Superconvergence of Discontinuous Galerkin and Local Discontinuous Galerkin Schemes for Linear Hyperbolic and Convection Diffusion Equations in One Space Dimension

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Abstract

In this paper, we study the superconvergence property for the discontinuous Galerkin (DG) and the local discontinuous Galerkin (LDG) methods, for solving one-dimensional time dependent linear conservation laws and convection-diffusion equations. We prove superconvergence towards a particular projection of the exact solution when the upwind flux is used for conservation laws and when the alternating flux is used for convection-diffusion equations. The order of superconvergence for both cases is proved to be $k+\frac{3}{2}$ when piecewise P^k polynomials with $k \geq 1$ are used. The proof is valid for arbitrary non-uniform regular meshes and for piecewise P^k polynomials with arbitrary $k \geq 1$, improving upon the results in [8, 9] in which the proof based on Fourier analysis was given only for uniform meshes with periodic boundary condition and piecewise P^1 polynomials.

Keywords: discontinuous Galerkin method; local discontinuous Galerkin method; superconvergence; upwind flux; projection; error estimates.

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1 Introduction

In this paper, we consider one-dimensional linear hyperbolic conservation laws

$$u_t + cu_x = 0, \tag{1.1}$$

and convection-diffusion equations

$$u_t + cu_x = bu_{xx}, \tag{1.2}$$

where c, b are constants and $b > 0$. We study the the superconvergence of the discontinuous Galerkin (DG) solutions and the local DG (LDG) solutions towards a particular projection of the exact solution. Superconvergence requires upwind fluxes for the DG scheme and alternating fluxes for the LDG scheme. This superconvergence also implies a good control on the time evolution of the errors.

The DG method discussed here is a class of finite element methods using completely discontinuous piecewise polynomial space for the numerical solution and the test functions. It was originally devised to solve hyperbolic conservation laws containing only first order spatial derivatives, e.g. [14, 13, 12, 11, 15, 17]. It has the advantage of flexibility for arbitrarily unstructured meshes, with a compact stencil, and with the ability to easily accommodate arbitrary *h*-*p* adaptivity. The DG method was later generalized to the LDG method by Cockburn and Shu to solve the convection-diffusion equation [16]. Their work was motivated by the successful numerical experiments of Bassi and Rebay [5] for the compressible Navier-Stokes equations.

For ordinary differential equations and steady hyperbolic problems, Adjerid *et al.* [1, 4] proved the DG solution is superconvergent at Radau points. In [8], we proved superconvergence of the DG solution towards a particular projection of the exact solution in the case of piecewise linear polynomials on uniform meshes for the linear conservation law (1.1) and considered its impact on the time growth of the errors. We also demonstrated numerically that the conclusions hold true for very general cases, including higher order DG methods, nonlinear equations, systems, and two dimensions. For convection-diffusion equations, in [7], Celiker and Cockburn studied the steady state solution of (1.2), and proved that for a large class of DG methods, the numerical fluxes (traces) are superconvergent, see also [6] for related discussions on elliptic problems. In [3, 2], Adjerid *et al.* showed for convection or diffusion dominant time dependent equations, the LDG solution is superconvergent at Radau points. In [9], we discussed the superconvergence property of the LDG scheme for convection-diffusion equations. We proved the superconvergence result for the heat equation in the case of piecewise linear solutions on uniform meshes, and gave numerical tests to demonstrate the validity of the result for higher order schemes and nonlinear equations.

The proof in [8, 9] uses Fourier analysis and works only for piecewise linear approximation space (because of the algebraic complication for higher order polynomials), uniform meshes and periodic boundary conditions. In this paper, we use a different framework to prove the superconvergence results and do not rely on Fourier analysis. The proof now works for arbitrary non-uniform regular meshes and schemes of any order.

Even though the proof in this paper is given for the simple scalar equations (1.1) and (1.2), the same superconvergence results can be easily proved for one-dimensional linear systems along the same lines. The generalization to two space dimensions is more involved, see [8] for some discussion.

This paper is organized as follows: in Section 2, we consider the superconvergence of the DG method for the linear conservation law (1.1). We prove our main superconvergence result in Theorem 2.2. In Section 3, we prove the superconvergence of the LDG method for the linear convection-diffusion equation (1.2), and discuss the effect of fluxes on superconvergence. Finally, conclusions and plans for future work are provided in Section 4. The proofs for some of the technical lemmas are collected in the Appendix.

2 Conservation laws

In this section, we consider, without loss of generality, the linear conservation law (1.1) with $c = 1$:

$$\begin{cases} u_t + u_x = 0, & 0 \le x \le 2\pi \\ u(x,0) = u_0(x) \end{cases} . \tag{2.1}$$

We will consider both the periodic boundary condition $u(0,t) = u(2\pi,t)$ and the initial-boundary value problem $u(0,t) = g(t)$.

The usual notation of the DG method is adopted. If we want to solve this equation on the interval $I = [0, 2\pi]$, first we divide it into N cells as follows

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{N+\frac{1}{2}} = 2\pi. \tag{2.2}$$

We denote

$$I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \qquad x_j = \frac{1}{2}\left(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}\right), \tag{2.3}$$

as the cells and cell centers respectively. $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ denotes length of each cell. We denote $h = \max_j h_j$ as length of the largest cell.

Define $V_h^k = \{v : v|_{I_j} \in P^k(I_j),\ j = 1, \cdots, N\}$ to be the approximation space, where $P^k(I_j)$ denotes all polynomials of degree at most k on I_j. The DG scheme using the upwind flux will become: find $u_h \in V_h^k$, such that

$$\int_{I_j} (u_h)_t v_h dx - \int_{I_j} u_h (v_h)_x dx + u_h^- v_h^-|_{j+\frac{1}{2}} - u_h^- v_h^+|_{j-\frac{1}{2}} = 0 \tag{2.4}$$

holds for any $v_h \in V_h^k$. Here and below $(v_h)^-_{j+\frac{1}{2}} = v_h(x^-_{j+\frac{1}{2}})$ denotes the left limit of the function v_h at the discontinuity point $x_{j+\frac{1}{2}}$. Likewise for v_h^+.

