New Second-Order Optimality Conditions for Variational Problems with C^2-Hamiltonians

by

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Abstract

The generalized problem of Bolza with C^2-Hamiltonian is considered. Necessary and sufficient conditions are obtained respectively in terms of the accessory problem, the existence of conjoined basis, and the existence of a solution to a Riccati equation with boundary conditions. No strengthened Legendre-Clebsch condition is required. When applied to a general optimal control problem, these results include and generalize known results.

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1 Introduction

Consider the generalized problem of Bolza:

$$\text{(P)} \qquad \text{minimize } J(x) = \ell(x(0), x(1)) + \int_0^1 L(t, x(t), \dot{x}(t))dt$$

over all absolutely continuous functions (arcs)$x(\cdot) \in W^{1,1}[0,1]$ satisfying

$$\text{(1.1)} \qquad \phi(x(0), x(1)) = 0,$$

where $\ell : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $L : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, and $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^r$, $r \leq 2n$. The norm in $W^{1,p}[0,1]$ is $\|x\|_{1,p} := |x(0)| + \|\dot{x}(\cdot)\|_p$. An arc is said to be *admissible* if it satisfies (1.1).

The Hamiltonian corresponding to the problem is:

$$H(t, x, p) := \sup\{\langle p, v\rangle - L(t, x, v) : v \in \mathbb{R}^n\}. \tag{1.2}$$

The problem (P) was introduced by R.T. Rockafellar. It is more general than it appears to be, due to the fact that L is allowed to take the value $+\infty$. While this problem subsumes diverse constrained problems (e.g., calculus of variations and optimal control problems), it is distinguished from the classical setting by the lack of regularity of L. Nevertheless, the Hamiltonian itself may actually be well-behaved for certain classes of problems. This was the reason for the program of studying the problem (P) from the point of view of the Hamiltonian H. In fact, existence theory involving conditions on H was obtained by Rockafellar [8], and first-order necessary conditions in terms of the Hamiltonian inclusions were developed by Clarke [2] and [3]. Sufficiency criteria for optimality of zero, first and second-order were derived in [13] and [15] for the case where $H(t, \cdot, \cdot)$ is not necessarily concave-convex. There, one state endpoint is assumed to be fixed, and the second-order sufficiency criterion assumes that $H(t, \cdot, \cdot)$ be $C^{1+}(C^{1,1})$, that is, has a Lipschitz gradient, and is phrased in terms of a certain Riccati inequality. However, the question of second-order necessary conditions in terms of the Hamiltonian remains an open question. By analogy with the classical calculus of variations setting, these conditions are expected to be expressed in terms of

(i) the accessory problem,

(ii) the coupled point theory,

(iii) a conjoined basis with appropriate boundary conditions,

(iv) a certain Riccati-type equation with appropriate boundary conditions.

As opposed to the classical setting, these conditions must be phrased in terms of the Hamiltonian, which can be "nice" even when L is "bad".

Now consider the *optimal control* problem:

(C) minimize $J(x, u) := \ell(x(0), x(1)) + \int_0^1 g(t, x(t), u(t))dt$
subject to:

$$\begin{aligned}
&\dot{x}(t) = f(t, x(t), u(t)) \text{ a.e.,}\\
&u(t) \in U \text{ a.e.,}\\
&\phi(x(0), x(1)) = 0,
\end{aligned}$$

where $x(\cdot) \in W^{1,1}[0,1]$ and $u(\cdot)$ measurable, ℓ and ϕ as in the Problem (P), $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$.

We shall assume:

$$\text{(A)} \qquad \begin{cases} \bullet\ f \text{ and } g \text{ are } \mathcal{L}\times\mathcal{B}\text{-measurable, and continuous in } (x,u) \\ \bullet\ U \text{ is closed} \\ \bullet\ \ell \text{ is lower semicontinuous} \end{cases}$$

The Hamiltonian associated with (C) is:

$$\mathcal{H}(t,x,p) := \sup\{p\cdot f(t,x,u) - g(t,x,u) \mid u \in U\}. \tag{1.3}$$

A pair (x,u) satisfying all the constraints is called *admissible.*

The program of studying the problem (C) from the point of view of the "true" Hamiltonian $\mathcal{H}$ is one step further along than that for (P). In fact, in addition to the existence theory of [8], the first-order necessary conditions in [3], and the sufficiency criteria up to second-order in [13] and [14] for the case where only one state endpoint varies, necessary conditions for (C) were obtained in [1, Theorem 5.1] in terms of the Riccati equation $(\mathcal{R})$ and (3.17). However, these conditions were derived for $W^{1,1}$-weak minimality and for the case where the initial state variable is *fixed* and the final one is free. There, a triplet $(\bar{x},\bar{u},\bar{p})$ satisfying the first-order necessary conditions is considered. It is assumed that $\nabla^2\ell$ is Lipschitz and $\nabla^2_{(x,p)}\mathcal{H}$ is Lipschitz at $(\bar{x},\bar{p})$ from L^∞ to L^1 (see Definition 2.2). The result states that if the Riccati equation $(\mathcal{R})$ and the boundary condition (3.17) have on $]t_c,t]$, for some $t_c \in (0,1)$, a solution that does not extend to t_c, and if the *strengthened Legendre-Clebsch* condition $(S\mathcal{L})$ holds near t_c, for $t < t_c$,

$$\text{for some } \lambda > 0, \quad \mathcal{H}_{pp}(t,\bar{x}(t),\bar{p}(t)) \geq \lambda I \quad \text{a.e.} \tag{$S\mathcal{L}$}$$

and

$$\sup_{t\in[t_c-\lambda,t_c]} \left|\nabla^2_{(x,p)}\mathcal{H}(t,\bar{x}(t),\bar{p}(t))\right| < \infty, \tag{1.4}$$

then $(\bar{x},\bar{u})$ is not a $W^{1,1}$-weak local minimum for (C). These assumptions (see Lemma 5.1 and Remark 5.2) imply that the problem (C) resembles the classical calculus of variations setting, which is quite restrictive. Also the question of necessary conditions of types (i)-(iii) remains open.

The aim of this paper is to *complete* the program of studying the problems (P) and (C) from the point of view of the "true" Hamiltonian. This is accomplished by answering the open necessity and sufficiency questions of types (i), (iii) and (iv). Conditions of type (ii) will be handled in a separate paper.

The paper is divided as follows. In Section 2, preliminary results are derived concerning the optimality notions used in the manuscript and the connection with the classical ones. In the same section we recall known results on the connection between (P) and (C) and on the notion of M-controllability.

For the Problem (P), necessary conditions in terms of the Hamiltonian H are obtained in Section 3 for $W^{1,1}$- and $W^{1,\infty}$-local optimality. The first condition takes the form of the accessory problem. The novelty of its proof lies in associating to (P) an optimal control problem (C_H) that inherits the optimality of (P). The accessory problem for (C_H) is what we call the accessory problem of (P). As a consequence and by means of the results in [16] we obtain necessary conditions of types (iii) and (iv) for the case when the endpoints costs and constraints are separable. In Section 4 we develop a sufficiency optimality criterion in (P) for general endpoints conditions and another one for separable costs and endpoints conditions. This latter enjoys the distinction that it is as close as possible to the necessary conditions developed in Section 3.

In Section 5 we apply the results of Sections 3 and 4 to the optimal control problem (C) to obtain necessary and sufficient conditions. Indeed, the necessary conditions are derived under two different sets of conditions depending on whether we write (C) in the framework of (P) or ($C_{\mathcal{H}}$). Neither of the two sets requires the strengthened Legendre-Clebsch conditions ($S\mathcal{L}$). Hence, when specialized to the case considered in [1] our results are much more general. More specifically, Part (1)(ii) of Theorem (5.2) of this paper extends [1, Theorem 5.1] to the case where the Hessians $\nabla^2\ell$ and $\nabla^2\mathcal{H}$ are not necessarily Lipschitz, (1.4) is not satisfied, or the strengthened Legendre-Clebsch ($S\mathcal{L}$) does not hold. This latter is replaced by a normality condition which is automatically satisfied when ($S\mathcal{L}$) holds. Example 5.1 shows that the normality condition can be satisfied when ($S\mathcal{L}$) fails and hence it illustrates the utility of our generalization. Furthermore, Part (1) of Theorem (5.2) applies equally to $W^{1,1}$- and $W^{1,\infty}$-weak local minimality. Moreover, Part (2) of Theorem 5.2 applies to L^1- and L^∞-weak local minimum. In addition to necessary conditions in terms of the Riccati equation ($\mathcal{R}$), Theorem 5.2 develop necessary conditions in terms of the accessory problem (Theorem 3.1) and a conjoint basis for (3.14)-(3.15) (Theorem 3.3).

In this paper we assume that $H(t,\cdot,\cdot)$ and $\mathcal{H}(t,\cdot,\cdot)$ are C^2. This assumption is appropriate for certain classes of optimal control problems (see e.g., [1]). Furthermore, the results of this paper will serve as a principal stepping stone for a subsequent paper where this regularity assumption on H and $\mathcal{H}$ is weakened.

2 Preliminary Results

For the problem (P) defined in the previous section, we are concerned with the following three notions of optimality.

Definition 2.1. An admissible arc $\bar{x}$ is said to be

(i) *strong local minimum* for (P) if there exists $\varepsilon > 0$ such that for any admissible arc x satisfying $\|x - \bar{x}\|_\infty < \varepsilon$ we have $J(x) \geq J(\bar{x})$,

(ii) $W^{1,1}$-*weak local minimum* for (P) if there exists $\varepsilon > 0$ such that for any admissible arc x satisfying $\|x - \bar{x}\|_{1,1} < \varepsilon$ we have $J(x) \geq J(\bar{x})$,

(iii) $W^{1,\infty}$-*weak local minimum* for (P) if there exists $\varepsilon > 0$ such that for any admissible arc x for which $\|x - \bar{x}\|_{1,\infty}$ is defined, and $\|x - \bar{x}\|_{1,\infty} < \varepsilon$ we have $J(x) \geq J(\bar{x})$.

Clearly, if $\bar{x}$ is a strong local minimum then $\bar{x}$ is an $W^{1,1}$-weak local minimum, and if $\bar{x}$ is $W^{1,1}$-weak local minimum then it is $W^{1,\infty}$-weak local minimum.

We shall need the following notions. A function K on $[0,1]\times\mathbb{R}^k$ is *$\mathcal{L}\times\mathcal{B}$-measurable* if it is measurable with respect to the σ-algebra generated by products of Lebesgue sets in $[0,1]$ and Borel sets in $\mathbb{R}^k$.

For $\bar{z}\in L^\infty[0,1]$ we make these definitions.

Definition 2.2. Let $K:[0,1]\times\mathbb{R}^k\to\mathbb{R}$ be $\mathcal{L}\times\mathcal{B}$-measurable. We say that K is *continuous at $\bar{z}$, from L^∞ to L^1* (resp. L^∞) if the map $z(\cdot)\to K(\cdot,z(\cdot))$ from L^∞ to L^1 (resp. L^∞) is continuous at $\bar{z}$.

The function K is said to be *Lipschitz* near $\bar{z}$, from L^∞ to L^1 (resp. L^∞) if the map $z(\cdot)\to K(\cdot,z(\cdot))$ from L^∞ to L^1 (resp. L^∞) is Lipschitz near $\bar{z}$.

Definition 2.3. Let $\bar{x}\in L^\infty$, $\bar{y}\in L^1$ and $K:[0,1]\times\mathbb{R}^r\times\mathbb{R}^s\to\mathbb{R}$. The function K is said to be *continuous at $(\bar{x},\bar{y})$, from $L^\infty\times L^1$ to L^1* if the map $(x(\cdot),y(\cdot))\to K(\cdot,x(\cdot),y(\cdot))$ from $L^\infty\times L^1\to L^1$ is continuous at $(\bar{x},\bar{y})$.

We say that K is *Lipschitz* near $(\bar{x},\bar{y})$, from $L^\infty\times L^1$ to L^1 if the map $(x(\cdot),y(\cdot))\to K(\cdot,x(\cdot),y(\cdot))$ from $L^\infty\times L^1$ to L^1 is Lipschitz near $(\bar{x},\bar{y})$.

It is clear that the Lipschitz property of K yields the corresponding continuity property of K. However, as we shall see below, the continuity of the gradient $\nabla_z K$ yields the Lipschitz condition on K.

Lemma 2.1. *Let $K(t,\cdot)$ be C^1 and $\bar{z}\in L^\infty$.*

(i) *Assume that $\nabla_z K(\cdot,\bar{z}(\cdot))$ is L^1. If $\nabla_z K$ is continuous at $\bar{z}$ from L^∞ to L^1, then K is Lipschitz near $\bar{z}$ from L^∞ to L^1.*

(ii) *Assume that $\nabla_z K(\cdot,\bar{z}(\cdot))$ is L^∞. If $\nabla_z K$ is continuous at $\bar{z}$ from L^∞ to L^∞, then K is Lipschitz near $\bar{z}$ from L^∞ to L^∞.*

Proof. Apply Definition 2.2(i) to $\nabla_z K$ and take $\varepsilon=1$.

For (i) (resp. (ii)). The continuity at $\bar{z}$ of $\nabla_z K$ from L^∞ to L^1 (resp. L^∞) yields the existence of $\delta>0$ such that for all $z:\|z-\bar{z}\|_\infty<\delta$ we have

$$\|\nabla_z K(\cdot,z(\cdot))-\nabla_z K(\cdot,\bar{z}(\cdot))\|_{1(\text{resp.}\infty)}<1.$$

Then, for all z and z' with $\|z-\bar{z}\|_\infty<\delta$, $\|z'-\bar{z}\|_\infty<\delta$, the mean value theorem implies that for almost all t there exists $\tilde{z}_t$ satisfying $|\tilde{z}_t-\bar{z}(t)|<\delta$ and

$$\begin{aligned}|K(t,z(t))-K(t,z'(t))| &\le |\nabla_z K(t,\tilde{z}_t)|\cdot|z(t)-z'(t)|\\ &\le |\nabla_z K(t,\tilde{z}_t)-\nabla_z K(t,\bar{z}(t))|\cdot\|z-z'\|_\infty\\ &\quad+|\nabla_z K(t,\bar{z}(t))|\cdot\|z-z'\|_\infty.\end{aligned}$$

Since $\nabla_z K(\cdot, \bar{z}(\cdot))$ is L^1 (resp. in L^∞), we get

$$\|K(\cdot, z(\cdot)) - K(\cdot, z'(\cdot))\|_1 \le (1 + \|\nabla_z K(\cdot, \bar{z}(\cdot))\|_1)\|z - z'\|_\infty$$

$$(\text{resp. } \|K(\cdot, z(\cdot)) - K(\cdot, z'(\cdot))\|_\infty \le (1 + \|\nabla_z K(\cdot, \bar{z}(\cdot))\|_\infty)\|z - z'\|_\infty).$$

Therefore, the result follows.

Optimality notions for the optimal control problem (C) similar to those given for (P) can be defined. Note that the notion of strong local minimality coincides with the classical one. On the other hand, the following *classical* notions of L^1-weak and L^∞-weak local minima are known for (C).

Definition 2.4. An admissible pair $(\bar{x}, \bar{u})$ for (C) is *classically* L^1-weak, respectively, *classically* L^∞-weak local minimum for (C) if for some $\varepsilon > 0$,

$$J(x, u) \ge J(\bar{x}, \bar{u})$$

for all admissible pairs (x, u) satisfying, respectively,

$$\|x - \bar{x}\|_\infty + \|u - \bar{u}\|_1 < \varepsilon, \text{ and } \|x - \bar{x}\|_\infty + \|u - \bar{u}\|_\infty < \varepsilon.$$

The next result shows that, under natural hypotheses on the dynamics f, the $W^{1,1}$ (resp. $W^{1,\infty}$)-weak local minimality in (C) implies the *classical* L^1 (resp. L^∞)-weak local minimality. This implication will play an important role in developing the results of this paper. The converse is shown to hold under more restrictive hypotheses on f which we do not use in this paper.

Lemma 2.2. *Let (A) be satisfied and $(\bar{x}, \bar{u})$ be an admissible pair for* (C) *such that $\bar{u} \in L^1$.*

(a) *Assume that $(\bar{x}, \bar{u})$ is $W^{1,1}$-weak local minimum for* (C). *If $\bar{u} \in L^\infty$ and f is continuous at $(\bar{x}, \bar{u})$ from $L^\infty \times L^\infty$ to L^1, then $(\bar{x}, \bar{u})$ is classically L^∞-weak local minimum for* (C). *If f is continuous at $(\bar{x}, \bar{u})$ from $L^\infty \times L^1$ to L^1, then $(\bar{x}, \bar{u})$ is classically L^1-weak local minimum for* (C).

