.

How do the conditions of Theorem 1 translate in terms of $U(\cdot)$, and its associated pointwise objects: $T(\bar{u}(t) \mid U(t)), N(\bar{u}(t) \mid U(t)), \delta^*(\cdot \mid U(t))$?

- Lemma (P/Z): Int $\sigma_\infty(U) \neq \phi$ iff $\exists \; r \geq \rho > 0$ and

  $(*)$ for a.a. $t, \exists \; u_t \in \mathbb{R}^m : B_\rho(u_t) \subset U(t) \cap B_r$.

• For $\bar{u} \in \sigma_\infty(U)$, define the meas. set-valued maps:

$$T(\bar{u} \mid U)(t) := T(\bar{u}(t) \mid U(t)) \text{ a.e.,}$$

$$N(\bar{u} \mid U)(t) := N(\bar{u}(t) \mid U(t)) \text{ a.e.,}$$

Where T is the adjacent or Clarke tangent cone. Their connection with $T(\bar{u} \mid \sigma_\infty(U))$ and $N(\bar{u} \mid \sigma_\infty(U))$ require the following notion.

Definition: – A set $\mathbb{Q} \subseteq L^\infty$ is L^1-*closed in* L^∞ if whenever $\{x_n\}$ in $\mathbb{Q}$ converges in the L^1-norm to $x_0 \in L^\infty$ then $x_0 \in \mathbb{Q}$.

– The L^1-closure in L^∞ of $\mathbb{Q}$ is denoted by $c\ell_1\mathbb{Q}$.

– Note that $\mathbb{Q} := \sigma_\infty(U)$ is L^1-closed in L^∞. But $T(\bar{u} \mid \sigma_\infty(U))$ which is a closed subset in L^∞ but is not necessarily L^1-closed in L^∞. This is the case for either the adjacent or Clarke tangent cone.

Theorem 2 $\bullet cl_1 T(\bar{u} \mid \sigma_\infty(U)) = \sigma_\infty(T(\bar{u} \mid U))$

$\bullet N(\bar{u} \mid \sigma_\infty(U)) \cap L^1 = \sigma_1(N(\bar{u} \mid U))$.

$y(\cdot) \in L^\infty, y(t) \in T(\bar{u}(t) \mid U(t))$ a.e. $\not\Rightarrow y \in T(\bar{u} \mid \sigma_\infty(U))$.

But $\mathbb{V} \neq \quad \phi \quad \Rightarrow \quad y \in T(\bar{u} \mid \sigma_\infty(U))$.

Also, for $x^* \in L^1$,

$x^* \in N(\bar{u} \mid \sigma_\infty(U)) \quad \Leftrightarrow \quad x^*(t) \in N(\bar{u}(t) \mid U(t))$ a.e.

Needed in terms of $\delta^*(\cdot \mid U(t))$:

– an expression of $\delta^*(x^* \mid \mathbb{V})$ when $x^* \in L^1$

– a characterization of $\mathbb{V} \neq \phi$ ($\mathbb{V} := V(\bar{u}, y, \sigma_\infty(U))$).

When we apply the result to a specific problem we need conditions on the data guaranteeing $x^* \in L^1$.

- The calculation of $\delta^*(x^* \mid \mathbb{V})$ when $x^* \in L^1$.

Define the measurable set-valued map $t \to \mathcal{V}(t)$,

$$\mathcal{V}(t) := V(\bar{u}(t), y(t), U(t)) \subset \mathbb{R}^m.$$

Is there a connection between $\mathbb{V}$ and $\mathcal{V}(\cdot)$?

Theorem 3 *If* $\mathbb{V} \neq \phi$ *then*

$$c\ell_1 \mathbb{V} = \sigma_\infty(\bar{\mathcal{V}}).$$

Hence,

$$\delta^*(x^* \mid \mathbb{V}) = \delta^*(x^* \mid \sigma_\infty(\bar{\mathcal{V}})) = \int_0^1 \delta^*(x^*(t) \mid \mathcal{V}(t))dt.$$

What is $\delta^*(\cdot \mid \mathcal{V}(t))$ in terms of $\delta^*(\cdot \mid U(t))$ when $y(t) \in T(\bar{u}(t) \mid U(t))$?

Assume $U(t)$ is also convex. Set

$$\mathbb{C}(t) := N(\bar{u}(t) \mid U(t)) \cap \{y(t)\}^\perp$$

$$\mathbb{S}(t) := \{\xi \in \mathbb{R}^m : \langle \xi, y(t) \rangle > 0\}$$

$$\mathbb{S}'(t) := \{\xi \in \mathbb{R}^m : \exists\ \xi_n \to \xi, \xi_n \in \mathbb{S}(t)\ \forall\ n \in \mathbb{N}\}$$

and define on $[0,1] \times \mathbb{R}^m$,

$$\sigma(t,\xi) := \begin{cases} \liminf_{\substack{\beta\to\xi \\ \beta\in\mathbb{S}(t)}} \frac{b^2(t,\beta)}{-4a(t,\beta)} & \text{if } \xi \in \mathbb{C}(t) \cap \mathbb{S}'(t) \\ 0 & \text{if } \xi \in \mathbb{C}(t) \backslash \mathbb{S}'(t) \\ +\infty & \text{otherwise,} \end{cases}$$

$$a(t,\beta) := \delta^*(\beta \mid U(t)) - \langle \beta, \bar{u}(t) \rangle$$

$$b(t,\beta) := \langle \beta, y(t) \rangle.$$

$\sigma(t,\cdot)$ is positively homogeneous, ℓsc on $\mathbb{R}^m \backslash \{0\}$. It is in terms of $\delta^*(\cdot \mid U(t))$.