In addition, if $k \ge 1$, we can define $P_h^- u$ to be a projection of u into V_h^k, such that

$$\int_{I_j} P_h^- u\, v_h\, dx = \int_{I_j} u\, v_h\, dx \tag{2.5}$$

for any $v_h \in P^{k-1}$ on I_j, where k is the polynomial degree of the DG solution, and

$$(P_h^- u)^- = u^- \qquad \text{at } x_{j+1/2}. \tag{2.6}$$

Notice that this special projection is used in the error estimates of the DG methods to derive optimal L^2 error bounds in the literature, e.g. in [18]. We are going to show that indeed the numerical solution is closer to this special projection of the exact solution than to the exact solution itself, extending the results in [8]. Let us denote $e = u - u_h$ to be the error between the exact solution and numerical solution, $\varepsilon = u - P_h^- u$ to be the projection error, and $\bar{e} = P_h^- u - u_h$ to be the error between the numerical solution and the projection of the exact solution.

We introduce two functionals which are essential to our estimates. We prove in Lemma 2.1 that they are related to the L^2 norm of a function on I_j.

$$\mathcal{B}_j^-(M) = \int_{I_j} M(x) \frac{x - x_{j-1/2}}{h_j} \frac{d}{dx} \left(M(x) \frac{x - x_j}{h_j} \right) dx,$$

$$\mathcal{B}_j^+(M) = \int_{I_j} M(x) \frac{x - x_{j+1/2}}{h_j} \frac{d}{dx} \left(M(x) \frac{x - x_j}{h_j} \right) dx.$$

Lemma 2.1. For any function $M(x) \in C^1$ on I_j,

$$\mathcal{B}_j^-(M) = \frac{1}{4h_j} \int_{I_j} M^2(x) dx + \frac{M^2(x_{j+1/2})}{4}, \tag{2.7}$$

$$\mathcal{B}_j^+(M) = -\frac{1}{4h_j} \int_{I_j} M^2(x) dx - \frac{M^2(x_{j-1/2})}{4}. \tag{2.8}$$

The proof of this lemma is given in the Appendix.

Theorem 2.2. Let u be the exact solution of the equation (2.1). If $k \geq 1$, define u_h to be the DG solution of (2.4) with the initial condition $u_h(\cdot, 0) = P_h^- u_0$. We have the following error estimate:

$$||\bar{e}(\cdot, t)||_{L^2} \leq C_1 \, (t + 1) \, h^{k+3/2}, \tag{2.9}$$

and

$$||e(\cdot, t)||_{L^2} \leq C_1 \, t \, h^{k+3/2} + C_2 \, h^{k+1}, \tag{2.10}$$

where $C_1 = C_1(||u||_{k+3})$, $C_2 = C_2(||u||_{k+3})$.

Proof: Since u satisfies (2.1), we can easily check that

$$\int_{I_j} u_t v_h dx - \int_{I_j} u(v_h)_x dx + u^- v_h^-|_{j+\frac{1}{2}} - u^- v_h^+|_{j-\frac{1}{2}} = 0 \qquad (2.11)$$

holds for any $v_h \in V_h^k$. Combined with (2.4), we have the error equation

$$\int_{I_j} e_t v_h dx - \int_{I_j} e(v_h)_x dx + e^- v_h^-|_{j+\frac{1}{2}} - e^- v_h^+|_{j-\frac{1}{2}} = 0 \qquad (2.12)$$

which holds true for any $v_h \in V_h^k$. By the property (2.5) of the projection P_h^-, we have

$$\int_{I_j} \varepsilon(v_h)_x dx = 0$$

since $(v_h)_x$ is a polynomial of degree at most $k-1$ in I_j. By the property (2.6) of the projection P_h^-, we have

$$e^-_{j+\frac{1}{2}} = \varepsilon^-_{j+\frac{1}{2}} + \bar{e}^-_{j+\frac{1}{2}} = \bar{e}^-_{j+\frac{1}{2}}.$$

Thus,

$$\int_{I_j} e_t v_h dx - \int_{I_j} \bar{e}(v_h)_x dx + \bar{e}^- v_h^-|_{j+\frac{1}{2}} - \bar{e}^- v_h^+|_{j-\frac{1}{2}} = 0, \qquad (2.13)$$

and by integration by parts,

$$\int_{I_j} e_t v_h dx + \int_{I_j} (\bar{e})_x v_h dx + [\bar{e}] v_h^+|_{j-\frac{1}{2}} = 0, \qquad (2.14)$$

where $[\bar{e}] = \bar{e}^+ - \bar{e}^-$ denotes the jump of $\bar{e}$.

Taking $v_h = \bar{e}$ in (2.13), since $e = \bar{e} + \varepsilon$, we obtain

$$\int_{I_j} (\bar{e})_t \, \bar{e} dx + \int_{I_j} \varepsilon_t \, \bar{e} dx - \int_{I_j} \bar{e}\bar{e}_x dx + \bar{e}^- \bar{e}^-|_{j+\frac{1}{2}} - \bar{e}^- \bar{e}^+|_{j-\frac{1}{2}} = 0,$$

or

$$\int_{I_j} (\bar{e})_t \, \bar{e} dx + \int_{I_j} \varepsilon_t \, \bar{e} dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} + \frac{1}{2}[\bar{e}]^2_{j+\frac{1}{2}} = 0 \qquad (2.15)$$

with

$$\hat{F}_{j+\frac{1}{2}} = -\frac{1}{2}(\bar{e}^+_{j+\frac{1}{2}})^2 + \bar{e}^-_{j+\frac{1}{2}} \bar{e}^+_{j+\frac{1}{2}}.$$

Summing the equality (2.15) over j, with periodic boundary conditions, we have

$$\int_I (\bar{e})_t \, \bar{e} dx + \frac{1}{2} \sum_{j=1}^{N} [\bar{e}]^2_{j+\frac{1}{2}} + \int_I \varepsilon_t \, \bar{e} dx = 0.$$

For initial-boundary value problems, since $\bar{e}^-_{\frac{1}{2}} = 0$, the equality becomes

$$\int_I (\bar{e})_t \, \bar{e} dx + \frac{1}{2} \sum_{j=1}^{N-1} [\bar{e}]^2_{j+\frac{1}{2}} + \frac{1}{2} (\bar{e}^+_{\frac{1}{2}})^2 + \frac{1}{2} (\bar{e}^-_{N+\frac{1}{2}})^2 + \int_I \varepsilon_t \, \bar{e} dx = 0.$$

In both cases,

$$\frac{d}{dt} ||\bar{e}||^2_{L^2} \leq 2 \left| \int_I \varepsilon_t \, \bar{e} dx \right|. \tag{2.16}$$

Now, let us return to the error equation (2.14). If $v_h^+(x_{j-\frac{1}{2}}) = 0$, then the equation reduces to

$$\int_{I_j} e_t v_h dx + \int_{I_j} (\bar{e})_x v_h dx = 0.$$

Notice that this is a completely local equality inside the cell I_j. Throughout this paper we will repeatedly use such special test functions to obtain similar local equalities to facilitate our analysis.