(b) *If $\bar{u} \in L^\infty$, f is continuous at $(\bar{x}, \bar{u})$ from $L^\infty \times L^\infty$ to L^∞, and if $(\bar{x}, \bar{u})$ is $W^{1,\infty}$-weak local minimum, then $(\bar{x}, \bar{u})$ is also classically L^∞-weak local minimum for* (C).

Conversely,

(c) *Let $(\bar{x}, \bar{u})$ be an L^1-weak local minimum for* (C). *Assume that the map $x(\cdot) \to f(\cdot, x(\cdot), u(\cdot))$ is continuous at $\bar{x}(\cdot)$ from L^∞ to L^1 uniformly in $u(\cdot) \in L^1$ with $u(t) \in U$ and that there exists $m > 0$ such that, for all $u(\cdot) \in L^1$ such that $u(t) \in U$ a.e.,*

$$\|f(\cdot, \bar{x}(\cdot), u(\cdot)) - f(\cdot, \bar{x}(\cdot), \bar{u}(\cdot))\|_1 \ge m\|u - \bar{u}\|_1.$$

Then $(\bar{x}, \bar{u})$ is a $W^{1,1}$-weak local minimum for (C).

(d) *Let* $(\bar{x}, \bar{u})$ *be an* L^∞*-weak local minimum for* (C)*. Assume that the map* $x(\cdot) \to f(\cdot, x(\cdot), u)$ *is continuous at* $\bar{x}(\cdot)$ *from* L^∞ *to* L^∞ *uniformly in* $u \in U$ *and that there exists* $m > 0$ *such that, for almost all* t *and for all* $u \in U$,

$$|f(t, \bar{x}(t), u) - f(t, \bar{x}(t), \bar{u}(t))| \geq m|u - \bar{u}(t)|.$$

Then $(\bar{x}, \bar{u})$ *is a* $W^{1,\infty}$*-weak local minimum for* (C).

Proof. For (a). Let $\varepsilon > 0$ be such that for any admissible pair (x, u) satisfying $\|x - \bar{x}\|_{1,1} < \varepsilon$, we have $J(x, u) \geq J(\bar{x}, \bar{u})$. If f is continuous at $(\bar{x}, \bar{u})$ from $L^\infty \times L^\infty$ to L^1, there exists $\delta_0 > 0$ such that for $\|(x, u) - (\bar{x}, \bar{u})\|_\infty < \delta_0$ we have $\|f(\cdot, x(\cdot), u(\cdot)) - f(\cdot, \bar{x}(\cdot), \bar{u}(\cdot))\|_1 < \dfrac{\varepsilon}{2}$. Set

$$\bar{\varepsilon} := \min\{\frac{\varepsilon}{2}, \delta_0\}.$$

Now, let (x, u) be admissible for (C) with

$$\|x - \bar{x}\|_\infty + \|u - \bar{u}\|_\infty < \bar{\varepsilon},$$

it follows that

$$\|x - \bar{x}\|_{1,1} \leq |x(0) - \bar{x}(0)| + \|f(\cdot, x(\cdot), u(\cdot)) - f(\cdot, \bar{x}(\cdot), \bar{u}(\cdot))\|_1 < \varepsilon.$$

Whence,

$$J(x, u) \geq J(\bar{x}, \bar{u}),$$

proving that $(\bar{x}, \bar{u})$ is *classically* L^∞-weak local minimum. On the other hand, if f is continuous at $(\bar{x}, \bar{u})$ from $L^\infty \times L^1$ to L^1, then there exists $\delta_0 > 0$ such that for $\|x - \bar{x}\|_\infty < \delta_0$ and $\|u - \bar{u}\|_1 < \delta_0$ we have $\|f(\cdot, x(\cdot), u(\cdot)) - f(\cdot, \bar{x}(\cdot), \bar{u}(\cdot))\|_1 < \dfrac{\varepsilon}{2}$. Set $\bar{\varepsilon} := \min\{\frac{\varepsilon}{2}, \delta_0\}$ and let (x, u) admissible for (C) with

$$\|x - \bar{x}\|_\infty + \|u - \bar{u}\|_1 < \bar{\varepsilon}.$$

Then, $\|x - \bar{x}\|_{1,1} < \varepsilon$. Thus, $J(x, u) \geq J(\bar{x}, \bar{u})$, that is, $(\bar{x}, \bar{u})$ is *classically* L^1-weak local minimum.

For (b). Since $(\bar{x}, \bar{u})$ is $W^{1,\infty}$-weak local minimum, then for some $\varepsilon > 0$ we have $J(x, u) \geq J(\bar{x}, \bar{u})$, for all admissible pairs (x, u) satisfying

$$\|x - \bar{x}\|_{1,\infty} < \varepsilon.$$

By the hypothesis, there exists $\delta_0 > 0$ such that for all (x, u) such that $\|x - \bar{x}\|_\infty < \delta_0$, $\|u - \bar{u}\|_\infty < \delta_0$ we have

$$\|f(\cdot, x(\cdot), u(\cdot)) - f(\cdot, \bar{x}(\cdot), \bar{u}(\cdot))\|_\infty < \frac{\varepsilon}{2}.$$

Let (x, u) admissible with

$$\|x - \bar{x}\|_\infty + \|u - \bar{u}\|_\infty < \bar{\varepsilon},$$

where $\bar{\varepsilon} := \min\{\delta_0, \frac{\varepsilon}{2}\}$. Then,

$$\|x-\bar{x}\|_{1,\infty} \leq \bar{\varepsilon} + \|f(\cdot, x(\cdot), u(\cdot)) - f(\cdot, \bar{x}(\cdot), \bar{u}(\cdot))\|_\infty < \varepsilon.$$

Hence,

$$J(x,u) \geq J(\bar{x}, \bar{u})$$

and $(\bar{x}, \bar{u})$ is *classically* an L^∞-weak local minimum for (C).

For (c) (resp. (d)). The L^1 (resp. L^∞)-weak local minimality of $(\bar{x}, \bar{u})$ for (C) yields the existence of $\varepsilon > 0$ such that $J(x,u) \geq J(\bar{x}, \bar{u})$, for all admissible pairs (x,u) satisfying

$$\|x-\bar{x}\|_\infty + \|u-\bar{u}\|_{1(\text{resp. } \infty)} < \varepsilon.$$

By the continuity assumption on f, there exists $\delta_0 > 0$, such that for $\|x - \bar{x}\|_\infty < \delta_0$,

$$\|f(\cdot, x(\cdot), u(\cdot)) - f(\cdot, \bar{x}(\cdot), u(\cdot))\|_{1(\text{resp. } \infty)} < \frac{\varepsilon m}{4}$$

for all $u(\cdot) \in L^1$ (resp. L^∞) with $u(t) \in U$ a.e., where m is the constant in the assumption.

Define $\bar{\varepsilon} = \min\{\frac{m\varepsilon}{4}, \frac{\varepsilon}{2}, \delta_0\}$. Let (x,u) be admissible for (C) with $\|x-\bar{x}\|_{1,1} < \bar{\varepsilon}$ (resp. $\|x-\bar{x}\|_{1,\infty} < \bar{\varepsilon}$). Then $\|x-\bar{x}\|_\infty < \frac{\varepsilon}{2}$ and,

$$\|f(\cdot, x(\cdot), u(\cdot)) - f(\cdot, \bar{x}(\cdot), u(\cdot))\|_1 < \frac{m\varepsilon}{4}$$
$$(\text{resp. } |f(t, x(t), u(t)) - f(t, \bar{x}(t), u(t))| < \frac{m\varepsilon}{4} \text{ a.e. } t).$$

It results that

$$\begin{aligned}
\frac{m\varepsilon}{4} &> \|f(\cdot, \bar{x}(\cdot), u(\cdot)) - f(\cdot, \bar{x}(\cdot), \bar{u}(\cdot)) + f(\cdot, x(\cdot), u(\cdot)) - f(\cdot, \bar{x}(\cdot), u(\cdot))\|_{1(\text{resp. } \infty)} \\
&\geq \|f(\cdot, \bar{x}(\cdot), u(\cdot)) - f(\cdot, \bar{x}(\cdot), \bar{u}(\cdot))\| - |f(\cdot, x(\cdot), u(\cdot)) - f(\cdot, \bar{x}(\cdot), u(\cdot))\|_{1(\text{resp. } \infty)} \\
&> m\|u-\bar{u}\|_{1(\text{resp. } \infty)} - \frac{m\varepsilon}{4}.
\end{aligned}$$

Thus, $\|u-\bar{u}\|_{1(\text{resp. } \infty)} < \frac{\varepsilon}{2}$. Whence, $J(x,u) \geq J(\bar{x}, \bar{u})$ and therefore, $(\bar{x}, \bar{u})$ is a $W^{1,1}$-(resp. $W^{1,\infty}$-)weak local minimum for (C).

Let $\bar{x} \in C[0,1]$ and $\varepsilon > 0$ be given. Define

$$T(\bar{x};\varepsilon) := \{(t,x) \in [0,1] \times \mathbb{R}^n : |x - \bar{x}(t)| < \varepsilon\}. \tag{2.1}$$

We say that an arc $x(\cdot) \in T(\bar{x};\varepsilon)$ when $(t, x(t)) \in T(\bar{x};\varepsilon)$, for all t.

There are several ways to write (C) in the form of (P). Below we exhibit a general one. Set

$$L_C(t,x,v) := \begin{cases} \inf\{g(t,x,u) : v = f(t,x,u) \text{ and } u \in U\} & \text{if } (t,x) \in T(\bar{x};\varepsilon) \\ +\infty & \text{otherwise .} \end{cases} \tag{2.2}$$

Clearly that $L_C(t,x,v) = +\infty$ when $\{u : v = f(t,x,u), u \in U\} = \emptyset$. Furthermore, even if $L_C(t,x,v)$ is finite, it can be discontinuous or nonsmooth. To Problem (C) we can associate the following *generalized Bolza Problem*

$(\mathrm{P_C})$ minimize $J_C(x) := \ell(x(0), x(1)) + \int_0^1 L_C(t, x(t), \dot{x}(t))dt$

subject to:

$$\phi(x(0), x(1)) = 0.$$

The Hamiltonian corresponding to $(\mathrm{P_C})$ on $T(\bar{x};\varepsilon) \times \mathbb{R}^n$ is

$$\begin{aligned} H_C(t,x,p) &:= \sup\{\langle p, v\rangle - L_C(t,x,v) : v \in \mathbb{R}^n\} \\ &= \sup_{u \in U}\{\langle p, f(t,x,u)\rangle - g(t,x,u)\} \\ &= \mathcal{H}(t,x,p), \end{aligned}$$

the Hamiltonian corresponding to (C).

Remark 2.1. Since $\int_0^1 L_C(t,x(t),\dot{x}(t))dt \le \int_0^1 g(t,x(t),u(t))dt$ for all (x,u) admissible for (C), it results that if $\bar{x}$ is strong, $W^{1,1}$-weak or $W^{1,\infty}$-weak local minimum for $(\mathrm{P_C})$ and if there exists $\bar{u}$ such that $(\bar{x},\bar{u})$ is admissible for (C) and

$$L_C(t,\bar{x}(t),\dot{\bar{x}}(t)) = g(t,\bar{x}(t),\bar{u}(t)) \text{ a.e.},$$

then, it follows that $(\bar{x},\bar{u})$ is, respectively, strong, $W^{1,1}$-weak or $W^{1,\infty}$-weak local minimum for (C).

For the converse, extra hypothesis is needed. Assume:

$$(\bar{\mathrm{A}}) \qquad \begin{cases} \bullet \text{ For } (t,x) \in T(\bar{x};\varepsilon) \text{ and } v \in f(t,x,U), \\ \qquad \text{the infimum in (2.2) is attained.} \\ \bullet\ L_C \text{ is } \mathcal{L}\times\mathcal{B}\text{-measurable and lower semicontinuous in } (x,v). \end{cases}$$

With minor modification of the proof of the Equivalence Theorem in [8] we obtain the following.

Theorem 2.1 [8]. *If assumptions* (A) *and* $(\bar{\mathrm{A}})$ *are satisfied, then for all arcs* $x(\cdot)$ *in* $T(\bar{x};\varepsilon)$ *that are admissible for* $(\mathrm{P_C})$ *we have* $J_C(x)$ *is well-defined. Furthermore, if* $J_C(x) < +\infty$ *then*

$$J_C(x) = \min_{u(\cdot)\text{ measurable}} \{J(x,u) : \dot{x}(t) = f(t,x(t),u(t)) \text{ a.e., and } u(t) \in U \text{ a.e.}\}$$

That is, $\bar{x}$ *is strong,* $W^{1,1}$*-weak or* $W^{1,\infty}$*-weak local minimum for* $(\mathrm{P_C})$ *iff there exists a control* $\bar{u}$ *corresponding to* $\bar{x}$ *such that* $(\bar{x},\bar{u})$ *is, respectively, strong,* $W^{1,1}$*-weak or* $W^{1,\infty}$*-weak local minimum for* (C).

In applications it is important to have more verifiable conditions guaranteeing $(\bar{\mathrm{A}})$. For, we shall use the following assumptions:

$(\mathcal{L}_C)$ For $(t,x) \in T(\bar{x};\varepsilon)$, $\{(f(t,x,u), g(t,x,u)+r) : u \in U,\ r \ge 0\}$ is convex and closed.

($\mathcal{B}$) For every *bounded* set $V \subseteq \mathbb{R}^n$, the following set is bounded: $\{u \in U \mid \exists\,(t,x,v) \in T(\bar{x};\varepsilon) \times V :\ v = f(t,x,u)\}$.

Remark 2.2. Assume that $\mathcal{H}$ is $\mathcal{L} \times \mathcal{B}$-measurable and, for $(t,x,p) \in T(\bar{x},\bar{p};\varepsilon)$, $\mathcal{H}$ is upper semicontinuous in x and less than $+\infty$. Then conditions ($\mathcal{L}_C$) and ($\mathcal{B}$) yield that ($\bar{\text{A}}$) is satisfied and $L_C(t,x,\cdot)$ is convex. In fact ($\mathcal{L}_C$) implies that $L(t,x,\cdot)$ is convex, since for $(t,x) \in T(\bar{x};\varepsilon)$ the set defined in ($\mathcal{L}_C$) is the epigraph of $L_C(t,x,\cdot)$. Condition ($\bar{\text{A}}$)(ii) follows by [8, Proposition 4], and condition ($\bar{\text{A}}$)(i) follows from (A) and ($\mathcal{B}$). When U is compact, ($\mathcal{B}$) holds automatically.

In the rest of this section we present notions of controllability and normality that we shall use throughout this paper. For given L^1-matrix functions $A(\cdot)$ and $R(\cdot)$ on $[0,1]$ and an $r \times 2n$-matrix M, consider the linear system

$$\left\{ \begin{array}{ll} \dot{\eta}(t) = A(t)\eta(t) + R(t)v(t) & \text{a.e.,} \\ M \begin{pmatrix} \eta(0) \\ \eta(1) \end{pmatrix} = 0, & v(\cdot) \in L^\infty[0,1]. \end{array} \right. \tag{2.3}$$

The case, when $r = 0$ indicates that η has free endpoints.

Definition 2.5. The system (2.3)(i) is *M-controllability* on $[a,b] \subset [0,1]$ means that for all $\beta \in \mathbb{R}^n$ there exists $\alpha_\beta \in \mathbb{R}^n$ and $v_\beta(\cdot) \in L^\infty[a,b]$ such that the solution on $[a,b]$ of

$$\dot{\eta}(t) = A(t)\eta(t) + R(t)v_\beta(t) \quad \text{a.e.,}$$

$$\eta(a) = \alpha_\beta$$

satisfies

$$M \begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix} = \beta.$$

The M-controllability on $[a,b]$ amounts to saying that

$$\operatorname{Im} \Lambda = \mathbb{R}^r,$$

where

$$\Lambda \begin{pmatrix} \alpha \\ v(\cdot) \end{pmatrix} := M \begin{bmatrix} I \\ Z_b(a) \end{bmatrix} \alpha + M \begin{bmatrix} 0 \\ I \end{bmatrix} \int_a^b Z_b(\tau) R(\tau) v(\tau)\, d\tau,$$

and $Z_b(\cdot)$ is defined by $-\dot{Z}(t) = Z(t)A(t)$ a.e., $Z(b) = I$, here I is the $n \times n$-identity matrix. When $b = 1$ we denote Z_1 by Z.

Remark 2.3. The M-controllability on $[a,b]$ of the system (2.3)(i) is known to be equivalent to its *M-normality*, that is: $(y,\gamma) \equiv 0$ is the only solution on $[a,b]$ to

$$\left\{ \begin{array}{ll} \dot{y}(t) = -A^T(t)y(t) & \text{a.e.} \\ \begin{pmatrix} y(a) \\ -y(b) \end{pmatrix} = M^T\gamma & \\ R^T(t)y(t) = 0 & \text{a.e.} \end{array} \right. \tag{2.4}$$

Hence, when $r = 0$, the system (2.3)(i) is automatically M-controllable.