Set $\overline{\text{co}}\ \sigma(t,\cdot)$ to be the largest ℓsc, convex function below $\sigma(t,\cdot)$.

Theorem 4 *If $U(t)$ is closed and convex. Then*

$$\delta^*(\cdot \mid \mathcal{V}(t)) = \overline{\mathit{co}}\ \sigma(t,\cdot).$$

Hence, if $\mathbb{V} \neq \phi$ then

$$\delta^*(x^* \mid \mathbb{V}) = \int_0^1 \overline{\mathit{co}}\ \sigma(t, x^*(t))dt.$$

- Characterization of $\mathbb{V} \neq \phi$ in terms of $\delta^*(\cdot \mid U(t))$.

Note: $\mathbb{V} \neq \phi \Rightarrow \operatorname{int} \sigma_\infty(U) \neq \phi$ & $y \in T(\bar{u} \mid \sigma_\infty(U))$.

Theorem 5 *Let $U(\cdot)$ have convex images and $(*)$ holds. Let $\bar{u} \in \sigma_\infty(U)$ and $y \in L^\infty$. Then $\mathbb{V} := V(\bar{u}, y, \sigma_\infty(U)) \neq \phi$ iff*

(1) $y(t) \in T(\bar{u}(t) \mid U(t))$ *a.e., and*

(2) $\exists K > 0 : t \in [0,1]$ *a.e.,* $\forall \xi : b(t,\xi) > 0$,

$$b^2(t,\xi) \leq K|\xi| a(t,\xi),$$

where $a(t,\xi) := \delta^*(\xi \mid U(t)) - \langle \xi, \bar{u}(t) \rangle$

$$b(t,\xi) := \langle \xi, y(t) \rangle.$$

- (1) and (2) $\Rightarrow y \in T(\bar{u} \mid \sigma_\infty(U))$.

- When $\sigma_\infty(U) = \{u \in L^\infty_m : u(t) \leq 0 \text{ a.e.}\}$, Conditions (1) and (2) of Theorem 5 yield that for $x^* \in L^1$ with $\langle x^*(t), \bar{u}(t) \rangle = 0$ and $\langle x^*(t), y(t) \rangle = 0$ a.e., then $\delta^*(x^* \mid \mathbb{V}) = 0$.

• When does x^*, in Theorem 1, belong to L^1?

(P) min $F_0(z)$ over $F(z) \in \sigma_\infty(U)$, $E(z) = 0$, $z \in Z$.

Proposition (P/Z): If there exists a mapping $\mathcal{A} : L^\infty_m \to \mathbb{Z}$ such that $\mathcal{A}(0) = 0$, $F_0'(\bar{z}) \circ \mathcal{A}$ is L^1-continuous, $E'(\bar{z}) \circ \mathcal{A}$ is weakly L^1-continuous and

$$(**) \qquad F'(\bar{z})(\mathcal{A}w) - w \in T(F(\bar{z}) \mid \sigma_\infty(U)).$$

Then x^* in Theorem 1 is in L^1.

Example. Consider the setting of Problem (C), where

$$\sigma_\infty(U) := \{u \in L^\infty_m[0,1] : u(t) \le 0 \quad \text{a.e.}\},$$

and $F(z)(t) := g(t, \alpha(t))$ for $z = (x, \alpha)$. Let $(\bar{x}, \bar{\alpha})$ be a local minimum. Assume that for some $\gamma > 0$,

$$\det J(t) J^T(t) \ge \gamma > 0 \text{ a.e. } t, \text{ where}$$

$$J(t) = \begin{bmatrix} \partial_\alpha \bar{g}_1(t) & \sqrt{-\bar{g}_1(t)} & & 0 \\ \vdots & & \ddots & \\ \partial_\alpha \bar{g}_m(t) & 0 & & \sqrt{-\bar{g}_m(t)} \end{bmatrix},$$

Then $(**)$ is satisfied.

Set

$$\begin{bmatrix} A(t) \\ C(t) \end{bmatrix} := J^T(t)(J(t)J^T(t))^{-1}.$$

Define $\mathcal{A} : L_m^\infty \to L_r^\infty$ by $\mathcal{A}(w)(t) = A(t)w(t)$.

4. Open Problems

I. How does $(**)$ translate in terms of the data, when the constraint is $g(t, \alpha(t)) \in U(t)$ a.e., and $U(\cdot)$ is measurable, with convex closed images?

II. Is it possible to enlarge the set of critical directions d so that the condition $b^2(t,\xi) \leq K|\xi|a(t,\xi)$ a.e.t, $\forall\ \xi : b(t,\xi) > 0$, can be eliminated? If so for what choice of $U(\cdot)$?

III. Identify all the cases when $\delta^*(x^* \mid \mathbb{V}) = 0$. (This is true when $U(t) := \{u \in \mathbb{R}^m : u \leq 0\}$.)

IV. Develop the conjugate point theory for control problems wtih General Control Constraints:

$$g(t, \alpha(t)) \in U(t) \text{ a.e.}$$