Define $\bar{e} = e_j + w_j(x)(x - x_j)/h_j$ on I_j, with $e_j = \bar{e}(x_j)$ and $w_j(x) = (\bar{e} - e_j)h_j/(x - x_j) \in P^{k-1}$, then

$$\int_{I_j} e_t v_h dx + \int_{I_j} (w_j(x)(x - x_j)/h_j)_x v_h dx = 0$$

as long as $v_h^+(x_{j-\frac{1}{2}}) = 0$, $v_h \in P^k$. Clearly, $v_h = w_j(x)(x - x_{j-1/2})/h_j$ is a legitimate choice, so using the definition of $\mathcal{B}_j^-(M)$, we have

$$\int_{I_j} e_t w_j(x)(x - x_{j-1/2})/h_j dx + \mathcal{B}_j^-(w_j) = 0.$$

From Lemma 2.1, this is equivalent to

$$\int_{I_j} e_t \, w_j(x)(x - x_{j-1/2})/h_j \, dx + \frac{1}{4h_j} \int_{I_j} w_j^2(x) dx + \frac{w_j^2(x_{j+1/2})}{4} = 0. \tag{2.17}$$

Hence,

$$\int_{I_j} w_j(x)^2 dx \leq -4h_j \int_{I_j} e_t \, w_j(x)(x - x_{j-1/2})/h_j \, dx = -4 \int_{I_j} e_t \, w_j(x)(x - x_{j-1/2}) \, dx.$$

We define piecewise polynomials $w(x)$ and $\phi_1(x)$, such that $w(x) = w_j(x)$, $\phi_1(x) = x - x_{j-1/2}$ on I_j. Clearly $||\phi_1||_{L^\infty} = \max_j h_j = h$, hence

$$||w||^2_{L^2} \leq 4||e_t||_{L^2}||w||_{L^2}||\phi_1||_{L^\infty} \leq 4h||e_t||_{L^2}||w||_{L^2},$$

thus

$$||w||_{L^2} \leq 4h||e_t||_{L^2}. \tag{2.18}$$

The bound of $||e_t||_{L^2}$ can be obtained from the following lemma.

Lemma 2.3. Under the same condition as in Theorem 2.2, we have

$$||e_t||_{L^2} \leq Ch^{k+1}(t+1) \tag{2.19}$$

where $C = C(||u||_{k+3})$.

The proof of this lemma is given in the Appendix. We now resume the proof of Theorem 2.2. Combining (2.18) and (2.19), we have

$$||w||_{L^2} \leq Ch^{k+2}(t+1),$$

where $C = C(||u||_{k+3})$.

Next, we look back at the right hand side of (2.16)

$$\int_I \varepsilon_t \, \bar{e} dx = \sum_j \int_{I_j} \varepsilon_t \, (e_j + w_j(x)(x - x_j)/h_j) dx = \sum_j \int_{I_j} \varepsilon_t \, w_j(x)(x - x_j)/h_j dx$$

where we have used the fact that $k \geq 1$ and hence the definition of the projection P_h^- ensures that ε, as well as ε_t, are orthogonal to piecewise constant functions. Define a new function $\phi(x) = (x - x_j)/h_j$ on I_j, then $||\phi(x)||_{L^\infty} = \frac{1}{2}$, and

$$\left| \int_I \varepsilon_t \, \bar{e} dx \right| \leq ||\varepsilon_t||_{L^2} \; ||\phi||_{L^\infty} \; ||w||_{L^2} \leq C'||u||_{k+2} h^{k+1} \frac{1}{2} C h^{k+2}(t+1) = C_1 h^{2k+3}(1+t)$$

where $C_1 = C_1(||u||_{k+3})$. Plugging into (2.16), we have

$$\frac{d}{dt}||\bar{e}||_{L^2}^2 \leq C_1 h^{2k+3}(1+t).$$

Since $\bar{e}(x, 0) = 0$, we have, after an integration in t,

$$||\bar{e}||_{L^2} \leq C_1 h^{k+3/2}(1+t).$$

Combined with $||\varepsilon||_{L^2} \leq C'||u||_{k+1} h^{k+1}$, we have finished the proof for Theorem 2.2. ■

We remark that the error estimate (2.10) implies that the error does not grow with time for a long time $t = O(\frac{1}{\sqrt{h}})$. See [8] for numerical experiment results to show this non-growth of error for a long time for linear as well as nonlinear scalar and systems of hyperbolic conservation laws, for both periodic boundary conditions and initial-boundary value problems. This is a major advantage of DG methods for solving hyperbolic wave equations over long time.

3 Convection-diffusion equations

We are interested in the linear convection-diffusion equation with periodic boundary conditions,

$$\begin{cases} u_t + cu_x = bu_{xx} \\ u(x,0) = u_0(x) \\ u(0,t) = u(2\pi,t). \end{cases} \tag{3.1}$$

Here, $u_0(x)$ is a smooth 2π-periodic function, c and b are constants and $b > 0$. We consider only the periodic boundary conditions for simplicity. Since we do not use Fourier analysis, this assumption is not essential. The LDG scheme for (3.1) uses the same mesh and approximation space as in Section 2 and is formulated based on rewriting (3.1) into

$$\begin{cases} u_t + cu_x = aq_x \\ q - au_x = 0. \end{cases} \tag{3.2}$$

Here $a = \sqrt{b}$, and we introduce a new variable $z = cu - aq$, that will be used later in the proof. Then the scheme becomes, to find $u_h, q_h \in V_h^k$, such that

$$\begin{aligned} \int_{I_j} (u_h)_t v_h dx - \int_{I_j} cu_h (v_h)_x dx + c\tilde{u}_h v_h^-|_{j+\frac{1}{2}} - c\tilde{u}_h v_h^+|_{j-\frac{1}{2}} \\ + \int_{I_j} aq_h (v_h)_x dx - a\hat{q}_h v_h^-|_{j+\frac{1}{2}} + a\hat{q}_h v_h^+|_{j-\frac{1}{2}} = 0, \\ \int_{I_j} q_h w_h dx + \int_{I_j} au_h (w_h)_x dx - a\hat{u}_h w_h^-|_{j+\frac{1}{2}} + a\hat{u}_h w_h^+|_{j-\frac{1}{2}} = 0 \end{aligned} \tag{3.3}$$