When the endpoints of η in (2.3) are separated, then for some $r_0 \times n$-matrix M_0 and $r_1 \times n$-matrix M_1 we have $M = \begin{bmatrix} M_0 & 0 \\ 0 & M_1 \end{bmatrix}$. In this case, the M-normality is denoted by $(M_0 : M_1)$-normality and the vector γ in (2.4) takes the form $\gamma = \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}$ where $\gamma_0 \in \mathbb{R}^{r_0}$ and $\gamma_1 \in \mathbb{R}^{r_1}$.

3 Necessary Conditions for (P)

In this section we present the first part of the main results of this paper. It concerns deriving three types of second-order necessary conditions for (weak or strong) optimality in the generalized problem of Bolza (P). The key feature resides in the fact that all these conditions are phrased in terms of the Hamiltonian H given in (1.2). The first condition takes the form of the accessory problem, that is, the nonnegativity of a certain quadratic functional. It plays a fundamental role in obtaining the other two.

Let $\bar{z} := (\bar{x}, \bar{p}) \in W^{1,1}[0,1]$ be given. Throughout the paper we denote by $\bar{\Phi}(t)$ the evaluation $\Phi(t, \bar{z}(t))$, and Φ_x and Φ_p are the partial derivatives of Φ with respect to x and p, respectively. Elements in $\mathbb{R}^n$ are considered to be column vectors, and gradient of real-valued maps evaluated at a point are considered row vectors. The notation $B_n(x_0; \varepsilon)$ refers to the ball in $\mathbb{R}^n$ centered at x_0 of radius ε. When $x_0 = 0$ we write $B_n(\varepsilon)$.

In this section we assume that ℓ and ϕ are C^1 on $B_{2n}((\bar{x}(0), \bar{x}(1); \varepsilon)$ and L is $\mathcal{L} \times \mathcal{B}$-measurable. Set

$$\bar{M} := \nabla\phi(\bar{x}(0), \bar{x}(1)).$$

In stating our main results, we will refer to the following assumptions concerning (P).

There exists $\varepsilon > 0$ such that

($\mathcal{L}$) $L(t, x, \cdot)$ is convex and lower semicontinuous for $(t, x) \in T(\bar{x}; \varepsilon)$.

$$(H_1) \qquad \begin{cases} \bullet\ \forall\, t,\ H(t, \cdot) \text{ is } C^1 \text{ on } B_{2n}(\bar{z}(t); \varepsilon) \\ \bullet\ \bar{H}(\cdot) \text{ and } \overline{\nabla_z H}(\cdot) \text{ are integrable} \\ \bullet\ \nabla_z H \text{ is continuous at } \bar{z} \text{ from } L^\infty \text{ to } L^1. \end{cases}$$

$$(H_\infty) \qquad H_p \text{ is continuous at } \bar{z} \text{ from } L^\infty \text{ to } L^\infty.$$

$$(H_2) \qquad \begin{cases} \bullet\ \forall\, t,\ H(t, \cdot) \text{ is } C^2 \text{ on } B_{2n}(\bar{z}(t); \varepsilon) \text{ with partial derivatives measurable in } t \\ \bullet\ \bar{H}(\cdot),\ \overline{\nabla_z H}(\cdot),\ \overline{\nabla_z^2 H}(\cdot) \text{ are integrable} \\ \bullet\ \nabla_z^2 H \text{ is continuous at } \bar{z}, \text{ from } L^\infty \text{ to } L^1. \\ \bullet\ \ell \text{ and } \phi \text{ are } C^2 \text{ on } B_{2n}((\bar{x}(0), \bar{x}(1)); \varepsilon) \\ \bullet\ \bar{M} \text{ is of full rank.} \end{cases}$$

¿From Lemma 2.1, (H_2) (iii) yields (H_1) (iii). Hence, (H_2) implies (H_1). Also, if we have that $\nabla_z H_p$ is continuous from L^∞ to L^∞, and $\overline{\nabla_z H_p}(\cdot) \in L^\infty$, by Lemma 2.1 we get that (H_∞) holds true.

When no endpoints constraint is present, that is, when $\phi : \mathbb{R}^{2n} \to \{0\}$, then $\bar{M}$ is automatically of full rank.

Definition 3.1. Let $\bar{z} := (\bar{x}, \bar{p}) \in W^{1,1}[0,1]$ at which (H_1) holds. The arc $\bar{z}$ is called a *normal extremal* for (P) if for some $\bar{\gamma} \in \mathbb{R}^r$, it satisfies the Hamiltonian equations and the transversality conditions:

$$\tag{3.1} \begin{cases} -\dot{\bar{p}}^T(t) = \bar{H}_x(t) \quad \text{a.e.}, \\ \dot{\bar{x}}^T(t) = \bar{H}_p(t) \quad \text{a.e.}, \\ (\bar{p}^T(0), -\bar{p}^T(1)) = \nabla \ell(\bar{x}(0), \bar{x}(1)) + \bar{\gamma}^T \bar{M}. \end{cases}$$

Note that when the matrix $\bar{M}$ is of full rank, at most one vector $\bar{\gamma} \in \mathbb{R}^r$ can satisfy (3.1).

Clarke showed in [2] that when problem (P) is calm, $L(t, \cdot, \cdot)$ satisfies certain assumptions including ($\mathcal{L}$), and when H satisfies the strong Lipschitz condition:

$\exists\, K(\cdot) \in L^1[0,1]$, $\exists\, \varepsilon > 0 : \forall p \in \mathbb{R}^n$, $\forall\, (t, y_1)$ and $(t, y_2) \in T(\bar{x}; \varepsilon)$ one has $|H(t, y_1, p) - H(t, y_2, p)| \leq K(t)(1 + |p|)|y_1 - y_2|$,

then a necessary condition for the strong minimality of $\bar{x}$ is the existence of $\bar{p} \in W^{1,1}[0,1]$ satisfying the Hamiltonian Inclusions and the transversality conditions. Hence, if $H(t, \cdot, \cdot)$ is C^1 near $\bar{z} := (\bar{x}, \bar{p})$ then $\bar{z}$ satisfies (3.1). The result in [2] was generalized by Clarke in [3] and also by Loewen and Rockafellar in [5]. When $\bar{x}$ is an $W^{1,\infty}$-weak local minimum for (P), it is proved that there exists an arc $\bar{p} \in W^{1,1}[0,1]$ satisfying the *Euler equations* and the *transversality conditions.* Under extra hypotheses, and if $H(t, \cdot, \cdot)$ is C^1 near $\bar{z} := (\bar{x}, \bar{p})$ these conditions are equivalent to equations (3.1), as was proved by Rockafellar in [9].

Definition 3.2. For a given normal extremal $(\bar{x}, \bar{p})$ with associated $\bar{\gamma}$ the *accessory problem corresponding to* (P) at $(\bar{x}, \bar{p}, \bar{\gamma})$ is defined to be

$$\text{(AP)} \quad \text{minimize } J_2(\eta, v) := \frac{1}{2}\left\langle \Gamma \begin{pmatrix} \eta(0) \\ \eta(1) \end{pmatrix}, \begin{pmatrix} \eta(0) \\ \eta(1) \end{pmatrix} \right\rangle + \frac{1}{2}\int_0^1 \left\{ \langle \bar{H}_{pp}(t) v(t), v(t) \rangle - \langle \bar{H}_{xx}(t) \eta(t), \eta(t) \rangle \right\} dt$$

over $(\eta, v) \in W^{1,1}[0,1] \times L^\infty[0,1]$ *solution of*

$$\tag{3.2} \dot{\eta}(t) = \bar{H}_{px}(t)\eta(t) + \bar{H}_{pp}(t) v(t) \quad \text{a.e.}$$

$$\bar{M} \begin{pmatrix} \eta(0) \\ \eta(1) \end{pmatrix} = 0,$$

where $\Gamma := \nabla^2(\ell + \phi^T\bar{\gamma})|_{(\bar{x}(0),\bar{x}(1))}$.

Necessary conditions for (P) involving the accessory problem are given by the following result.

Theorem 3.1. *Assume that* $\bar{z} := (\bar{x}, \bar{p}) \in W^{1,1}[0,1]$ *satisfy* (H_2) *and* ($\mathcal{L}$), *and form together with* $\bar{\gamma} \in \mathbb{R}^r$ *a normal extremal for* (P). *Suppose that* $\bar{x}$ *is either a* $W^{1,1}$*-weak local minimum or a* $W^{1,\infty}$*-weak local minimum with* (H_∞) *satisfied. Then either the minimum value of the accessory problem is zero or else the* $\bar{M}$*-controllability of (3.2) fails on* $[0,1]$.

Let $Z : [0,1] \to \mathbb{R}^{n\times n}$ denote the fundamental matrix of the system

$$\text{(3.3)} \qquad \begin{cases} -\dot{Z}(t) = Z(t)\bar{H}_{px}(t) & \text{a.e.,} \\ Z(1) = I = \text{Identity matrix in } \mathbb{R}^{n\times n}. \end{cases}$$

Remark 3.1. From Definition 2.5, system (3.2) is *not* $\bar{M}$-controllable on $[0,1]$ means that Im Λ is a proper linear subspace of $\mathbb{R}^r$, where $A(t) := \bar{H}_{px}(t)$, $R(t) := \bar{H}_{pp}(t)$, and $Z_b = Z$. Hence, there exists $\gamma \in \mathbb{R}^r$, $\gamma \neq 0$ satisfying

$$\gamma^T\bar{M}\begin{bmatrix} I \\ Z(0) \end{bmatrix} = 0$$

and

$$\gamma^T\bar{M}\begin{bmatrix} 0 \\ I \end{bmatrix} Z(t)\bar{H}_{pp}(t) = 0 \quad t \in [0,1] \quad \text{a.e.}$$

This is equivalent to saying that there exists $(y,\gamma) \not\equiv 0$ solution on $[0,1]$ of (2.4), where the equivalence is revealed by setting $y(t) := Z^T(t)\bar{M}_R^T\gamma$ with $\bar{M}_R$ is the $r\times n$-matrix in the partition $\bar{M} = (\bar{M}_L\ \bar{M}_R)$. Hence, if the system (3.2) is $\bar{M}$-normal, then the accessory problem must have a zero minimum value. This is the case, for instance, if we have that $\bar{H}_{pp}(t)$ is positive definite and $\bar{M}$ of full rank, since in this case $(y,\gamma) \equiv 0$ is the only solution to the corresponding system (2.4).

The proof of Theorem 3.1, which we present later, will involve the following variational problem.

($\mathrm{C_H}$) minimize $J_C(x,p) := \ell(x(0),x(1)) + \int_0^1\{\langle p(t),\dot{x}(t)\rangle - H(t,x(t),p(t))\}dt$

subject to:

$$\dot{x}^T(t) = H_p(t,x(t),p(t)) \quad \text{a.e.,}$$
$$\phi(x(0),x(1)) = 0,$$

where $x(\cdot) \in W^{1,1}[0,1]$, $p(\cdot) \in L^\infty[0,1]$ and H is the Hamiltonian given by (1.2).

Note that the problem ($\mathrm{C_H}$) is an *optimal control* problem where $x(\cdot)$ is the state and $p(\cdot)$ is the control. Hence, the optimality notions of strong, $W^{1,1}$-weak, and $W^{1,\infty}$-weak local minima introduced in Section 2 and the corresponding classical notions given in Definition 2.4 apply to ($\mathrm{C_H}$).

It is known that the objective function $J_C(x,p)$, when unconstrained, has no local minima nor maxima. However, as we shall see below, when constrained as in ($\mathrm{C_H}$), $J_C(x,p)$ admits a (local) minimum whenever the generalized Bolza problem (P) does.

Proposition 3.1. *Assume that there exists $\bar{z} := (\bar{x}, \bar{p}) \in W^{1,1}[0,1] \times L^{\infty}[0,1]$ that satisfy* ($\mathcal{L}$), ($\mathrm{H_1}$) *and equation* (3.1) (ii). *Then, for x near $\bar{x}$, $J(x)$ is well-defined (possibly $+\infty$), and if $\bar{x}$ is a strong, $W^{1,1}$-weak, $W^{1,\infty}$-weak local minimum for* (P) *then $(\bar{x}, \bar{p})$ is, respectively, a strong, $W^{1,1}$-weak, $W^{1,\infty}$-weak local minimum for* ($\mathrm{C_H}$).

Proof. From the definition of H in equation (1.2) and hypothesis ($\mathrm{H_1}$)(i) it follows that $H(t,x,\cdot)$ is convex and lower semicontinuous for all $(t,x) \in T(\bar{x};\varepsilon)$. Then, by ($\mathcal{L}$) we obtain, for $(t,x,v) \in T(\bar{x};\varepsilon) \times \mathbb{R}^n$,

$$L(t,x,v) = \sup\{\langle p, v\rangle - H(t,x,p) \mid p \in \mathbb{R}^n\}. \tag{3.4}$$

Whence, by ($\mathrm{H_1}$) (ii) - (iii) and Lemma 2.1 it results that there exists $\bar{\delta} > 0$ ($\bar{\delta} < \varepsilon$) and $\gamma(\cdot) \in L^1$ such that for all $(t,z) \in T(\bar{z};\bar{\delta})$,

$$|H(t,z)| \leq \gamma(t). \tag{3.5}$$

Then,

$$L(t,x,v) \geq \langle \bar{p}(t), v\rangle - \gamma(t)$$

and thus, by the $\mathcal{L} \times \mathcal{B}$-measurability of L we get that $J(x)$ is well-defined (possibly $+\infty$) near $\bar{x}$. Using (3.4) it follows that for all $x(\cdot) \in T(\bar{x};\bar{\delta})$

$$\begin{aligned} J(x) \;&:=\; \ell(x(0),x(1)) + \int_0^1 L(t,x(t),\dot{x}(t))dt \\ &=\; \ell(x(0),x(1)) + \int_0^1 \sup_{p\in\mathbb{R}^n}\{\langle p, \dot{x}(t)\rangle - H(t,x(t),p)\}dt. \end{aligned}$$

The convexity of $H(t,x,\cdot)$, and equation (3.1) (ii) yield that

$$\begin{aligned} J(\bar{x}) \;&=\; \ell(\bar{x}(0),\bar{x}(1)) + \int_0^1 \{\langle \bar{p}(t), \dot{\bar{x}}(t)\rangle - \bar{H}(t)\}dt \\ &=\; J_C(\bar{x},\bar{p}). \end{aligned}$$

Furthermore, for any (x,p) that is admissible for ($\mathrm{C_H}$) with $\|x - \bar{x}\|_{\infty} < \bar{\delta}$ we have

$$J(x) = J_C(x,p).$$

Therefore, if there exists $\varepsilon_0 > 0$ such that

$$J(x) \geq J(\bar{x})$$

for all arcs $x(\cdot)$ that are admissible for (P) and satisfying $\|x-\bar{x}\|_\infty < \varepsilon_0$, $\|x-\bar{x}\|_{1,1} < \varepsilon_0$, or $\|x-\bar{x}\|_{1,\infty} < \varepsilon_0$ then, for all (x,p) admissible in ($\mathrm{C_H}$) with $\|x-\bar{x}\|_\infty < \bar{\varepsilon}$, $\|x-\bar{x}\|_{1,1} < \bar{\varepsilon}$, or $\|x-\bar{x}\|_{1,\infty} < \bar{\varepsilon}$, we have

$$J_C(x,p) = J(x) \geq J(\bar{x}) = J_C(\bar{x},\bar{p}),$$

where $\bar{\varepsilon} = \min\{\bar{\delta}, \varepsilon_0\}$.

Combining Lemma 2.2 and Proposition 3.1, we obtain the connection between optimality in (P) and *classical* optimality in ($\mathrm{C_H}$).

Corollary 3.1. *Assume that $\bar{z} := (\bar{x},\bar{p})$ satisfy the conditions of Proposition 3.1. Then,*

(i) *if $\bar{x}$ is a strong local minimum for* (P) *then $(\bar{x},\bar{p})$ is classically L^∞-weak local for* ($\mathrm{C_H}$). *If in addition, for all t, $H(t,\cdot)$ is C^1 on $B_n(\bar{x}(t);\varepsilon)\times\mathbb{R}^n$ then $(\bar{x},\bar{p})$ is a strong local minimum for* ($\mathrm{C_H}$).

(ii) *If $\bar{x}$ is a $W^{1,1}$-weak local minimum for* (P) *then $(\bar{x},\bar{p})$ is classically an L^∞-weak local minimum for* ($\mathrm{C_H}$). *If in addition $H(t,\cdot)$ is C^1 on $B_n(\bar{x}(t);\varepsilon)\times\mathbb{R}^n$ and H_p is continuous at $(\bar{x},\bar{p})$, from $L^\infty \times L^1$ to L^1, then $(\bar{x},\bar{p})$ is classically L^1-weak local minimum for* ($\mathrm{C_H}$).