hold for any $v_h, w_h \in V_h^k$, where $\tilde{u}_h$ is the upwind flux depending on the sign of c. Without loss of generality we assume $c \ge 0$ and $\tilde{u}_h = u_h^-$. The alternating diffusion fluxes are taken as

$$\hat{q}_h = q_h^+, \qquad \hat{u}_h = u_h^-, \tag{3.4}$$

or

$$\hat{q}_h = q_h^-, \qquad \hat{u}_h = u_h^+. \tag{3.5}$$

The projection P_h^- is defined as before. Similarly, the projection P_h^+ is defined as follows: for any function u, $P_h^+ u \in V_h^k$ satisfies

$$\int_{I_j} P_h^+ u \, v_h \, dx = \int_{I_j} u \, v_h \, dx$$

for any $v_h \in P^{k-1}$ on I_j and

$$(P_h^+ u)^+ = u^+ \qquad \text{at } x_{j-1/2}.$$

In order to better control the errors of the initial condition, we define two operators P_h^1 and P_h^2, which will be used in the initial condition of the numerical scheme. P_h^1 is defined as: for any function u, $P_h^1 u \in V_h^k$, and suppose $q_h \in V_h^k$ is the unique solution to

$$\int_{I_j} q_h w_h dx + \int_{I_j} a P_h^1 u (w_h)_x dx - a(P_h^1 u)^- w_h^- |_{j+\frac{1}{2}} + a(P_h^1 u)^- w_h^+ |_{j-\frac{1}{2}} = 0 \tag{3.6}$$

for any $w_h \in V_h^k$, then we require

$$\int_{I_j} \left((P_h^- u - P_h^1 u) - a(P_h^+ q - q_h) \right) v_h \, dx = 0 \tag{3.7}$$

for any $v_h \in P^{k-1}$ on I_j and

$$u^- - (P_h^1 u)^- = a(q^+ - q_h^+) \qquad \text{at } x_{j-1/2}. \tag{3.8}$$

We recall that $q = a u_x$.

On the other hand, P_h^2 is needed only for the case of $c > 0$ and is defined as follows. For any function u, $P_h^2 u \in V_h^k$, and suppose $q_h \in V_h^k$ is the unique solution to

$$\int_{I_j} q_h w_h dx + \int_{I_j} a P_h^2 u (w_h)_x dx - a(P_h^2 u)^+ w_h^- |_{j+\frac{1}{2}} + a(P_h^2 u)^+ w_h^+ |_{j-\frac{1}{2}} = 0 \tag{3.9}$$

for any $w_h \in V_h^k$, then we require

$$c P_h^2 u - a q_h = P_h^- z \tag{3.10}$$

for $z = cu - aq$.

The definitions of the above operators are nontrivial. Proof for existence and uniqueness will be provided in Lemma 3.1. We remark that P_h^1 and P_h^2 are only introduced for technical purposes in the proof, to guarantee that the initial errors of the LDG solution are small enough to be compatible with the superconvergence error estimate. In the numerical experiments, we have used simply the L^2 projection of u or $P_h^- u$, $P_h^+ u$ as the initial condition, and still observed superconvergence, see [9].

In the discussion that follows, we will consider various measurements of errors. Let us denote $e_u = u - u_h$ to be the error between the exact solution and the numerical solution, $\varepsilon_u = u - P_h u$ to be the projection error, and $\bar{e}_u = P_h u - u_h$ to be the error between the numerical solution and the projection of the exact solution. Similarly, $e_q = q - q_h$, $e_z = z - z_h$ is the error between the exact solution and the numerical solution, $\varepsilon_q = q - P_h q$, $\varepsilon_z = z - P_h z$ is the projection error, and $\bar{e}_q = P_h q - q_h$, $\bar{e}_z = P_h z - z_h$ is the error between the numerical solution and the projection of the exact solution for q and z, respectively. Here, the projection P_h can be P_h^- or P_h^+ depending on the problem and will be specified later.

We introduce a new parameter $\lambda = \frac{\min_j h_j}{\max_j h_j}$. In the rest of the paper, if λ appears in the estimate, it means we require a lower bound for λ, i.e., the mesh needs to be regular.

Lemma 3.1. $P_h^1 u$, $P_h^2 u$ exist and are unique. Moreover, we have the following estimate

$$||P_h^- u - P_h^1 u|| \leq C(\lambda, ||u||_{k+2}) h^{k+3/2}, \tag{3.11}$$

$$||P_h^+ u - P_h^2 u|| \leq C(||u||_{k+2}) h^{k+3/2}. \tag{3.12}$$

The proof of this lemma is given in the Appendix.

We will next present the major result of this section. We first consider the case $c > 0$.

Theorem 3.2. If $k \geq 1$, let u, $q = au_x$ be the exact solution of the convection diffusion equation (3.1) when $c > 0$ and $\tilde{u}_h = u_h^-$, and u_h, q_h be the LDG solution of (3.3). If the fluxes (3.4) are used, then we define $P_h u = P_h^- u$, $P_h q = P_h^+ q$, and we choose the initial condition as $u_h(\cdot,0) = P_h^1 u_0$. We have the following error estimate:

$$||\bar{e}_u(\cdot,t)||_{L^2}^2 + \int_0^t ||\bar{e}_q(\cdot,s)||_{L^2}^2\, ds \le Ch^{2k+3}(t+1)^2,$$

and in particular

$$||\bar{e}_u(\cdot,t)||_{L^2} \le Ch^{k+3/2}(t+1).$$

where $C = C(||u||_{k+5}, \lambda, a/c)$.

Otherwise, if the fluxes (3.5) are used, we let $P_h u = P_h^+ u$, $P_h z = P_h^- z$ and $u_h(\cdot,0) = P_h^2 u_0$. We have the following error estimate:

$$||\bar{e}_u(\cdot,t)||_{L^2}^2 + \int_0^t ||\bar{e}_z(\cdot,s)||_{L^2}^2\, ds \le Ce^{2C_1 t} h^{2k+3},$$

and in particular

$$||\bar{e}_u(\cdot,t)||_{L^2} \le Ce^{C_1 t} h^{k+3/2}.$$

where $C = C(||u||_{k+5},\, a/c)$ and $C_1 = C_1(a/c) > 0$.

Proof: Without loss of generality, we will only prove for $c = 1$.

We first consider the case for the fluxes (3.4). Now, the scheme becomes,

$$\int_{I_j} (u_h)_t v_h dx + \mathcal{T}_j(u_h, q_h; v_h) = 0, \qquad \int_{I_j} q_h w_h dx + \mathcal{Q}_j(u_h; w_h) = 0, \tag{3.13}$$

for any $v_h, w_h \in V_h^k$, where

$$\mathcal{T}_j(u_h,q_h;v_h) = -\int_{I_j} u_h (v_h)_x dx + u_h^- v_h^-|_{j+\frac{1}{2}} - u_h^- v_h^+|_{j-\frac{1}{2}} + \int_{I_j} a q_h (v_h)_x dx - a q_h^+ v_h^-|_{j+\frac{1}{2}} + a q_h^+ v_h^+|_{j-\frac{1}{2}},$$

and

$$\mathcal{Q}_j(u_h; w_h) = \int_{I_j} a u_h (w_h)_x dx - a u_h^- w_h^-|_{j+\frac{1}{2}} + a u_h^- w_h^+|_{j-\frac{1}{2}}.$$

From (3.2), we have

$$\int_{I_j} u_t v_h dx + \mathcal{T}_j(u, q; v_h) = 0, \qquad \int_{I_j} q w_h dx + \mathcal{Q}_j(u; w_h) = 0$$

that hold for any $v_h, w_h \in V_h^k$. Combined with (3.13), we have the error equations

$$\int_{I_j} (e_u)_t v_h dx + \mathcal{T}_j(e_u, e_q; v_h) = 0, \qquad \int_{I_j} e_q w_h dx + \mathcal{Q}_j(e_u; w_h) = 0$$

that hold for any $v_h, w_h \in V_h^k$. Using the properties of the projections P_h^- and P_h^+, we have

$$\int_{I_j} (e_u)_t v_h dx + \mathcal{T}_j(\bar{e}_u, \bar{e}_q; v_h) = 0, \tag{3.14}$$

$$\int_{I_j} e_q w_h dx + \mathcal{Q}_j(\bar{e}_u; w_h) = 0, \tag{3.15}$$

or equivalently

$$\int_{I_j} (e_u)_t v_h dx - \int_{I_j} \bar{e}_u (v_h)_x dx + \bar{e}_u^- v_h^- |_{j+\frac{1}{2}} - \bar{e}_u^- v_h^+ |_{j-\frac{1}{2}} - a \int_{I_j} (\bar{e}_q)_x v_h dx - a[\bar{e}_q] v_h^- |_{j+\frac{1}{2}} = 0, \tag{3.16}$$

and

$$\int_{I_j} e_q w_h dx - a \int_{I_j} (\bar{e}_u)_x w_h dx - a[\bar{e}_u] w_h^+ |_{j-\frac{1}{2}} = 0 \tag{3.17}$$

for any $v_h, w_h \in V_h^k$. Taking $v_h = \bar{e}_u, w_h = \bar{e}_q$ in (3.14) and (3.15), summing (3.14) and (3.15) and then over all j, we obtain

$$\int_I (\bar{e}_u)_t \, \bar{e}_u dx + \int_I (\bar{e}_q)^2 dx + \int_I (\varepsilon_u)_t \, \bar{e}_u dx + \int_I \varepsilon_q \, \bar{e}_q dx + \frac{1}{2} \sum_j [\bar{e}_u]_{j+\frac{1}{2}}^2 = 0. \tag{3.18}$$

Thus,

$$\frac{1}{2} \frac{d}{dt} ||\bar{e}_u||_{L^2}^2 + ||\bar{e}_q||_{L^2}^2 \le \left| \int_I (\varepsilon_u)_t \, \bar{e}_u dx \right| + \left| \int_I \varepsilon_q \, \bar{e}_q dx \right|. \tag{3.19}$$

Now, we return to the error equation (3.17). If $w_h^+(x_{j-\frac{1}{2}}) = 0$, then the equation reduces to

$$\int_{I_j} e_q w_h dx - a \int_{I_j} (\bar{e}_u)_x w_h dx = 0.$$

Define $\bar{e}_u = r_j + d_j(x)(x - x_j)/h_j$ on I_j, where r_j is a constant and $d_j(x) \in P^{k-1}$, and let $w_h = d_j(x)(x - x_{j-1/2})/h_j$. Clearly, $w_h \in P^k$ and $w_h^+(x_{j-\frac{1}{2}}) = 0$, so

$$\int_{I_j} e_q \, d_j(x)(x - x_{j-1/2})/h_j \, dx - a\mathcal{B}_j^-(d_j) = 0.$$

By Lemma 2.1, we have

$$\int_{I_j} e_q \, d_j(x)(x - x_{j-1/2})/h_j \, dx - \frac{a}{4h_j} \int_{I_j} d_j^2(x) dx - a \frac{d_j^2(x_{j+1/2})}{4} = 0,$$

hence

$$\int_{I_j} d_j(x)^2 dx \le \frac{4}{a}\int_{I_j} e_q\, d_j(x)(x - x_{j-1/2})\, dx. \tag{3.20}$$

Introducing piecewise polynomials $\phi_1(x)$ and $d(x)$, such that $\phi_1(x) = x - x_{j-1/2}$ and $d(x) = d_j(x)$ on I_j, we know that $||\phi_1||_{L^\infty} = h$. We then have

$$||d||_{L^2}^2 \le \frac{4}{a}||e_q||_{L^2}||d||_{L^2}||\phi_1||_{L^\infty},$$

thus

$$||d||_{L^2} \le \frac{4h}{a}||e_q||_{L^2}. \tag{3.21}$$

For the other error equation (3.16), we follow the same procedure. If $v_h^-(x_{j+\frac{1}{2}}) = 0$, then the equation reduces to

$$\int_{I_j}(e_u)_t v_h dx - \int_{I_j}\bar{e}_u (v_h)_x dx - \bar{e}_u^- v_h^+|_{j-\frac{1}{2}} - a\int_{I_j}(\bar{e}_q)_x v_h dx = 0.$$