(iii) *If $\bar{x}$ is a $W^{1,\infty}$-weak local minimum for* (P) *and* (H_∞) *holds true, then $(\bar{x},\bar{p})$ is classically an L^∞-weak local minimum for* ($\mathrm{C_H}$).

The next result shows that if system (3.2) is $\bar{M}$-controllable (see Definition 2.5) and $(\bar{x},\bar{p})$ is a classically L^∞-weak local minimum for the optimal control problem ($\mathrm{C_H}$), then the adjoint variable $q(\cdot)$ corresponding to $(\bar{x},\bar{p})$ in the weak version of the Pontryagin maximum principle coincides with $\bar{p}$.

Lemma 3.1. *Assume that $(\bar{x},\bar{p},q,\gamma)$ satisfy the weak-version of the Pontryagin maximum principle applied to* ($\mathrm{C_H}$). *If the system (3.2) is $\bar{M}$-controllable, $\bar{M}$ is of full rank and $(\bar{x},\bar{p})$ satisfy equations (3.1), then the problem* ($\mathrm{C_H}$) *is normal and $(q,\gamma) = (\bar{p},\bar{\gamma})$, up to scalar multiplication.*

Proof. Let $(\bar{x},\bar{p},q,\gamma)$ satisfy the weak-version of the maximum principle. Then, there exists $\lambda_0 \geq 0$ such that λ_0, γ and q are not all zero, and

$$\begin{aligned}
&-\dot{q}(t) = \bar{H}_{xp}(t)q(t) - \lambda_0[\bar{H}_{xp}(t)\bar{p}(t) - \bar{H}_x(t)] \quad \text{a.e.}\\
&\bar{H}_{pp}(t)[q(t) - \lambda_0\bar{p}(t)] = 0 \quad \text{a.e.},\\
&(q^T(0), -q^T(1)) = \lambda_0\nabla\ell(\bar{x}(0),\bar{x}(1)) + \gamma^T\bar{M}.
\end{aligned}$$

If $\lambda_0 = 0$ then (q,γ) satisfies system (2.4), where $A = \bar{H}_{xp}$ and $R = \bar{H}_{pp}$. By the $\bar{M}$-controllability of the system (3.2) and Remark 2.3 we get that $q \equiv 0$ and $\gamma = 0$. Since λ_0, γ and q are not all zero, we conclude that $\lambda_0 \neq 0$.

Now, assume $\lambda_0 = 1$ and use in the above equations that $(\bar{x}, \bar{p})$ satisfy equations (3.1) we obtain:

$$
\begin{aligned}
&-(\dot{q}(t) - \dot{\bar{p}}(t)) = \bar{H}_{xp}(t)(q(t) - \bar{p}(t)) \quad \text{a.e.,}\\
&\bar{H}_{pp}(t)(q(t) - \bar{p}(t)) = 0 \quad \text{a.e.,}\\
&(q^T(0) - \bar{p}^T(0), -(q^T(1) - \bar{p}^T(1))) = (\gamma^T - \bar{\gamma}^T)\bar{M}.
\end{aligned}
$$

Using the $\bar{M}$-controllability it follows that $q \equiv \bar{p}$. The full rank condition on $\bar{M}$ yields that $\gamma = \bar{\gamma}$.

In order to prove Theorem 3.1 we shall establish the following result which says that the necessary conditions given by Theorem 3.1 are in fact necessary for L^∞-weak local minimality of $(\bar{x}, \bar{p})$ in $(\mathrm{C_H})$.

Theorem 3.2. *Assume that $(\bar{x}, \bar{p}) \in W^{1,1}[0,1]$ satisfies $(\mathrm{H_2})$. If $(\bar{x}, \bar{p})$ is a classically L^∞-weak local minimum for $(\mathrm{C_H})$, then the conclusions of Theorem 3.1 are satisfied.*

To prove Theorem 3.2 we intend to calculate the second variation of the optimal control problem $(\mathrm{C_H})$. For achieving this goal, one could attempt to apply known second-order necessary conditions. However, all known results (see [7], [11] and the references therein), require the dynamics and the integrand to be at least *twice directionally* differentiable. Since $H_p(t,\cdot,\cdot)$ is only C^1 and is involved in both the dynamics and the integrand of $(\mathrm{C_H})$, those results cannot be applied here. Nevertheless, as we shall see below in the proof of Theorem 3.2, the second variation of $(\mathrm{C_H})$ can be derived, by taking its special structure into consideration. We shall need the following result which says that the $\bar{M}$-controllability of system (3.2) allows us to associate with each pair (η, v) solving that system, an admissible family $(x(\cdot,\alpha), p(\cdot,\alpha))_{\alpha\in(-\delta,\delta)}$ for the problem $(\mathrm{C_H})$ that includes $(\bar{x}, \bar{p})$ at $\alpha = 0$ and has a first derivative w.r.t. α at $\alpha = 0$ equal to (η, v).

Proposition 3.2. *Assume $(\mathrm{H_2})$ to hold at the admissible pair $\bar{z} = (\bar{x}, \bar{p})$ and that the system (3.2) is $\bar{M}$-controllable. Then, for every $(\eta, v) \in W^{1,1}[0,1] \times L^\infty[0,1]$ satisfying system (3.2), there exists an admissible family $(x(\cdot;\alpha), p(\cdot;\alpha))_{\alpha\in(-\delta,\delta)}$ for $(\mathrm{C_H})$ such that $(x(t;\cdot),\ p(t;\cdot))$ is C^1, $\left(\frac{\partial x}{\partial \alpha}, \frac{\partial p}{\partial \alpha}\right)$ is continuous at 0 from $\mathbb{R}$ to L^∞ and*

(i) $x(t;0) = \bar{x}(t),\ p(t;0) = \bar{p}(t) \quad \forall\, t \in [0,1]$,

(ii) $\frac{\partial x}{\partial \alpha}(t;0) = \eta(t),\ \frac{\partial p}{\partial \alpha}(t;0) = v(t) \quad \forall\, t \in [0,1]$.

Proof. Let $\beta_1, \dots, \beta_r$ be r linearly independent vectors in $\mathbb{R}^r$. By the $\bar{M}$-controllability of system (3.2), there exist functions $v_1, \dots, v_r \in L^\infty[0,1]$ and $\eta_1, \dots, \eta_r$ in $W^{1,1}[0,1]$ satisfying

$$
\left\{
\begin{aligned}
&\dot{\eta}_j(t) = \bar{H}_{px}(t)\eta_j(t) + \bar{H}_{pp}(t)v_j(t) \quad \text{a.e.t.,}\\
&\bar{M}\begin{pmatrix} \eta_j(0) \\ \eta_j(1) \end{pmatrix} = \beta_j.
\end{aligned}
\right. \tag{3.6}
$$

Define on $[0,1]\times\mathbb{R}\times\mathbb{R}^r$

$$p(t;\alpha,\lambda) := \bar{p}(t) + \alpha v(t) + \sum_{j=1}^{r} \lambda_j v_j(t). \tag{3.7}$$

Consider the system

$$\begin{cases} \dot{x}^T(t) = H_p(t, x(t), p(t,\alpha,\lambda)) \\ x(0) = \bar{x}(0) + \alpha\eta(0) + \sum_{j=1}^{r} \lambda_j \eta_j(0). \end{cases} \tag{3.8}$$

Clearly when $(\alpha,\lambda) = (0,0)$, System (3.8) has $\bar{x}$ as the solution. Hypothesis (H_2) and Lemma 2.1 yield that for

$$F(t,x,\alpha,\lambda) := H_p(t,x,p(t,\alpha,\lambda))$$

F and $\nabla_{(x,\alpha,\lambda)}F$ are Carathéodory on some tube $T(\bar{x},(0,0);\varepsilon_0)$ for $\varepsilon_0 \le \varepsilon$. Hence, by the Embedding Theorem of Differential Equations [10, II.4.11], it results that there exists $\bar{\delta} > 0(\bar{\delta} < \varepsilon_0)$ such that for $(\alpha,\lambda) \in B_{r+1}(\bar{\delta})$ and for $x_0 \in B_n(\bar{x}(0);\bar{\delta})$, the system (3.8)(i) has a unique solution $x(\cdot;x_0,\alpha,\lambda)$ in $W^{1,1}[0,1]$ satisfying $x(0;x_0,\alpha,\lambda) = x_0$. Furthermore, on $B_{r+1}(\bar{\delta}) \times B_n(\bar{x}(0);\bar{\delta}), x(t,\cdot,\cdot,\cdot)$ has a derivative that is continuous, L^∞-uniformly in t. However, by (3.8)(ii), there exists $\delta_0 > 0$ ($\delta_0 \le \bar{\delta}$) such that for $(\alpha,\lambda) \in B_{r+1}(\delta_0)$ the unique solution to (3.8) $x(t;\alpha,\lambda) := x(t;x_{\alpha,\lambda},\alpha,\lambda)$, where $x_{\alpha,\lambda} := \bar{x}(0) + \alpha\eta(0) + \sum_{j=1}^{r}\lambda_j\eta_j(0)$, has a derivative with respect to (α,λ) that is continuous at 0 from $\mathbb{R}^{r+1}$ to L^∞.

Differentiate the system (3.8) with respect to (α,λ) and use (3.2), (3.6) and (3.7) to get

$$\begin{cases} \dfrac{\partial x}{\partial \alpha}(\cdot;0,0) = \eta(\cdot), \quad \dfrac{\partial p}{\partial \alpha}(\cdot;0,0) = v(\cdot), \\ \dfrac{\partial x}{\partial \lambda_j}(\cdot;0,0) = \eta_j(\cdot), \quad \dfrac{\partial p}{\partial \lambda_j}(\cdot;0,0) = v_j(\cdot) \quad \forall\, j = 1,\dots,r. \end{cases} \tag{3.9}$$

Defined $G : B(\delta_0)\times B_r(\delta_0) \to \mathbb{R}^r$ by

$$(\alpha,\lambda) \to \phi(x(0;\alpha,\lambda), x(1;\alpha,\lambda)).$$

We have: $G(0,0) = 0$, $\dfrac{\partial G(0,0)}{\partial \lambda_j} = \bar{M}\begin{pmatrix} \eta_j(0) \\ \eta_j(1) \end{pmatrix} = \beta_j$ and thus, $\dfrac{\partial G}{\partial \lambda}(0,0)$ is nonsingular. By the implicit function theorem, there exists $\delta < \delta_0$ and a C^1-function $\lambda(\cdot) : B(\delta) \to \mathbb{R}^r$ with $\lambda(0) = 0$, $\lambda(B(\delta)) \subset B(\delta_0)$ and

$$\phi(x(0;\alpha,\lambda(\alpha)), x(1;\alpha,\lambda(\alpha))) = 0. \tag{3.10}$$

Differentiate (3.10) with respect to α at $\alpha = 0$ and use (3.9), (3.6) and the fact the (η, v) satisfies (3.2) we get $\sum_{j=1}^{r}\dot{\lambda}_j(0)\beta_j = 0$, which yields $\dot{\lambda}_j(0) = 0$, for all j.

Now, set on $[0,1]\times(-\delta,\delta)$

$$p(t;\alpha) := p(t;\alpha,\lambda(\alpha))$$

$$x(t;\alpha) := x(t;\alpha,\lambda(\alpha)).$$

Using (3.7) and (3.9), we obtain the result of this proposition.

We are now ready to prove Theorem 3.2. As we shall see in the proof, the map

$$\alpha \to J_C(x(\cdot;\alpha), p(\cdot;\alpha))$$

is twice differentiable at 0, even though the functions defining $J_C(x,p)$ are not themselves twice differentiable.

Proof of Theorem 3.2. Let $(\bar{x}, \bar{p})$ satisfy (3.1).

Assume that system (3.2) is $\bar{M}$-controllable and that $(\bar{x}, \bar{p})$ provides a classically L^∞-weak local minimum for $(\mathrm{C_H})$. Thus, $(\bar{x}, \bar{p})$ must satisfy the weak version of the Pontryagin maximum principle. By Lemma 3.1, we deduce that $(\mathrm{C_H})$ is normal and the adjoint variable corresponding to $(\bar{x}, \bar{p})$ is $\bar{p}$.

Let (η, v) satisfy system (3.2). Then, by Proposition 3.2, there exists a family $(x(\cdot;\alpha), p(\cdot;\alpha))_{\alpha\in(-\delta,\delta)}$ that is admissible for $(\mathrm{C_H})$ and satisfies the conditions stated in Proposition 3.2. In particular, using Lemma 2.1 it follows that there exists $\gamma > 0 (\gamma \leq \delta)$ and $\mu > 0$ such that, for almost all t and for $\alpha, \alpha' \in (-\gamma, \gamma)$,

$$|(x(t,\alpha), p(t,\alpha)) - (x(t;\alpha'), p(t,\alpha'))| \leq \mu|\alpha - \alpha'| \tag{3.11}$$

Thus, choose γ small enough so that $(\bar{x}, \bar{p})$ is optimal for J_C on $T(\bar{x}, \bar{p}; \mu\gamma)$. Hence, by setting $\mathbb{H}(t;\alpha) := H(t, x(t;\alpha), p(t;\alpha))$ and $\mathbb{J}_C(\alpha) := J_C(x(\cdot;\alpha), p(\cdot;\alpha))$, we get that $\alpha = 0$ is a minimum for $\mathbb{J}_C(\alpha)$ over $(-\gamma, \gamma)$. On the other hand, Hypothesis $(\mathrm{H_2})$ and Lemma 2.1 yield that there exist a neighborhood of 0, $(-\bar{\gamma}, \bar{\gamma})$, where $\mathbb{H}, \mathbb{H}_p$ and $\mathbb{H}_x$ are Lipschitz from $\mathbb{R}$ to L^1 and $\mathbb{H}_p(t;\cdot)$ is bounded by an integrable function of t. Set $\gamma_0 := \min(\gamma, \bar{\gamma})$. Thus, for $\alpha \in (-\gamma_0, \gamma_0)$ we have

$$\begin{aligned}
\mathbb{J}_C(\alpha) &:= \ell(x(0;\alpha), x(1;\alpha)) + \int_0^1 \{\mathbb{H}_p(t;\alpha)p(t;\alpha) - \mathbb{H}(t;\alpha)\}dt \\
&= \ell(x(0;\alpha), x(1;\alpha)) + \int_0^1 \{\mathbb{H}_p(t;\alpha)(p(t;\alpha) - \bar{p}(t)) + \dot{x}^T(t;\alpha)\bar{p}(t) - \mathbb{H}(t;\alpha)\}dt \\
&= \ell(x(0;\alpha), x(1;\alpha)) + \bar{p}^T(1)x(1;\alpha) - \bar{p}^T(0)x(0;\alpha) \\
&\quad + \int_0^1 \{\mathbb{H}_p(t;\alpha)(p(t;\alpha) - \bar{p}(t)) - \mathbb{H}(t;\alpha) - \dot{\bar{p}}^T(t)x(t;\alpha)\}dt.
\end{aligned}$$

We shall show that $\mathbb{J}_C(\cdot)$ is twice differentiable at 0. Since on $(-\gamma_0, \gamma_0)$, x and p are Lipschitz from $\mathbb{R}$ to L^∞ (see (3.11)), $\mathbb{H}$ and $\mathbb{H}_p$ are Lipschitz near 0 from $\mathbb{R}$ to L^1, $\mathbb{H}_p(t;\cdot)$ is bounded by an L^1-function of t and ℓ is Lipschitz, it follows from the Dominated Convergence theorem and (3.1)(iii) that for $\bar{\alpha} \in (-\gamma_0, \gamma_0)$ and for $x_\alpha = \frac{\partial x}{\partial \alpha}$, $p_\alpha = \frac{\partial p}{\partial \alpha}$, we have

$$\begin{aligned}
\mathbb{J}'_C(\bar{\alpha}) &= (\nabla \ell(x(0;\bar{\alpha}), x(1;\bar{\alpha})) - \nabla \ell(\bar{x}(0), \bar{x}(1)) - \bar{\gamma}^T \bar{M}) \begin{pmatrix} x_\alpha(0;\bar{\alpha}) \\ x_\alpha(1;\bar{\alpha}) \end{pmatrix} \\
&\quad + \int_0^1 \big\{ (\mathbb{H}_{px}(t;\bar{\alpha}) x_\alpha(t;\bar{\alpha}) + \mathbb{H}_{pp}(t;\bar{\alpha}) p_\alpha(t;\bar{\alpha}))^T (p(t;\bar{\alpha}) - \bar{p}(t)) \\
&\quad - (\mathbb{H}_x(t;\bar{\alpha}) + \dot{\bar{p}}^T(t)) x_\alpha(t;\bar{\alpha}) \big\} dt.
\end{aligned}$$