Define $\bar{e}_q = b_j + s_j(x)(x - x_j)/h_j$ on I_j, and let $v_h = s_j(x)(x - x_{j+1/2})/h_j$ in the equation above. Clearly, $v_h \in P^k$ and $v_h^-(x_{j+\frac{1}{2}}) = 0$, so

$$\int_{I_j}(e_u)_t\, s_j(x)(x - x_{j+1/2})/h_j\, dx - Q_{1,j} + Q_{2,j} = 0$$

with

$$\begin{aligned} Q_{1,j} &= \int_{I_j}\bar{e}_u(v_h)_x dx + \bar{e}_u^- v_h^+|_{j-\frac{1}{2}} \\ &= \int_{I_j} d_j(x)\frac{x - x_j}{h_j}(v_h)_x dx + (d_{j-1}(x)\frac{x - x_{j-1}}{h_{j-1}} + r_{j-1} - r_j)v_h^+|_{j-\frac{1}{2}} \\ &= \int_{I_j} d_j(x)\frac{x - x_j}{h_j}(s_j(x)(x - x_{j+1/2})/h_j)_x dx - \left(\frac{1}{2}d_{j-1}(x_{j-\frac{1}{2}}) + r_{j-1} - r_j\right)\, s_j(x_{j-\frac{1}{2}}) \end{aligned}$$

and

$$Q_{2,j} = -a\int_{I_j}(\bar{e}_q)_x v_h dx = -a\mathcal{B}_j^+(s_j) = a\left(\frac{1}{4h_j}\int_{I_j} s_j^2(x)dx + \frac{s_j^2(x_{j-\frac{1}{2}})}{4}\right)$$

where we have used Lemma 2.1. Thus,

$$\int_{I_j} s_j^2(x)dx \le \frac{4h_j}{a}\left(Q_{1,j} - \int_{I_j}(e_u)_t\, s_j(x)(x - x_{j+1/2})/h_j\, dx\right)$$

and hence, if we define the piecewise polynomial $s(x)$ such that $s(x) = s_j(x)$ when $x \in I_j$, then

$$\int_I s^2(x)dx \le \frac{4}{a}\left(\left|\sum_j h_j Q_{1,j}\right| + \left|\sum_j \int_{I_j} (e_u)_t\, s_j(x)(x - x_{j+1/2})\, dx\right| \right). \tag{3.22}$$

To estimate the right hand side of the above inequality, we need the following lemma.

Lemma 3.3. Under the same condition as in Theorem 3.2, we have

$$||(e_u)_t||_{L^2} \le Ch^{k+1}(t+1), \tag{3.23}$$

$$||e_u||_{L^2} \le Ch^{k+1}(t+1), \tag{3.24}$$

$$||e_q||_{L^2} \le Ch^{k+1}(t+1), \tag{3.25}$$

where $C = C(||u||_{k+5})$ is a constant.

The proof of this lemma is given in the Appendix. From (3.21) and (3.25), we have

$$||d||_{L^2} \le Ch^{k+2}(t+1),$$

where $C = C(a)$. Our next goal is to bound the right hand side of (3.22), thus obtain a bound for $s(x)$. Define the piecewise polynomial $\phi_2(x)$, such that $\phi_2(x) = x - x_{j+1/2}$ on I_j. Then $||\phi_2||_{L^\infty} = h$ and

$$\left|\sum_j \int_{I_j} (e_u)_t\, s_j(x)(x - x_{j+1/2})\, dx\right| \le ||(e_u)_t||_{L^2}\, ||s||_{L^2}\, ||\phi_2||_{L^\infty} \le Ch^{k+2}(t+1)||s||_{L^2}.$$

The other term on the right hand side of (3.22) is

$$\begin{aligned} &\sum_j h_j Q_{1,j} \\ = \;& \sum_j \left(\int_{I_j} d_j(x)(x - x_j)(s_j(x)(x - x_{j+1/2})/h_j)_x dx - h_j \left(\frac{1}{2} d_{j-1}(x_{j-\frac{1}{2}}) + r_{j-1} - r_j\right) s_j(x_{j-\frac{1}{2}}) \right). \end{aligned}$$

We need to express $r_j - r_{j-1}$ in terms of d_j and $\int_{I_j} e_q dx$. In (3.15), let $w_h = 1$, we get

$$\int_{I_j} e_q dx - a\bar{e}_u^-|_{j+\frac{1}{2}} + a\bar{e}_u^-|_{j-\frac{1}{2}} = 0.$$

After plugging in $\bar{e}_u = r_j + d_j(x)\frac{x-x_j}{h_j}$ on I_j, we obtain

$$r_j - r_{j-1} = \frac{\int_{I_j} e_q dx}{a} - \frac{1}{2}d_j(x_{j+\frac{1}{2}}) + \frac{1}{2}d_{j-1}(x_{j-\frac{1}{2}}).$$

Thus

$$\begin{aligned}
& \sum_j h_j Q_{1,j} \\
= \ & \sum_j \int_{I_j} d_j(x)(x-x_j)s_j(x)/h_j dx + \sum_j \int_{I_j} d_j(x)(x-x_j)s_j'(x)(x-x_{j+1/2})/h_j dx \\
& - \sum_j \frac{1}{2}h_j\, d_j(x_{j+\frac{1}{2}})\, s_j(x_{j-\frac{1}{2}}) + \frac{1}{a}\sum_j h_j \left(\int_{I_j} e_q dx\right) s_j(x_{j-\frac{1}{2}}) \\
= \ & T_1 + T_2 - T_3 + T_4,
\end{aligned}$$

where

$$T_1 = \sum_j \int_{I_j} d_j(x)s_j(x)(x-x_j)/h_j dx, \qquad T_2 = \sum_j \int_{I_j} d_j(x)(x-x_j)s_j'(x)(x-x_{j+1/2})/h_j dx,$$

$$T_3 = \sum_j \frac{1}{2}h_j\, d_j(x_{j+\frac{1}{2}})\, s_j(x_{j-\frac{1}{2}}), \qquad T_4 = \frac{1}{a}\sum_j h_j \left(\int_{I_j} e_q dx\right) s_j(x_{j-\frac{1}{2}}).$$

We again introduce a piecewise polynomial $\phi_3 = (x-x_j)/h_j$ on I_j, then $||\phi_3||_{L^\infty} = \frac{1}{2}$, and

$$|T_1| \le ||d||_{L^2}\, ||s||_{L^2}\, ||\phi_3||_{L^\infty} \le Ch^{k+2}(t+1)||s||_{L^2}.$$

Similarly, for T_2, we introduce $\phi_4 = (x-x_j)(x-x_{j+1/2})/h_j$ on I_j. We have $||\phi_4||_{L^\infty} = \frac{h}{2}$, and

$$|T_2| \le ||d||_{L^2}\, ||s'||_{L^2}\, ||\phi_4||_{L^\infty} \le Ch^{k+3}(t+1)||s'||_{L^2}.$$

Since $s(x) \in V_h^{k-1}$, we have $||s'||_{L^2} \le C_{k-1}/h\, ||s||_{L^2}$ for a regular mesh. Here, C_{k-1} only depends on k. Thus,

$$|T_2| \le Ch^{k+2}(t+1)||s||_{L^2}.$$

For the two remaining terms, we have

$$
\begin{aligned}
|T_3| &\leq \frac{1}{2}\sum_j |h_j\, d_j(x_{j+\frac{1}{2}})\, s_j(x_{j-\frac{1}{2}})| \\
&\leq \frac{1}{2}\sum_j h_j \frac{C_k}{\sqrt{h_j}}\sqrt{\int_{I_j} d_j^2 dx}\frac{C_k}{\sqrt{h_j}}\sqrt{\int_{I_j} s_j^2 dx} \\
&\leq \frac{1}{2}C_k^2 \sum_j \sqrt{\int_{I_j} d_j^2 dx \int_{I_j} s_j^2 dx} \\
&\leq \frac{1}{2}C_k^2 \sqrt{\sum_j \int_{I_j} d_j^2 dx \sum_j \int_{I_j} s_j^2 dx} \\
&= \frac{1}{2}C_k^2 ||d||_{L^2}\, ||s||_{L^2} \\
&\leq Ch^{k+2}(t+1)||s||_{L^2},
\end{aligned}
$$

and

$$
\begin{aligned}
|T_4| &\leq \frac{1}{a}\sum_j \left| h_j \left(\int_{I_j} e_q dx\right) s_j(x_{j-\frac{1}{2}})\right| \\
&\leq \frac{1}{a}\sum_j h_j \sqrt{h_j}\sqrt{\int_{I_j} e_q^2 dx}\frac{C_k}{\sqrt{h_j}}\sqrt{\int_{I_j} s_j^2 dx} \\
&\leq \frac{1}{a}C_k h \sum_j \sqrt{\int_{I_j} e_q^2 dx \int_{I_j} s_j^2 dx} \\
&\leq \frac{1}{a}C_k h \sqrt{\sum_j \int_{I_j} e_q^2 dx \sum_j \int_{I_j} s_j^2 dx} \\
&= \frac{1}{a}C_k h ||e_q||_{L^2}\, ||s||_{L^2} \\
&\leq Ch^{k+2}(t+1)||s||_{L^2}.
\end{aligned}
$$

In the above derivation, we use the property that $d(x), s(x) \in V_h^k$, thus C_k is a constant that only depends on k.

Now, (3.22) yields

$$||s||_{L^2}^2 \leq Ch^{k+2}(t+1)||s||_{L^2},$$

and therefore

$$||s||_{L^2} \leq C(\lambda, a)h^{k+2}(t+1).$$

We are now ready for the final step of our proof. In (3.19),

$$\int_I (\varepsilon_u)_t\, \bar{e}_u dx = \sum_j \int_{I_j} (\varepsilon_u)_t\, (r_j + d_j(x)(x-x_j)/h_j) dx = \sum_j \int_{I_j} (\varepsilon_u)_t\, d_j(x)(x-x_j)/h_j dx,$$

$$\int_I \varepsilon_q\, \bar{e}_q dx = \sum_j \int_{I_j} \varepsilon_q\, (b_j + s_j(x)(x-x_j)/h_j) dx = \sum_j \int_{I_j} \varepsilon_q\, s_j(x)(x-x_j)/h_j dx.$$

Recall $\phi_3(x) = (x-x_j)/h_j$, and $||\phi_3(x)||_{L^\infty} = \frac{1}{2}$, thus

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\bar{e}_u||_{L^2}^2 + ||\bar{e}_q||_{L^2}^2 &\le \left|\int_I (\varepsilon_u)_t\, \bar{e}_u dx\right| + \left|\int_I \varepsilon_q\, \bar{e}_q dx\right| \\
&\le ||(\varepsilon_u)_t||_{L^2}\, ||\phi_3||_{L^\infty}\, ||d||_{L^2} + ||\varepsilon_q||_{L^2}\, ||\phi_3||_{L^\infty}\, ||s||_{L^2} \\
&\le Ch^{2k+3}(t+1)
\end{aligned}$$

where $C = C(||u||_{k+5}, \lambda, a)$. Using the fact that $||\bar{e}_u||_{L^2} \le C(||u||_{k+3}, \lambda) h^{k+3/2}$ at $t = 0$, which is due to the special choice of the initial condition and Lemma 3.1, we have proved

$$||\bar{e}_u(\cdot, t)||_{L^2}^2 + \int_0^t ||\bar{e}_q(\cdot, s)||_{L^2}^2\, ds \le Ch^{2k+3}(t+1)^2,$$

and in particular

$$||\bar{e}_u(\cdot, t)||_{L^2} \le Ch^{k+3/2}(t+1).$$

Next, we consider the flux choice (3.5). This is the case that the choices for $\hat{u}$ for the diffusion part and $\tilde{u}$ for the convection part do not coincide. The scheme becomes,

$$\begin{aligned}
&\int_{I_j} (u_h)_t v_h dx - \int_{I_j} u_h (v_h)_x dx + u_h^- v_h^-|_{j+\frac{1}{2}} - u_h^- v_h^+|_{j-\frac{1}{2}} \\
&\qquad + \int_{I_j} a q_h (v_h)_x dx - a q_h^- v_h^-|_{j+\frac{1}{2}} + a q_h^- v_h^+|_{j-\frac{1}{2}} = 0, \\
&\int_{I_j} q_h w_h dx + \int_{I_j} a u_h (w_h)_x dx - a u_h^+ w_h^-|_{j+\frac{1}{2}} + a u_h^+ w_h^+|_{j-\frac{1}{2}} = 0
\end{aligned} \tag{3.26}$$

for any $v_h, w_h \in V_h^k$. The error equations are now,

$$\begin{aligned}
&\int_{I_j} (e_u)_t v_h dx - \int_{I_j} e_u (v_h)_x dx + e_u^- v_h^-|_{j+\frac{1}{2}} - e_u^- v_h^+|_{j-\frac{1}{2}} \\
&\qquad + \int_{I_j} a e_q (v_h)_x dx - a e_q^- v_h^-|_{j+\frac{1}{2}} + a e_q^- v_h^+|_{j-\frac{1}{2}} = 0, \\
&\int_{I_j} e_q w_h dx + \int_{I_j} a e_u (w_h)_x dx - a e_u^+ w_h^-|_{j+\frac{1}{2}} + a e_u^+ w_h^+|_{j-\frac{1}{2}} = 0
\end{aligned}$$