Using equations (3.1), system (3.2) and Proposition 3.2 we obtain

$$\mathbb{J}'_C(0) = 0.$$

Let us calculate $\mathbb{J}''_C(0)$. For $h \in (-\gamma_0, \gamma_0)$ and $i \in \{1, \ldots, r\}$ we have

$$\phi^i(x(0;h), x(1;h)) = 0,$$

hence,

$$\nabla \phi^i(x(0;h), x(1;h)) \begin{bmatrix} x_\alpha(0;h) \\ x_\alpha(1;h) \end{bmatrix} = 0.$$

Thus, by the Mean Value Theorem applied to each partial derivative $\partial_{x_j} \phi^i$ and $\partial_{x_j} \ell$, where $j = 1, \ldots, 2n$, it results that there exist h^i_j and $\tilde{h}_j$ in between 0 and h such that, for $\tilde{h}_j$, $\bar{M} = \begin{bmatrix} \bar{M}^1 \\ \vdots \\ \bar{M}^r \end{bmatrix}$, and $x_\alpha = \begin{pmatrix} x^1_\alpha \\ \vdots \\ x^n_\alpha \end{pmatrix}$ we have

$$\begin{aligned}
-\frac{\bar{M}^i}{h} \begin{pmatrix} x_\alpha(0;h) \\ x_\alpha(1;h) \end{pmatrix} &= \left[\frac{\nabla \phi^i(x(0;h), x(1;h)) - \bar{M}^i}{h} \right] \begin{pmatrix} x_\alpha(0;h) \\ x_\alpha(1;h) \end{pmatrix} \\
&= \sum_{j=1}^{n} x^j_\alpha(0;h) \nabla \partial_{x_j} \phi^i(x(0,h^i_j), x(1,h^i_j)) \begin{pmatrix} x_\alpha(0;h^i_j) \\ x_\alpha(1;h^i_j) \end{pmatrix} \\
&\quad + \sum_{j=n+1}^{2n} x^{j-n}_\alpha(1;h) \nabla \partial_{x_j} \phi^i(x(0,h^i_j), x(1,h^i_j)) \begin{pmatrix} x_\alpha(0;h^i_j) \\ x_\alpha(1;h^i_j) \end{pmatrix}
\end{aligned}$$

and

$$\frac{\partial_{x_j} \ell(x(0;h), x(1;h)) - \partial_{x_j} \ell(\bar{x}(0), \bar{x}(1))}{h} = \nabla \partial_{x_j} \ell(x(0;\tilde{h}_j), x(1;\tilde{h}_j)) \begin{pmatrix} x_\alpha(0;\tilde{h}_j) \\ x_\alpha(1;\tilde{h}_j) \end{pmatrix}.$$

Whence

$$\lim_{h\to 0} \frac{(\nabla \ell(x(0;h), x(1;h)) - \nabla \ell(\bar{x}(0), \bar{x}(1)) - \bar{\gamma}^T \bar{M})}{h} \begin{pmatrix} x_\alpha(0;h) \\ x_\alpha(1;h) \end{pmatrix}$$

$$= \left\langle \Gamma \begin{pmatrix} \eta(0) \\ \eta(1) \end{pmatrix}, \begin{pmatrix} \eta(0) \\ \eta(1) \end{pmatrix} \right\rangle$$

where $\Gamma := \nabla^2(\ell + \phi^T \bar{\gamma})\Big|_{(\bar{x}(0), \bar{x}(1))}$.

On the other hand, from (H_2) and the Lipschitz property of x and p in h obtained in Proposition 3.2, it results that the integrand

$$\int_0^1 \{(\mathbb{H}_{px}(t;h)x_\alpha(t;h) + \mathbb{H}_{pp}(t;h)p_\alpha(t;h))^T \frac{(p(t;h) - \bar{p}(t))}{h} - \frac{(\mathbb{H}_x(t;h) - \mathbb{H}_x(t;0))}{h} x_\alpha(t;h)\} dt$$

is bounded by an integrable function. Therefore, by the Dominated Convergence Theorem we get

$$\begin{aligned} \mathbb{J}_C''(0) := \lim_{h\to 0} \frac{\mathbb{J}'(h)}{h} &= \left\langle \Gamma \begin{pmatrix} \eta(0) \\ \eta(1) \end{pmatrix}, \begin{pmatrix} \eta(0) \\ \eta(1) \end{pmatrix} \right\rangle \\ &\quad \int_0^1 \{v^T(t)\bar{H}_{px}(t)\eta(t) + v^T(t)\bar{H}_{pp}(t)v(t) - \eta^T(t)\bar{H}_{xx}(t)\eta(t) \\ &\qquad - \eta^T(t)\bar{H}_{xp}(t)v(t)\}dt \\ &= J_2(\eta, v) \end{aligned}$$

Now, since $\min\{\mathbb{J}_C(\alpha) : \alpha \in (-\gamma_0, \gamma_0)\} = \mathbb{J}_C(0)$ and $\mathbb{J}_C'(0) = 0$, then

$$\mathbb{J}_C''(0) \geq 0.$$

Therefore the proof of the theorem is completed.

Proof of Theorem 3.1. Let $\bar{x}$ be, for (P), either L^1-weak local minimum, or L^∞-weak local minimum with (H_∞) satisfied. Consider $\bar{z} = (\bar{x}, \bar{p})$ as in the Theorem. Then, by Corollary 3.1, it results that $(\bar{x}, \bar{p})$ provides a classically L^∞-weak local minimum for ($\mathrm{C_H}$). Therefore, by Theorem 3.2 the proof is completed.

Now, for the rest of this section, assume that the endpoints costs and constraints in (P) are separable, that is,

$$\ell(x, y) := \ell_0(x) + \ell_1(y)$$

$$\phi(x,y) := \begin{pmatrix} \phi_0(x) \\ \phi_1(y) \end{pmatrix}$$

where $\phi_i : \mathbb{R}^n \to \mathbb{R}^{r_i}$ for $i = 0,1$, and $r_0 + r_1 = r$.

In this case, the matrices $\bar{M}$ and Γ in the accessory problem (AP) take the form

$$\bar{M} = \begin{bmatrix} \bar{M}_0 & 0 \\ 0 & \bar{M}_1 \end{bmatrix} \text{ and } \Gamma = \begin{bmatrix} \Gamma_0 & 0 \\ 0 & \Gamma_1 \end{bmatrix},$$

where

$$\begin{aligned} \bar{M}_0 &:= \nabla\phi_0(\bar{x}(0)), & \bar{M}_1 &= \nabla\phi_1(\bar{x}(1)) \\ \Gamma_0 &:= \nabla^2(\ell_0 + \phi_0^T\bar{\gamma}_0)\big|_{x=\bar{x}(0)}, & \Gamma_1 &= \nabla^2(\ell_1 + \phi_1^T\bar{\gamma}_1)\big|_{x=\bar{x}(1)} \end{aligned}$$

and

$$\begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix} := \gamma.$$

Here $\bar{M}_0 \in \mathbb{R}^{r_0\times n}$, $\bar{M}_1 \in \mathbb{R}^{r_1\times n}$, $\gamma_0 \in \mathbb{R}^{r_0}$ and $\gamma_1 \in \mathbb{R}^{r_1}$.

Note that when no final constraint is present that is, when $\phi_1 : \mathbb{R}^n \to \{0\}$, then $\bar{M}_1$ is automatically of full rank.

Without loss of generality we assume that $\bar{M}_0$ and $\bar{M}_1$ are in fact orthogonal projections in $\mathbb{R}^n$. This can easily be done by taking instead of $\bar{M}_i$ the orthogonal projections

$$\bar{M}_i^T(\bar{M}_i\bar{M}_i^T)^{-1}\bar{M}_i, \quad i = 0,1,$$

where the $\bar{M}_i$'s are of full rank, by (H_2)(v).

When system (3.2) is $(\bar{M}_0 : \bar{M}_1)$-normal, the optimality of (η, v) in (AP) yields the existence of an arc q and vectors γ_0 and γ_1 satisfying on $[0,1]$

$$\left\{ \begin{aligned} \dot{\eta}(t) &= A(t)\eta(t) + R(t)q(t) \quad \text{a.e.} \\ -\dot{q}(t) &= A^T(t)q(t) + P(t)\eta(t) \quad \text{a.e.} \end{aligned} \right. \tag{3.12}$$

$$\left\{ \begin{aligned} q(0) &= \Gamma_0\eta(0) + \bar{M}_0\gamma_0 \\ -q(1) &= \Gamma_1\eta(1) + \bar{M}_1\gamma_1, \end{aligned} \right.$$

where

$$A(t) := \bar{H}_{px}(t), \quad R(t) := \bar{H}_{pp}(t) \quad \text{and} \quad P(t) := \bar{H}_{xx}(t). \tag{3.13}$$

Consider (U_1, V_1) the solution of the matrix system corresponding to (3.12):

$$\left\{ \begin{aligned} \dot{U}(t) &= A(t)U(t) + R(t)V(t) \quad \text{a.e.} \\ -\dot{V}(t) &= A^T(t)V(t) + P(t)U(t) \quad \text{a.e.} \end{aligned} \right. \tag{3.14}$$

with final conditions

$$\begin{cases} U(1) = I - \bar{M}_1 \\ V(1) = -\Gamma_1(I - \bar{M}_1) - \bar{M}_1. \end{cases} \tag{3.15}$$

The following result states that the existence of a conjoined basis on $(0,1)$ for (3.14) and (3.15) with certain initial conditions is *necessary* for the optimality in (P) or in $(\mathrm{C_H})$.

Theorem 3.3. *Let $\bar{z} := (\bar{x}, \bar{p})$ satisfy the conclusions of Theorem 3.1. If, in addition, for all $c \in (0,1)$ the system (3.2) is $(\bar{M}_0 : I)$-normal on $[0,c]$ and $(I : \bar{M}_1)$-normal on $[c,1]$, then (U_1, V_1) satisfies*

(i) $U^T V = V^T U$ *on* $[0,1]$,

(ii) $\det U(t) \neq 0$ *on* $(0,1)$

(iii) $U^T(0)(\Gamma_0 U(0) - V(0)) \geq 0$ *on* $K := \{\alpha : \bar{M}_0 U(0)\alpha = 0\}$,

and

(iv) $V(1) + \Gamma_1(I - \bar{M}_1) + \bar{M}_1 = 0.$

Proof. From the normality hypotheses it follows that the accessory problem (AP) has a minimum value equal to zero. Thus, by [16, Theorem 4.1] the result of the theorem follows.

An immediate consequence of the above theorem is an important result on necessary conditions concerning the Riccati equation.

Corollary 3.2. *Under the conditions of Theorem 3.3, there exists on $(0,1)$ a symmetric absolutely continuous solution W of*

$$\dot{W}(t) + A^T(t)W(t) + W(t)A(t) + W(t)R(t)W(t) + P(t) = 0 \tag{$\mathcal{R}$}$$

satisfying

$$\begin{cases} \lim_{t\to 1} W(t)U_1(t) = -\Gamma_1(I - \bar{M}_1) - \bar{M}_1, \\ U_1^T(0) \lim_{t\to 0^+} (\Gamma_0 - W(t))U_1(t) \geq 0, \\ \qquad\qquad \text{on } K := \{\alpha \mid \bar{M}_0 U_1(0)\alpha = 0\}. \end{cases} \tag{3.16}$$

Remark 3.2. When the final state endpoint is free, then $\bar{M}_1 = 0$ and (3.15) yield that $U_1(1) = I$. In this case the function W in Corollary 3.2 is defined on $(0,1]$ and equation (3.16)(i) reduces to

$$W(1) = -\Gamma_1. \tag{3.17}$$

If also the initial condition is fixed, (3.16)(ii) is trivially satisfied and thus, (3.16) becomes simply (3.17).

By symmetry, we could define (U_0, V_0) as the solution of (3.14) and, instead of (3.15),

$$U(0) = I - \bar{M}_0$$
$$V(0) = \Gamma_0(I - \bar{M}_0) + \bar{M}_0.$$

Then, necessary conditions for (P) and ($\mathrm{C_H}$) parallel to Theorem 3.3 and Corollary 3.2 could be phrased in terms of (U_0, V_0).

The following example illustrates the utility of the results in this section.

Example 3.1. Define on $[0,1] \times \mathbb{R} \times \mathbb{R}$ the function

$$L(t,x,v) := \begin{cases} x^3 + \dfrac{v^2}{(t-\frac{1}{2})^{2/3}} & \text{if } t \neq \dfrac{1}{2} \\ x^3 & \text{if } t = \dfrac{1}{2} \& v = 0 \\ +\infty & \text{if } t = \dfrac{1}{2} \& v \neq 0. \end{cases}$$

Consider the problem

$$(\mathrm{P}_0) \qquad \text{minimize} \qquad -\frac{5}{3}2^{5/3}x^2(1) + \int_0^1 L(t,x(t),\dot{x}(t))dt$$
$$\text{s.t.} \qquad x(0) = 0.$$

The Hamiltonian corresponding to (P_0) is

$$H(t,x,p) = \sup_{v\in\mathbb{R}}\{pv - L(t,x,v)\} = -x^3 + \frac{p^2}{4}(t-\frac{1}{2})^{2/3}.$$

Take $\bar{x} \equiv 0$ and $\bar{p} \equiv 0$. Then L satisfies $(\mathcal{L})$, and H satisfies Hypotheses (H_2) and (H_∞). Here, $\bar{M}_0 = I$, $\bar{M}_1 = 0$. For $\bar{\gamma} = \binom{0}{0}$, $(\bar{x}, \bar{p})$ with $\bar{\gamma}$ form a normal extremal for (P_0). In this problem, $A(t) \equiv 0$, $R(t) = \dfrac{(t-\frac{1}{2})^{2/3}}{2}$ and $P(t) = 0$. Since the system $\dot{y}(t) \equiv 0$ and $(t-\frac{1}{2})^{2/3}y(t) \equiv 0$ only admits $y \equiv 0$ as a solution on any internal $[a,b] \subset [0,1]$, then the normality assumptions in Theorem 3.3 are satisfied.

The Riccati equation $(\mathcal{R})$ and the boundary condition (3.17) are equivalent here to

$$(3.18) \qquad \dot{W}(t) + \frac{(t-\frac{1}{2})^{2/3}}{2}W^2(t) = 0, \qquad \text{and} \qquad W(1) = \frac{10}{3}2^{5/3}.$$

The solution is $W(t) = -\dfrac{10}{3(t-\frac{1}{2})^{5/3}}$, which exists on $(\frac{1}{2}, 1]$ and not on $(0,1]$. Hence, the conclusion of Corollary 3.2 fails. Therefore, by Corollary 3.2, $\bar{x} \equiv 0$ is not a $W^{1,1}$-weak local minimum for (P_0). Since (H_∞) holds, also $\bar{x} \equiv 0$ is not a $W^{1,\infty}$-weak local minimum for (P_0).

4 Sufficient Conditions for (P)

In this section we first derive for the generalized problem of Bolza (P) a sufficiency criterion for strong local optimality. The conditions therein involve the "Weierstrass-type" condition, and the Riccati equation $(\mathcal{R})$ of Corollary 3.2 whose coefficients are given by (3.13), together with the (joint) boundary conditions

$$\Gamma + \begin{bmatrix} -W(0) & 0 \\ 0 & W(1) \end{bmatrix} > 0 \quad \text{on} \quad \{(\alpha,\beta) : \bar{M}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0\}, \tag{4.1}$$

where $\Gamma = \nabla^2(\ell + \bar{\gamma}^T\phi)\Big|_{(\bar{x}(0),\bar{x}(1))}$.

This result extends the corresponding results in [13] and [15] to the case when both endpoints of x varies (i.e., $\phi(x(0), x(1)) = 0$).

Theorem 4.1. *Assume that L is $\mathcal{L}\times\mathcal{B}$-measurable and that, for a vector $\bar{\gamma}\in\mathbb{R}^r$, $\bar{z} := (\bar{x},\bar{p})$ is a normal extremal for* (P) *at which* (H_2) *holds true. Suppose that*

(i) $L(t,\bar{x}(t),\dot{\bar{x}}(t)+v) - L(t,\bar{x}(t),\dot{\bar{x}}(t)) \ge \langle\bar{p}(t), v\rangle$ *for almost all* $t\in[0,1]$ *and for all* $v\in\mathbb{R}^n$;

(ii) *there exists a symmetric absolutely continuous solution on* $[0,1]$ *to the Riccati equation* $(\mathcal{R})$ *with the boundary conditions* (4.1).

Then $J(x)$ is well defined (possibly $+\infty$) near $\bar{x}$, and $\bar{x}$ provides a strict strong local minimum for (P). *Furthermore, there exist $\bar{\varepsilon} > 0$ and $\bar{\delta} > 0$ such that for all admissible arcs x satisfying $\|x-\bar{x}\|_\infty < \bar{\delta}$ we have*

$$J(x) - J(\bar{x}) \ge \bar{\varepsilon}\|x-\bar{x}\|_2^2.$$

Proof. Define $\bar{L}(t,x,v) := \sup\{\langle p, v\rangle - H(t,x,p) : p\in\mathbb{R}^n\}$. By Hypotheses (H_2) and Lemma 2.1, it results that there exists $\bar{\delta} > 0$ and $\gamma(\cdot)\in L^1[0,1]$ with

$$|H(t,z)| \le \gamma(t) \quad \forall\,(t,z)\in T(\bar{z};\bar{\delta}),$$

and hence,

$$\bar{L}(t,x,v) \ge \langle\bar{p}(t), v\rangle - \gamma(t).$$

Since

$$L(t,x,v) \ge \bar{L}(t,x,v),$$

we conclude that $J(x)$ is well-defined (possibly $+\infty$) near $\bar{x}$.

Now, by the Embedding Theorem of Differential Equations there exists $\varepsilon_0 > 0$ and $\bar{W}$ on $[0,1]$ such that

$$\dot{\bar{W}} + A^T\bar{W} + \bar{W}A + \bar{W}R\bar{W} + P = -\varepsilon_0 I \quad \text{for} \quad t\in[0,1] \text{ a.e.}$$

and

$$\Gamma + \begin{pmatrix} -\bar{W}(0) & 0 \\ 0 & \bar{W}(1) \end{pmatrix} > \varepsilon_0 I \quad \text{on} \quad \{(\alpha,\beta) : \bar{M}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0\}. \tag{4.2}$$

Set

$$p(t,x) := \bar{p}(t) + \bar{W}(t)(x - \bar{x}(t)),$$

and

$$\begin{aligned} F(x(\cdot)) \;\; := \;\; & \int_0^1 \{H(t,x(t),p(t,x(t))) - \bar{H}(t) \\ & + \langle \dot{\bar{p}}(t), x(t) - \bar{x}(t)\rangle - \langle \dot{\bar{x}}(t), \bar{W}(t)(x(t) - \bar{x}(t))\rangle \\ & + \frac{1}{2}\langle x(t) - \bar{x}(t), \dot{\bar{W}}(t)(x(t) - \bar{x}(t))\rangle\}dt, \end{aligned} \tag{4.3}$$

where $x(\cdot)$ is continuous. We have that F is twice Gateaux differentiable near $\bar{x}$ with first and second derivatives in the direction $h(\cdot)$ are

$$\begin{aligned} \delta F(x(\cdot);h(\cdot)) \;\; = \;\; & \int_0^1 \langle H_x(t,x(t),p(t,x(t))) + H_p(t,x(t),p(t,x(t)))\bar{W}(t) \\ & + \dot{\bar{p}}^T(t) - \dot{\bar{x}}^T(t)\bar{W}(t) + (x(t) - \bar{x}(t))^T\dot{\bar{W}}(t), h(t)\rangle dt, \end{aligned}$$

and

$$\begin{aligned} \delta^2 F(x(\cdot);h(\cdot)) \;\; = \;\; & \int_0^1 \langle\{H_{xx}(t,x(t),p(t,x(t))) + H_{xp}(t,x(t),p(t,x(t)))\bar{W}(t) \\ & + \bar{W}(t)H_{px}(t,x(t),p(t,x(t))) + \bar{W}(t)H_{pp}(t,x(t),p(t,x(t)))\bar{W}(t) \\ & + \dot{\bar{W}}(t)\}h(t), h(t)\rangle dt. \end{aligned}$$

Also, using (3.1) it results that, for all continuous $h(\cdot)$,

$$F(\bar{x}(\cdot)) = 0, \quad \delta F(\bar{x}(\cdot);h(\cdot)) = 0 \quad \text{and}$$

$$\delta^2 F(\bar{x}(\cdot);h(\cdot)) = -\varepsilon_0 \int_0^1 |h(t)|^2 dt.$$

Hypotheses (H_2) yields that there exists $\delta_0 > 0$ such that for all continuous $x(\cdot)$ and $h(\cdot)$ satisfying $\|x - \bar{x}\|_\infty < \delta_0$ we have

$$\delta^2 F(x(\cdot);h(\cdot)) \le -\frac{\varepsilon_0}{2}\int_0^1 |h(t)|^2 dt.$$

Therefore, by [17, Theorem 40 A], for all continuous $x(\cdot) : \|x - \bar{x}\|_\infty < \delta_0$, we have

$$F(x(\cdot)) - F(\bar{x}(\cdot)) \le -\frac{\varepsilon_0}{4}\|x - \bar{x}\|_2^2. \tag{4.4}$$

Now, define

$$Q(t,x) := \langle \bar{p}(t), x-\bar{x}(t)\rangle + \frac{1}{2}\langle x-\bar{x}(t), \bar{W}(t)(x-\bar{x}(t))\rangle. \tag{4.5}$$

Let $\|x-\bar{x}\|_\infty < \delta_0$ with x admissible for (P). Then, using (4.3), (4.5), (1.2) and condition (i) of the Theorem we obtain:

$$\begin{aligned}&\int_0^1 \{(L(t,x(t),\dot{x}(t)) - L(t,\bar{x}(t),\dot{\bar{x}}(t))\}dt \\ &\quad \geq F(\bar{x}(\cdot)) - F(x(\cdot)) + \int_0^1 \{\frac{d}{dt}\{Q(t,x(t))\}dt.\end{aligned}$$

Thus, from (4.4), (4.5) and by integrating the last term we get:

$$\begin{aligned} J(x) - J(\bar{x}) \;\; \geq \;\; & \ell(x(0),x(1)) - \ell(\bar{x}(0),\bar{x}(1)) \\ & + \langle \bar{p}(1), x(1)-\bar{x}(1)\rangle + \frac{1}{2}\langle x(1)-\bar{x}(1), \bar{W}(1)(x(1)-\bar{x}(1))\rangle \\ & - \langle \bar{p}(0), x(0)-\bar{x}(0)\rangle - \frac{1}{2}\langle x(0)-\bar{x}(0), \bar{W}(0)(x(0)-\bar{x}(0))\rangle \\ & + \frac{\varepsilon_0}{4}\|x-\bar{x}\|_2^2. \end{aligned}$$

Consider the problem

$$\begin{aligned} \text{minimize } f(x,y) \;\; := \;\; & \ell(x,y) - \ell(\bar{x}(0),\bar{x}(1)) \\ & + \langle \bar{p}(1), y-\bar{x}(1)\rangle + \frac{1}{2}\langle y-\bar{x}(1), \bar{W}(1)(y-\bar{x}(1))\rangle \\ & - \langle \bar{p}(0), x-\bar{x}(0)\rangle - \frac{1}{2}\langle x-\bar{x}(0), \bar{W}(0)(x-\bar{x}(0))\rangle \end{aligned}$$

over $\phi(x,y) = 0$.

Using (3.1)(iii) it follows that

$$\nabla f(\bar{x}(0),\bar{x}(1)) + \bar{\gamma}^T\bar{M} = 0.$$

Furthermore, (4.2) is equivalent to saying that

$$\nabla^2(f+\bar{g}^T\phi)\Big|_{(\bar{x}(0),\bar{x}(1))} > 0 \quad \text{on} \quad \big\{(\alpha,\beta) : \bar{M}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0\big\}.$$

Hence, by standard sufficiency criterion for finite dimensional optimization problem (e.g. [6, p. 307]), we conclude that $(\bar{x}(0),\bar{x}(1))$ is a local minimum for $f(x,y)$ over $\phi(x,y)=0$. Thus, there exists $\bar{\delta} \leq \delta_0$ such that for any admissible x with $\|x-\bar{x}\|_\infty < \bar{\delta}$ we have,

$$J(x) - J(\bar{x}) \geq \bar{\varepsilon}\|x - \bar{x}\|_2^2,$$

where $\bar{\varepsilon} := \frac{\varepsilon_0}{4}$.

Remark 4.1. When the endpoints costs and constraints are separable we have $\Gamma = \begin{bmatrix} \Gamma_0 & 0 \\ 0 & \Gamma_1 \end{bmatrix}$ and $\bar{M} = \begin{bmatrix} \bar{M}_0 & 0 \\ 0 & \bar{M}_1 \end{bmatrix}$ and in this case, the boundary conditions (4.1) become

$$\begin{cases} \Gamma_0 - W(0) > 0 \text{ on } \{\alpha \neq 0 \big| \bar{M}_0\alpha = 0\} \\ \\ \Gamma_1 + W(1) > 0 \text{ on } \{\alpha \neq 0 \big| \bar{M}_1\alpha = 0\}. \end{cases} \tag{4.6}$$

Note that in this setting the proof of Theorem 4.1 remains valid when (4.6) (ii) is replaced by

$$W(1) = -\Gamma_1 - \beta\bar{M}_1$$

for some β. In fact, by the Embedding Theorem of Differential Equations, a solution of $(\mathcal{R})$ satisfying (4.6) (i) and the above condition leads to a solution of $(\mathcal{R})$ satisfying (4.6). By symmetry, we can replace in Theorem 4.1 condition (4.6) (i) by $W(0) = \Gamma_0 + \beta\bar{M}_0$.

In the remaining of this section we study the connection between the sufficiency criterion presented in Theorem 4.1 and the necessary conditions obtained in Section 3 in terms of the accessory problem (i.e., Theorem 3.1), the existence of a conjoined basis (i.e., Theorem 3.3), and a solution to the Riccati equation $(\mathcal{R})$ with boundary conditions (3.16) (i.e., Corollary 3.2). Our study below works for any L^1-matrix functions $A(\cdot), R(\cdot)$ and $P(\cdot)$ from $[0,1]$ to $\mathbb{R}^{n\times n}$ and matrices $\Gamma \in \mathbb{R}^{n\times n}$ and $\bar{M} \in \mathbb{R}^{r\times 2n}$ and not only for the choice given by (3.13) through the data of the problem (P).

Define

$$J_2(\eta, v) := \frac{1}{2}\langle\Gamma\begin{pmatrix}\eta(0)\\ \eta(1)\end{pmatrix}, \begin{pmatrix}\eta(0)\\ \eta(1)\end{pmatrix}\rangle + \frac{1}{2}\int_0^1 \{\langle R(t)v(t), v(t)\rangle - \langle P(t)\eta(t), \eta(t)\rangle\}dt.$$

In [16, Theorem 5.1] it is shown that the existence of a solution to $(\mathcal{R})$ in $[0,1]$ satisfying Condition (4.1), which is Condition (ii) of Theorem (4.1), is in fact sufficient for the coercivity in η of the quadratic form J_2, that is, for some $\gamma_0 > 0$,

$$J_2(\eta, v) \geq \frac{\gamma_0}{2}\int_0^1 |\eta(t)|^2 dt, \tag{4.7}$$

for all (η, v) satisfying (2.3) (i) and its boundary conditions $\bar{M}\begin{pmatrix}\eta(0)\\ \eta(1)\end{pmatrix} = 0$. Condition (4.7) is a natural strengthening of the necessary condition in Theorem 3.1. However, for the general case of joint boundary conditions on the state, the existence of a solution to $\mathcal{R}$ and (4.1) is sufficient but not necessary for the coercivity of $J_2(\eta, v)$. This latter condition or its equivalent ones are the best to be used in the sufficiency criteria, since the gap between necessary and sufficient conditions would be minimal. This can also be said about natural strengthening of the necessary conditions given by Theorem 3.3 and Corollary 3.2.

Now consider the case of separable state endpoints and costs conditions. Hence we have

$$\bar{M} = \begin{bmatrix} \bar{M}_0 & 0 \\ 0 & \bar{M}_1 \end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} \Gamma_0 & 0 \\ 0 & \Gamma_1 \end{bmatrix}$$

It is important to find out whether in this case the existence of a solution to $(\mathcal{R})$ on $[0,1]$ satisfying (4.6) is only sufficient for the coercivity in (4.7) or that the two conditions are rather equivalent. As we shall see below, under a certain controllability assumption, they are indeed equivalent. Observe that the coercivity of J_2 is in η and not in v, due to the lack of the strengthened Legendre-Clebsch hypothesis. However, under the latter hypothesis this result is known. It is worth mentioning that in [16, Theorem 5.2] a result parallel to Theorem 4.2 is obtained for the *positivity* of J_2 in η and *not* for its *coercivity.*

Theorem 4.2. *Assume that the system (2.3) (i) is either $(I : \bar{M}_1)$-normal on $[c, 1]$ for all $c \in [0, 1)$, or $(\bar{M}_0 : I)$-normal on $[0, c]$ for all $c \in (0, 1]$. Then the following are equivalent.*

(1) *$J_2(\eta, v)$ is coercive in η for all (η, v) solving (2.3)(i) and $\bar{M}_0\eta(0) = \bar{M}_1\eta(1) = 0$.*

(2) *There exists a solution (U, V) of (3.14) satisfying*

   (i) $U^TV = V^TU$ *on* $[0,1]$,

   (ii) $\det U(t) \neq 0$ *on* $[0,1]$,

   (iii) $U^T(0)[\Gamma_0U(0) - V(0)] > 0$ *on* $\{\alpha \neq 0 \mid \bar{M}_0U(0)\alpha = 0\}$

   (iv) $U^T(1)[\Gamma_1U(1) + V(1)] > 0$ *on* $\{\alpha \neq 0 \mid \bar{M}_1U(1)\alpha = 0\}$.

(3) *There exists a symmetric absolutely continuous solution W on $[0,1]$ of $(\mathcal{R})$ satisfying (4.6).*

Proof. By taking $W = VU^{-1}$ it follows that condition (2) implies (3). Also from [16, Theorem 5.1], condition (3) implies (1). thus, it remains to show that condition (1) implies condition (2). Assume that (2.3) (i) is $(I : \bar{M}_1)$-normal on $[c, 1]$ for all $c \in [0, 1)$. By symmetry, a similar argument holds when the $(\bar{M}_0 : I)$-normality occurs. From [16, Theorem 5.2], condition (1) implies that the solution (U_1, V_1) of (3.14) and (3.15) satisfies conditions 2(i), 2(iii),

$$\det U(t) \neq 0 \quad \text{on} \quad [0, 1)$$

and

$$U^T(1)[\Gamma_1U(1) + V(1)] = 0.$$

By [4, Theorem 3.1.2 (v)], there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$, the matrix

$$U_\delta := (I - \bar{M}_1) + \delta(-\Gamma_1(I - \bar{M}_1) - \bar{M}_1) \tag{4.8}$$

is invertible. Consider the system (3.14) with boundary conditions

$$\begin{cases} U(1) = U_\delta \\ V(1) = V_1(1) + \delta(\Gamma_1\Gamma_1(I - \bar{M}_1) + I). \end{cases} \tag{4.9}$$

Since for $\delta = 0$ the system (3.14) and (4.9) has (U_1, V_1) as a solution on $[0,1]$ and since from (4.9) we have

$$U^T(1)[V(1) + \Gamma_1 U(1)] = \delta I + \delta^2 \bar{S}$$

for $\bar{S} = [(I - \bar{M}_1)\Gamma_1 + \bar{M}_1][\Gamma_1 \bar{M}_1 - I]$, then by the Embedding Theorem, there exists $\delta_1 > 0$ ($\delta_1 < \delta_0$) such that for all $\delta \in (0, \delta_1]$ there exists a solution $(U_\delta(\cdot), V_\delta(\cdot))$ of (3.14) and (4.9) satisfying parts (i), (iii) and (iv) of condition (2).

It remains to show that for some $\delta \in (0, \delta_1]$ the solution to (3.14) and (4.9) also satisfies part (ii) of condition (2). Note that since U_δ, given by (4.8), is invertible, then for each $\delta \in (0, \delta_1]$, there exists $t_\delta \in [0,1)$ (maximal) such that $\det U_\delta(t) \neq 0$ on $(t_\delta, 1)$. Now, suppose that our claim is not true, then there exists $t_m \in [0,1)$ (maximal) such that

$$\det U_{1/m}(t) \neq 0 \quad \text{on} \quad (t_m, 1] \quad \text{and } \det U_{1/m}(t_m) = 0.$$

Using (4.9) it follows that $(U_{1/m}, V_{1/m}) \underset{m\to\infty}{\longrightarrow} (U_1, V_1)$. But since U_1 is invertible on $[0,1)$, we conclude that $t_m \to 0$. Let $\alpha_m \in \mathbb{R}^n$ with $|\alpha_m| = 1$ and

$$U_{1/m}(t_m)\alpha_m = 0. \tag{4.10}$$

By passing to a convergent subsequence, say $\alpha_m \underset{m\to\infty}{\longrightarrow} \alpha$, and upon taking the limit in (4.10) it results that

$$U_1(0)\alpha = 0 \quad \text{and} \quad |\alpha| = 1.$$

This contradicts that $\det U_1(0) \neq 0$. Therefore the proposition is proved.

Note that in this setting, condition (ii) of Theorem 4.1 can be replaced by any of the equivalent conditions in Theorem 4.2, as long as the normality assumption of the Theorem holds. Another consequence of this theorem is the following connection between Condition (ii) of Theorem 4.1, that is Condition (3) of Theorem 4.2, and the necessary conditions in Theorem 3.3 and Corollary 3.2.

Corollary 4.1. *Assume that system (2.3) (i) is $(I : \bar{M}_1)$-normal on $[c,1]$ for all $c \in [0,1)$. Then each of the three equivalent conditions of Theorem 4.2 is also equivalent to each of the following conditions.*

(a) *The solution (U_1, V_1) of (3.14) and (3.15) satisfies* (i) $U^T V = V^T U$, (ii) $\det U(t) \neq 0$ *on* $[0,1)$, (iii) $U^T(0)(\Gamma_0 U(0) - V(0)) > 0$ *on* $\{\alpha \neq 0 \mid \bar{M}_0 U(0)\alpha = 0\}$ *and* $V(1) + \Gamma_1(I - \bar{M}_1) + \bar{M}_1 = 0$.

(b) *There exists a symmetric solution W to $(\mathcal{R})$ on $[0,1)$ satisfying (4.6) (i) and*

$$\lim_{t\to 1^-} W(t)U_1(t) = -\Gamma_1(I - \bar{M}_1) - \bar{M}_1.$$

Proof. From [16, Theorem 5.2] we know that conditions (a) and (b) above are equivalent. Also in the proof of Theorem 4.2 we showed the following implications:

$$(1) \Rightarrow \text{condition (a)} \Rightarrow (2) \Rightarrow (3) \Rightarrow (1),$$

which proves the corollary.

Remark 4.2. The above result states that in Theorem 4.1 we can replace condition (ii) by either of conditions (a) or (b) of Corollary 4.1, since they imply the $(I : \bar{M}_1)$-normality required. Hence, Corollary 4.1 shows that the sufficiency criterion of Theorem 4.1 is obtained as natural strengthening of the necessary conditions in either Theorem 3.3 or Corollary 3.2.

5 Application to Optimal Control Problems

In this section we consider the optimal control problem (C) defined in the Introduction. First, we intend to apply the results of the previous section to the generalized Bolza $(\mathrm{P_C})$ problem defined in Section 2 and associated with (C), and hence sufficient conditions for strong local minimality in (C) will be obtained. Next, we plan to derive second order necessary conditions for optimality in (C). In this regard we shall obtain results parallel to Theorem 3.1, Theorem 3.3 and Corollary 3.2 by either applying Theorem 3.1 to the generalized problem of Bolza $(\mathrm{P_C})$ associated to (C), or by using Theorem 3.2 directly. The former approach yields necessary conditions for $W^{1,1}$- and $W^{1,\infty}$-local minimum and requires assuming conditions $(\mathcal{L}_\mathrm{C})$ and $(\mathcal{B})$. On the other hand the second approach produces necessary conditions for $W^{1,1}$-, $W^{1,\infty}$-, L^1-, and L^∞-weak local minimality. For the first two types of optimality this approach requires only that the supremum in (1.3) be attained at some $\mathbf{u}(t,x,p)$. While for the last two types of optimality, regularity assumptions are needed on $\mathbf{u}$. Of course, all the results of this section apply for the (classical) strong local minimality in (C).

Assume throughout this section that f, g, ℓ and U satisfy assumption (A) given in Section 2, for $t \in [0,1]$ and $u \in U$, $f(t,\cdot,u)$ and $g(t,\cdot,u)$ are differentiable on $B_n(\bar{x}(t);\varepsilon)$, and ℓ is differentiable on $B_{2n}(\bar{x}(0),\bar{x}(1);\varepsilon)$.

Let $(\bar{x},\bar{u},\bar{p},\bar{\gamma})$ be given. We say that $(\bar{x},\bar{u},\bar{p},\bar{\gamma})$ satisfies the *normal* version of the Pontryagin maximum principle if they satisfy

$$(5.1)\qquad \begin{cases} -\dot{\bar{p}} = f_x^T(t,\bar{x}(t),\bar{u}(t))\bar{p}(t) - g_x^T(t,\bar{x}(t),\bar{u}(t)) \\ (\bar{p}^T(0), -\bar{p}^T(1)) = \nabla\ell(\bar{x}(0),\bar{x}(1)) + \bar{\gamma}^T\bar{M} \\ \max\{\bar{p}^T(t) f(t,\bar{x}(t),u) - g(t,\bar{x}(t),u) \mid u \in U\} \\ \text{is attained at } \bar{u}(t), \text{ for almost all } t. \end{cases}$$

Let the Hamiltonian $\mathcal{H}(t,\cdot,\cdot)$ be differentiable at $(\bar{x},\bar{p})$. If $(\bar{x},\bar{u})$ is admissible for (C) and (5.1) holds, then $(\bar{x},\bar{u},\bar{p},\bar{\gamma})$ satisfies

$$\begin{aligned} -\dot{\bar{p}}(t) &= \bar{\mathcal{H}}_x(t) \quad \text{a.e.} \\ \dot{\bar{x}}(t) &= \bar{\mathcal{H}}_p(t) \quad \text{a.e.} \\ (\bar{p}^T(0), -\bar{p}^T(1)) &= \nabla\ell(\bar{x}(0),\bar{x}(1)) + \bar{\gamma}^T\bar{M} \\ \mathcal{H}(t,\bar{x}(t),\bar{p}(t)) &= \langle \bar{p}(t), f(t,\bar{x}(t),\bar{u}(t))\rangle - g(t,\bar{x}(t),\bar{u}(t)) \quad \text{a.e.} \end{aligned}$$

(see e.g., [1, Proposition 3.2]).

The following result is an application of Theorem 4.1 to the optimal control setting. It extends the corresponding result in [13], [14] and [1] to the case where $\bar{x}$ is not necessarily C^1 and both endpoints of x vary (i.e., $\phi(x(0), x(1)) = 0$). When we specialize to the case considered in [1], that is a fixed initial and a free final state constraints, our result assumes less regularity assumptions on the Hamiltonian $\mathcal{H}$ defined by (1.3) and on $\mathcal{H}_p$.

Theorem 5.1. *Let $(\bar{x}, \bar{u})$ be admissible for* (C) *$\bar{p} \in W^{1,1}[0,1]$ and $\bar{\gamma} \in \mathbb{R}^r$ such that $(\bar{x}, \bar{u}, \bar{p}, \bar{\gamma})$ satisfies the normal version of the Pontryagin maximum principle (equations (5.1)). Let $\mathcal{H}$, the Hamiltonian of* (C), *satisfy* (H_2), *for some $\varepsilon > 0$. Assume that there exists a symmetric, absolutely continuous solution on $[0,1]$ of (4.1) and the Riccati equation $(\mathcal{R})$, where H is replaced by $\mathcal{H}$. Then, there exist $\bar{\varepsilon} > 0$ and $\bar{\delta} > 0$ such that for all admissible pairs (x, u) with $\|x - \bar{x}\|_\infty < \bar{\delta}$ we have*

$$J(x, u) \geq J(\bar{x}, \bar{u}) + \bar{\varepsilon}\|x - \bar{x}\|_2^2,$$

and $(\bar{x}, \bar{u})$ is a strong local minimum for (C).

Proof. Define L_C via equation (2.2), where $T(\bar{x}; \varepsilon)$ is used. In order to prove this theorem, we use Remark 2.1. The maximality condition in (5.1) yields that

$$L_C(t, \bar{x}(t), \dot{\bar{x}}(t)) = g(t, \bar{x}(t), \bar{u}(t)) \quad \text{a.e.t.}$$

Let's show that $\bar{x}$ is a strong local minimum for $(\mathrm{P_C})$. Again the maximum principle yields that

$$L_C(t, \bar{x}(t), \dot{\bar{x}}(t) + v) - L_C(t, \bar{x}(t), \dot{\bar{x}}(t)) \geq \langle \bar{p}(t), v \rangle$$

for almost all t and for all $v \in \mathbb{R}^n$. Equations (5.1) yield that $(\bar{x}, \bar{p})$ satisfies (3.1), where $H := \mathcal{H}$. Since the Hamiltonian $\mathcal{H}$ of the control problem (C) is the same as the Hamiltonian corresponding to $(\mathrm{P_C})$, then all the conditions of Theorem 4.1 are satisfied for the problem $(\mathrm{P_C})$. It results that there exist $\bar{\varepsilon} > 0$ and $\bar{\delta} > 0$ such that for any x admissible for $(\mathrm{P_C})$ with $\|x - \bar{x}\|_\infty < \bar{\delta}$ we have

$$J_C(x) - J_C(\bar{x}) \geq \bar{\varepsilon}\|x - \bar{x}\|_2^2.$$

For (x, u) admissible for (C) Remark 2.1 implies that

$$J_C(x) \leq J(x, u).$$

We also know that

$$J_C(\bar{x}) = J(\bar{x}, \bar{u}),$$

hence, $(\bar{x}, \bar{u})$ is a strong local minimum for (C) and for all admissible pairs (x, u) with $\|x - \bar{x}\|_\infty < \bar{\delta}$

$$J(x, u) - J(\bar{x}, \bar{u}) \geq \bar{\varepsilon}\|x - \bar{x}\|_2^2.$$

Remark 5.1. Consider the case where the state endpoints costs and constraints are separable, that is,

$$\ell(x,y) := \ell_0(x) + \ell_1(y) \quad \text{and} \quad \phi(x,y) := \begin{pmatrix} \phi_0(x) \\ \phi_1(y) \end{pmatrix}. \tag{5.2}$$

Then, Theorem 4.2 and Corollary 4.1 yield that the result of Theorem 5.1 remains valid for Problem (C) when the assumption regarding the existence of a solution to $(\mathcal{R})$ and (4.1) is replaced by either condition (a) or condition (b) of Corollary 4.1, or by any of the equivalent conditions given in Theorem 4.2, provided that the normality hypothesis of that theorem hold.

Now, we intend to derive necessary conditions for (C). This is done either by ensuring that the problem (C), the function $\mathrm{L_C}$, and the Hamiltonian $\mathcal{H}$ satisfy assumptions $(\bar{A})$, $(\mathcal{L})$, and (H_2), and hence Theorem 3.1 will apply to $(\mathrm{P_C})$, or by connecting the two optimal control problems (C) and $(\mathrm{C}_\mathcal{H})$, and then ensuring that $H := \mathcal{H}$ satisfies the conditions of Theorem 3.2. Here, $(\mathrm{C}_\mathcal{H})$ is exactly the problem $(\mathrm{C_H})$ introduced in Section 3 with $H := \mathcal{H}$. The first method leads to Part (1)(i) of the theorem and the second method is behind the rest of the theorem.

Theorem 5.2. *Let f, g and U satisfy* (A). *Suppose that $(\bar{x}, \bar{u}, \bar{p}, \bar{\gamma})$ and $\mathcal{H}$ satisfy (5.1) and* (H_2). *Then for $H := \mathcal{H}$, the conclusions of Theorem 3.1 hold and Theorem 3.3 and Corollary 3.2 are valid if either of the conditions* (1) *or* (2) *is satisfied.*

(1) $(\bar{x}, \bar{u})$ *is either $W^{1,1}$-weak local minimum or, $W^{1,\infty}$-weak local minimum with* (H_∞) *satisfied. Suppose that either condition* (i) *or condition* (ii) *is in effect:*

- (i) *Assumptions* $(\mathcal{L}_\mathrm{C})$ *and* $(\mathcal{B})$ *hold.*
- (ii) *The supremum in (1.3) is attained for $(t, x, p) \in T(\bar{x}, \bar{p}; \varepsilon)$.*

(2) $(\bar{x}, \bar{u})$ *is classically L^1-weak local minimum (resp. L^∞-weak local minimum with* (H_∞) *satisfied) and, the supremum in condition* (ii) *is attained at $\mathbf{u}(t, x, p)$ such that $\mathbf{u}(t, \bar{x}(t), \bar{p}(t)) = \bar{u}(t)$ and $\mathbf{u}$ is continuous at $\bar{z} := (\bar{x}, \bar{p})$ from L^∞ to L^1* (resp. L^∞).

Remark 5.2. When U is compact, $(\mathcal{B})$ and condition (ii) of Theorem 5.2 hold automatically and hence condition (i) is more restrictive than (ii). Thus, in this case, Part (1) of the theorem disposes of conditions (i) and (ii). This shows that in this case the second-order necessary conditions obtained for (C) by passing through the associated generalized Bolza problem $(\mathrm{P_C})$ are weaker than those obtained by passing through $(\mathrm{C}_\mathcal{H})$.

The existence of a feedback-type function $\mathbf{u}$ as in Part (2) of the theorem has been employed in [12] when deriving sufficient conditions for strong local minimum.

Proof. Assumptions (A) yield that $\mathcal{H}$ is $\mathcal{L} \times \mathcal{B}$-measurable.

For (1): Assume that (i) holds. Then, by Remark 2.2, assumptions $(\bar{A})$ are satisfied. Thus, by applying Theorem 2.1, it results that if $(\bar{x}, \bar{u})$ is $W^{1,1}$(resp. $W^{1,\infty}$)-weak local minimum for (C), then $\bar{x}$ is $W^{1,1}$(resp. $W^{1,\infty}$)-weak local for $(\mathrm{P_C})$. We shall check whether the assumptions of Theorem 3.1 are met. As mentioned earlier, equations (5.1) yield that $(\bar{x}, \bar{p}, \bar{\gamma})$ is a normal extremal for ($\mathrm{P_C}$), whose Hamiltonian is $\mathcal{H}$, and Assumption ($\mathcal{L}_\mathrm{C}$) implies that Assumption ($\mathcal{L}$) is satisfied. Whence, the conclusions of Theorem 3.1 hold. Assume that ϕ and ℓ are separable, as in (5.2), then, under the normality assumptions therein, the results of Theorem 3.3 and Corollary 3.2 are valid.

To prove Part (1)(ii) and Part (2) of the theorem we shall use the following result that connects (C) with ($\mathrm{C}_\mathcal{H}$).

Proposition 5.1. *Assume that f, g and U satisfy* (A). *Suppose that condition (ii) of Theorem 5.2 holds and that $\mathcal{H}(t, \cdot, \cdot)$ is differentiable on $B_{2n}(\bar{x}(t), \bar{p}(t); \varepsilon)$. Let $(\bar{x}, \bar{u}, \bar{p})$ satisfy (5.1) (iii). If*

(a) $(\bar{x}, \bar{u})$ *is a $W^{1,1}$(resp. $W^{1,\infty}$)-weak local minimum for* (C) *and $\mathcal{H}_p$ is continuous at $\bar{z} := (\bar{x}, \bar{p})$, from L^∞ to L^1 (resp. L^∞),*

or

(b) $(\bar{x}, \bar{u})$ *is a classically L^1(resp. L^∞)-weak local minimum and the supremum in condition* (ii) *of Theorem 5.2 is attained at $\mathbf{u}(t, x, p))$ such that $\mathbf{u}(t, \bar{x}(t), \bar{p}(t)) = \bar{u}(t)$, and $\mathbf{u}$ and $\mathcal{H}_p$ are continuous at $(\bar{x}, \bar{p})$ from L^∞ to L^1 (resp. L^∞).*

Then, $(\bar{x}, \bar{p})$ is classically an L^∞-weak local minimum for ($\mathrm{C}_\mathcal{H}$).

Proof. For (a): Suppose that there exists $0 < \varepsilon_0 < \varepsilon$ such that $J(x, u) \geq J(\bar{x}, \bar{u})$ for all admissible pairs (x, u) with $\|x - \bar{x}\|_{1,1} < \varepsilon_0$ (resp. $\|x - \bar{x}\|_{1,\infty} < \varepsilon_0$). Let (x, p) be admissible for ($\mathrm{C}_\mathcal{H}$) with $(x, p) \in T(\bar{x}, \bar{p}; \varepsilon)$ and $\|x - \bar{x}\|_{1,1} < \varepsilon_0$ (resp. $\|x - \bar{x}\|_{1,\infty} < \varepsilon_0$). Condition (ii) of Theorem 5.2 and the differentiability of $\mathcal{H}(t, \cdot, \cdot)$ yield that

$$\mathcal{H}_p(t, x(t), p(t)) = f(t, x(t), A(t, x(t), p(t))),$$

where

$$A(t, x, p) := \{u \in U : \langle p, f(t, x, u)\rangle - g(t, x, u) = \mathcal{H}(t, x, p)\}$$

and $f(t, x, A(t, x, p))$ is a singleton. Hence, by the Measurable Selection Theorem, there exists a measurable function u such that $u(t) \in A(t, x(t), p(t))$, for almost all t. Thus,

$$\dot{x}(t) = f(t, x(t), u(t)) = \mathcal{H}_p(t, x(t), p(t)) \quad \text{a.e.,} \tag{5.3}$$

and

$$\langle p(t), f(t, x(t), u(t))\rangle - g(t, x(t), u(t)) = \mathcal{H}(t, x(t), p(t)) \quad \text{a.e.} \tag{5.4}$$

It follows that (x, u) is admissible for (C) and then,

$$J(x, u) \geq J(\bar{x}, \bar{u}).$$

However, using (5.3) and (5.4) we get

$$J_C(x, p) = J(x, u) \quad \text{and} \quad J_C(\bar{x}, \bar{p}) = J(\bar{x}, \bar{u}).$$

Therefore, $(\bar{x}, \bar{p})$ is an $W^{1,1}$-weak (resp. $W^{1,\infty}$-weak) local minimum for ($\mathrm{C}_\mathcal{H}$). Now apply Lemma 2.2 to the problem ($\mathrm{C}_\mathcal{H}$) we conclude the result.

For (b): Suppose for $0 < \varepsilon_0 < \varepsilon$ we have $J(x,u) \geq J(\bar{x},\bar{u})$ for $\|x-\bar{x}\|_\infty + \|u-\bar{u}\|_{1(\text{resp. }\infty)} < \varepsilon_0$. By continuity of $\mathbf{u}$, there exists $0 < \delta_0 < \dfrac{\varepsilon_0}{2}$ such that for $\|x-\bar{x}\|_\infty + \|p-\bar{p}\|_\infty < \delta_0$, we have

$$\|\mathbf{u}(\cdot,x(\cdot),p(\cdot)) - \mathbf{u}(\cdot,\bar{x}(\cdot),\bar{p}(\cdot))\|_{1(\text{resp. }\infty)} < \varepsilon_0/2.$$

Now, let (x,p) such that $\|x-\bar{x}\|_\infty + \|p-\bar{p}\|_\infty < \delta_0$ and (x,p) admissible for (C). In this case, $\mathbf{u}(t,x(t),p(t))$ belongs to $A(t,x(t),p(t))$ defined above, and hence, for $u(\cdot) = \mathbf{u}(\cdot,x(\cdot),p(\cdot))$, (x,u) is admissible for (C). Since

$$\|x-\bar{x}\|_\infty + \|u-\bar{u}\|_{1(\text{resp. }\infty)} < \delta_0 + \frac{\varepsilon_0}{2} < \varepsilon_0$$

it results that $J(x,u) \geq J(\bar{x},\bar{u})$. Arguments similar to those used for (a) yield that $J_{\mathrm{C}}(x,p) \geq J_{\mathrm{C}}(\bar{x},\bar{p})$. That is, $(\bar{x},\bar{p})$ is an L^∞-weak local minimum for $(\mathrm{C}_{\mathcal{H}})$.

Return to the proof of Part (1) of Theorem 5.2. Assume condition (ii) of the theorem holds. Then, Proposition 5.1(a) and Theorem 3.2 yield that the conclusions of Theorem 3.1 hold, and thus each of Theorem 3.3 and Corollary 3.2 is valid. Part (2) of the theorem follows from Proposition 5.1(b) and Theorem 3.2.

Remark 5.3. When the initial state value is fixed and the final state value is free, the problem (C) reduces to the one studied in [1]. In this case, ℓ and ϕ in (5.2) take the form $\ell(x,y) = \ell_1(y)$ and $\phi(x,y) = \phi_0(x) = x - x_0$. In this special setting necessary conditions in terms of the Riccati equation $(\mathcal{R})$ were derived in [1, Theorem 5.1] for $W^{1,1}$-weak local minimum. On the other hand, specialized to this setting, Part (1) of Theorem 5.2 here yields that, under Hypotheses (A), (5.1), (H_2) and condition (ii) of Theorem 5.2, Corollary 3.2 holds true. Hence, by Remark 3.2, these assumptions yield necessary conditions in terms of a solution on $(0,1]$ to $(\mathcal{R})$ and (3.17). However, these assumptions are considerably weaker than those imposed in [1, Theorem 5.1]. This is so since we do *not* assume (1.4), $\nabla_z^2\bar{\mathcal{H}}(\cdot)$ to be L^∞, $\nabla_z^2\mathcal{H}$ to be Lipschitz near $\bar{z} = (\bar{x},\bar{p})$ from L^∞ to L^1, nor that the strengthened Legendre-Clebsch condition $(S\mathcal{L})$ to hold. Also, our necessary conditions apply for $W^{1,\infty}$-, classically L^1- or L^∞-weak local minimum and not only for $W^{1,1}$-weak local minimum.

As we shall see below, the assumption $(S\mathcal{L})$ in [1] is very strong. In fact, it renders the corresponding function L_{C} to be locally C^1 in (x,v) and hence, the control problem (C) resembles to a calculus of variations problem.

Lemma 5.1. Let $(\bar{x},\bar{p}) \in W^{1,\infty}[0,1] \times L^\infty[0,1]$ satisfy $\dot{\bar{x}}(t) = \bar{\mathcal{H}}_p(t)$ a.e., and $\mathcal{H}$ satisfy (H_2), (H_∞) and $(S\mathcal{L})$. Assume that $\mathcal{H}_{pp}$ is continuous at $(\bar{x},\bar{p})$, from L^∞ to L^∞, and $\bar{\mathcal{H}}_p(\cdot)$ and $\mathcal{H}_{pp}(\cdot)$ are in L^∞. If, for $(t,x) \in T(\bar{x};\varepsilon)$, $L_C(t,x,\cdot)$ defined by (2.2), is convex, then L_C is $\mathcal{L} \times \mathcal{B}$-measurable, lower semicontinuous in (x,v), and there exists $\tilde{\varepsilon} > 0$ ($\tilde{\varepsilon} \leq \varepsilon$) and $p(t,x,v)$ such that $p(t,\cdot,\cdot)$ is C^1 on $T(\bar{x},\dot{\bar{x}};\tilde{\varepsilon})$ and ∇p is continuous at $(\bar{x},\dot{\bar{x}})$, from L^∞ to L^∞, such that

$$L_C(t,x,v) = p(t,x,v)\cdot v - H(t,x,p(t,x,v)).$$

Proof. Since $\mathcal{H}$ is also the Hamiltonian corresponding to L_C which is convex in v, then

$$L_C(t,x,v) = \sup\{\langle p, v\rangle - \mathcal{H}(t,x,p) \mid p \in \mathbb{R}^n\}$$

for $(t,x) \in T(\bar{x};\varepsilon)$. Hence by [8], L_C is $\mathcal{L}\times\mathcal{B}$-measurable and lower semicontinuous in (x,v). Consider the equation

$$v = \mathcal{H}_p(t,x,p) \tag{5.5}$$

We have that (5.5) is satisfied for almost all t, for $v = \dot{\bar{x}}(t),\ x = \bar{x}(t),\ p = \bar{p}(t)$. Set

$$N(z(\cdot);\varepsilon) := \{x(\cdot) \in L^\infty[0,1] \mid \|z - x\|_\infty < \varepsilon\},$$

and define

$$\begin{gathered}\mathcal{F} : N(\bar{x};\varepsilon) \times N(\bar{p};\varepsilon) \to L^\infty[0,1]\\ (x(\cdot),p(\cdot)) \to v(\cdot) = \mathcal{F}(x(\cdot),p(\cdot)).\end{gathered}$$

where

$$\mathcal{F}(x(\cdot),p(\cdot))(t) = \mathcal{H}_p(t,x(t),p(t)).$$

Consider the equation

$$v(\cdot) = \mathcal{F}(x(\cdot),p(\cdot)).$$

(i) For $x(\cdot) \in N(\bar{p};\varepsilon)$, $\mathcal{F}(x(\cdot),\cdot)$ is differentiable on $N(\bar{p};\varepsilon)$:

Let $p(\cdot) \in N(\bar{p};\varepsilon)$ and $\bar{\varepsilon} > 0$ given. The continuity of $\mathcal{H}_{pp}$ from L^∞ to L^∞ yields the existence of $\delta > 0$ such that for $\|q - p\|_\infty < \delta$ we have

$$|\mathcal{H}_{pp}(t,x(t),q(t)) - \mathcal{H}_{pp}(t,x(t),p(t))| < \frac{\bar{\varepsilon}}{2} \text{ a.e.t.}$$

Let $\bar{q}(\cdot) \in N(p(\cdot);\delta)$. We shall show

$$\|\mathcal{F}(x(\cdot),\bar{q}(\cdot)) - \mathcal{F}(x(\cdot),p(\cdot)) - \mathcal{F}_p(x(\cdot),p(\cdot))(\bar{q}(\cdot) - p(\cdot))\|_\infty \le \bar{\varepsilon}\|\bar{q} - p\|_\infty.$$

By the Mean Value Theorem for vector-valued functions applied to $\mathcal{H}_p(t,x(t),\cdot) - \mathcal{H}_{pp}(t,x(t),p(t))(\cdot)$ on the line segment joining $p(t)$ and $\bar{q}(t)$, there exists $\tilde{q}$ with $|\tilde{q}(t) - p(t)| < \delta$ such that:

$$\begin{aligned}&|\mathcal{H}_p(t,x(t),\bar{q}(t)) - \mathcal{H}_p(t,x(t),p(t)) - \mathcal{H}_{pp}(t,x(t),p(t))(\bar{q}(t) - p(t))|\\ &\quad\le \left|\big(\mathcal{H}_{pp}(t,x(t),\tilde{q}(t)) - \mathcal{H}_{pp}(t,x(t),p(t))\big)(\bar{q}(t) - p(t))\right|\\ &\quad\le \frac{\bar{\varepsilon}}{2}\|\bar{q} - p\|_\infty.\end{aligned}$$

Set

$$\mathcal{F}_p(x(\cdot),p(\cdot))(t) := \mathcal{H}_{pp}(t,x(t),p(t)).$$

The result follows

(ii) Let $\mathcal{B}(L^\infty, L^\infty)$ be the normed space of all bounded linear operators from L^∞ into L^∞. Then

$$\mathcal{F}_p : N(\bar{x}, \varepsilon) \times N(\bar{p}, \varepsilon) \to \mathcal{B}(L^\infty, L^\infty)$$

is continuous, since $\mathcal{H}_{pp}(t, \cdot, \cdot)$ is continuous on $N(\bar{x}, \varepsilon) \times N(\bar{p}, \varepsilon)$, L^∞-uniformly in t.

(iii) $\mathcal{F}_p(\bar{x}, \bar{p})$ is a homeomorphism of L^∞ onto L^∞: The strengthened Legendre condition $(S\mathcal{L})$ yields that $\mathcal{F}_p(\bar{x}, \bar{p})$ is a bijection and that $\bar{\mathcal{H}}_{pp}^{-1}(t)$ is essentially bounded. Since $\bar{\mathcal{H}}_{pp}(\cdot) \in L^\infty[0,1]$ we get that $\mathcal{F}_p(\bar{x}, \bar{p})$ is a homeomorphism.

(i)-(iii) allow us to apply the Implicit function theorem [10, II.3.8] to deduce the existence of $\varepsilon_0, \varepsilon_1 > 0$ and a unique C^1-function $\mathcal{P}$ such that

$$\mathcal{P} : N(\bar{x}, \varepsilon_0) \times N(\dot{\bar{x}}, \varepsilon_0) \to N(\bar{p}, \varepsilon_1)$$

$$v(\cdot) = \mathcal{F}(x(\cdot), \mathcal{P}(x(\cdot), v(\cdot)))$$

and

$$\mathcal{P}(\bar{x}, \dot{\bar{x}}) = \bar{p}.$$

Let us construct $p(t, x, v)$ on $T(\bar{x}, \dot{\bar{x}}; \frac{\varepsilon_0}{2})$. For given $(t, x, v) \in T(\bar{x}, \dot{\bar{x}}; \frac{\varepsilon_0}{2})$, set

$$x(s) := \bar{x}(s) + (\dot{\bar{x}}(t) - v)(t - s) + x - \bar{x}(t).$$

We have $x(t) = x$, and $\dot{x}(t) = v$. Define

$$p(t, x, v) := \mathcal{P}(x(\cdot), \dot{x}(\cdot))(t).$$

Then, on $T(\bar{x}, \dot{\bar{x}}; \frac{\varepsilon_0}{2})$, $p(\cdot, \cdot, \cdot)$ is well defined with values in $N(\bar{p}, \varepsilon_1)$,

$$v = \mathcal{H}_p(t, x, p(t, x, v)), \tag{5.6}$$

and

$$p(t, \bar{x}(t), \dot{\bar{x}}(t)) = \bar{p}(t).$$

Condition $(S\mathcal{L})$ yields that $p(\cdot, \cdot, \cdot)$ is unique. The C^1-property of $\mathcal{P}$ on $N(\bar{x}; \varepsilon_0) \times N(\dot{\bar{x}}, \varepsilon_0)$ implies that p is continuously differentiable from L^∞ to L^∞. Thus, by convexity of $\mathcal{H}(t, x, \cdot)$, for (t, x, v) in $T(\bar{x}, \dot{\bar{x}}; \frac{\varepsilon_0}{2})$,

$$L_C(t, x, v) = p(t, x, v) \cdot v - \mathcal{H}(t, x, p(t, x, v)).$$

Whence the result follows.

The following example shows the power of the results of this section, as it produces information where previous work offered none.

Example 5.1. Consider the optimal control problem

$$(\mathrm{C}_0) \qquad \text{minimize } J(x,u) := -\frac{5}{3}2^{5/3}x^2(1) + \int_0^1 (x^3(t) + u^2(t))dt$$

subject to:

$$\dot{x}(t) = \left(t - \frac{1}{2}\right)^{1/3} u(t)$$

$$x(0) = 0$$

$$u(t) \in \mathbb{R} \quad \text{a.e.},$$

The Hamiltonian corresponding to (C_0) is

$$\begin{aligned}\mathcal{H}(t,x,p) &:= \sup_{u\in\mathbb{R}}\{p\left(t-\frac{1}{2}\right)^{1/3} u - x^3 - u^2\} \\ &= -x^3 + \frac{p^2}{4}\left(t-\frac{1}{2}\right)^{2/3},\end{aligned}$$

and hence the supremum in $\mathcal{H}$ is attained for all $(t,x,p) \in [0,1]\times\mathbb{R}\times\mathbb{R}$ at

$$\mathbf{u}(t,x,p) = \frac{p}{2}\left(t-\frac{1}{2}\right)^{1/3},$$

with $\mathbf{u}$ continuous everywhere from $L^\infty \times L^\infty$ to L^∞. Moreover, $\mathcal{H}$ is the same Hamiltonian H of the problem (P_0) in Example 3.1.

Let $\bar{x} \equiv 0, \bar{u} \equiv 0, \bar{p} \equiv 0$ and $\bar{\gamma} = \binom{0}{0}$. Then, $(\bar{x},\bar{p},\bar{u},\bar{\gamma})$ satisfy the normal version of Pontryagin maximum principle (equations (5.1)). Furthermore, Assumptions (A) and (H_2) hold, and as seen in Example 3.1 the normality requirements of Theorem 3.3 are satisfied.

The Riccati equation $(\mathcal{R})$ for this problem with the boundary condition (3.17) are exactly given by (3.18). The solution $W(t) = -\dfrac{10}{3(t-\frac{1}{2})^{5/3}}$ exists on $(\frac{1}{2},1]$ and cannot be extended (by continuity) on $[\frac{1}{2},1]$. Even though (1.4) holds, the strengthened Legendre-Clebsch condition $(S\mathcal{L})$ required in the necessity theorem [1, Theorem 5.1] fails to hold near t_c, since $\bar{H}_{pp}(t) = \dfrac{1}{2}\left(t-\frac{1}{2}\right)^{2/3}$ is not uniformly bounded below for t near t_c $(t < t_c)$. Therefore, the result in [1, Theorem 5.1] does not apply to this problem and no information can be produced from [1] about the optimality of $(\bar{x},\bar{u})$ for (C_0).

On the other hand, as seen in Example 3.1 the conclusion of Corollary 3.2 does not hold, since the solution W does not exist on $(0,1]$. Given that condition (ii) of Part(1) of Theorem 5.2 is satisfied, it results from applying Theorem 5.2(1) that $(\bar{x},\bar{u})$ is not a $W^{1,1}$-weak local minimum for (C_0). Furthermore, since (H_∞) holds, $(\bar{x},\bar{u})$ is not even a $W^{1,\infty}$-weak local minimum for (C_0). Since the assumptions for Part (2) of Theorem 5.2 hold we also conclude that $(\bar{x},\bar{u})$ is nether L^1- nor L^∞-weak local minimum for (C_0).

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