for any $v_h, w_h \in V_h^k$. Since $z = u - aq$, $e_z = e_u - ae_q$, $e_q = (e_u - e_z)/a$, we have

$$\int_{I_j} (e_u)_t v_h dx - \int_{I_j} e_z (v_h)_x dx + e_z^- v_h^- |_{j+\frac{1}{2}} - e_z^- v_h^+ |_{j-\frac{1}{2}} = 0,$$

$$\int_{I_j} \frac{e_u - e_z}{a} w_h dx + \int_{I_j} a e_u (w_h)_x dx - a e_u^+ w_h^- |_{j+\frac{1}{2}} + a e_u^+ w_h^+ |_{j-\frac{1}{2}} = 0$$

for any $v_h, w_h \in V_h^k$. Using the properties of the projections P_h^- and P_h^+, we have

$$\int_{I_j} (e_u)_t v_h dx - \int_{I_j} \bar{e}_z (v_h)_x dx + \bar{e}_z^- v_h^- |_{j+\frac{1}{2}} - \bar{e}_z^- v_h^+ |_{j-\frac{1}{2}} = 0, \tag{3.27}$$

$$\int_{I_j} \frac{e_u - e_z}{a} w_h dx + \int_{I_j} a \bar{e}_u (w_h)_x dx - a \bar{e}_u^+ w_h^- |_{j+\frac{1}{2}} + a \bar{e}_u^+ w_h^+ |_{j-\frac{1}{2}} = 0 \tag{3.28}$$

or

$$\int_{I_j} (e_u)_t v_h dx + \int_{I_j} (\bar{e}_z)_x v_h dx + [\bar{e}_z] v_h^+ |_{j-\frac{1}{2}} = 0, \tag{3.29}$$

$$\int_{I_j} \frac{e_u - e_z}{a} w_h dx - a \int_{I_j} (\bar{e}_u)_x w_h dx - a [\bar{e}_u] w_h^- |_{j+\frac{1}{2}} = 0 \tag{3.30}$$

for any $v_h, w_h \in V_h^k$. Letting $w_h = \bar{e}_z$, $v_h = \bar{e}_u$, multiplying (3.27) by a, subtracting (3.28), and summing over all j, we obtain

$$a \int_I (\bar{e}_u)_t \, \bar{e}_u dx + \int_I \frac{(\bar{e}_z)^2}{a} dx + a \int_I (\varepsilon_u)_t \, \bar{e}_u dx + \frac{1}{a} \int_I \varepsilon_z \, \bar{e}_z dx - \frac{1}{a} \int_I \varepsilon_u \, \bar{e}_z dx - \frac{1}{a} \int_I \bar{e}_u \, \bar{e}_z dx = 0, \tag{3.31}$$

hence we have

$$\begin{aligned} \frac{a}{2} \frac{d}{dt} ||\bar{e}_u||_{L^2}^2 + \int_I \frac{(\bar{e}_z)^2}{a} dx &\le a \left| \int_I (\varepsilon_u)_t \, \bar{e}_u dx \right| + \frac{1}{a} \left| \int_I \varepsilon_z \, \bar{e}_z dx \right| + \frac{1}{a} \left| \int_I \varepsilon_u \, \bar{e}_z dx \right| + \frac{1}{a} \left| \int_I \bar{e}_u \, \bar{e}_z dx \right| \\ &\le a \left| \int_I (\varepsilon_u)_t \, \bar{e}_u dx \right| + \frac{1}{a} \left| \int_I \varepsilon_z \, \bar{e}_z dx \right| + \frac{1}{a} \left| \int_I \varepsilon_u \, \bar{e}_z dx \right| + \frac{1}{2a} ||\bar{e}_u||_{L^2}^2 + \frac{1}{2a} \int_I (\bar{e}_z)^2 dx. \end{aligned}$$

Thus,

$$\frac{d}{dt} ||\bar{e}_u||_{L^2}^2 + \frac{1}{a^2} ||\bar{e}_z||_{L^2}^2 \le 2 \left| \int_I (\varepsilon_u)_t \, \bar{e}_u dx \right| + \frac{2}{a^2} \left| \int_I \varepsilon_z \, \bar{e}_z dx \right| + \frac{2}{a^2} \left| \int_I \varepsilon_u \, \bar{e}_z dx \right| + \frac{1}{a^2} ||\bar{e}_u||_{L^2}^2. \tag{3.32}$$

Now, we return to the error equation (3.29) and (3.30). If $v_h^+(x_{j-\frac{1}{2}}) = 0$ and $w_h^-(x_{j+\frac{1}{2}}) = 0$, then the equations reduce to

$$\int_{I_j} (e_u)_t v_h dx + \int_{I_j} (\bar{e}_z)_x v_h dx = 0,$$

$$\int_{I_j} \frac{e_u - e_z}{a} w_h dx - a \int_{I_j} (\bar{e}_u)_x w_h dx = 0.$$

Define $\bar{e}_u = r_j + d_j(x)(x - x_j)/h_j$, $\bar{e}_z = b_j + s_j(x)(x - x_j)/h_j$ on I_j, with r_j, b_j being constants and $d_j(x), s_j(x) \in P^{k-1}$. If we choose the test functions as $v_h = s_j(x)(x - x_{j-1/2})/h_j$, $w_h = d_j(x)(x - x_{j+1/2})/h_j$, then we have

$$\int_{I_j} (e_u)_t \, s_j(x)(x - x_{j-1/2})/h_j \, dx + \mathcal{B}_j^-(s_j) = 0,$$
$$\int_{I_j} e_q \, d_j(x)(x - x_{j+1/2})/h_j \, dx - a\mathcal{B}_j^+(d_j) = 0.$$

By Lemma 2.1, we have

$$\int_{I_j} (e_u)_t \, s_j(x)(x - x_{j-1/2})/h_j \, dx + \frac{1}{4h_j} \int_{I_j} s_j^2(x) dx + \frac{s_j^2(x_{j+1/2})}{4} = 0,$$
$$\int_{I_j} e_q \, d_j(x)(x - x_{j+1/2})/h_j \, dx + \frac{a}{4h_j} \int_{I_j} d_j^2(x) dx + \frac{a d_j^2(x_{j-1/2})}{4} = 0.$$

Then,

$$\int_{I_j} s_j^2(x) dx \leq -4 \int_{I_j} (e_u)_t \, s_j(x)(x - x_{j-1/2}) \, dx,$$
$$\int_{I_j} d_j^2(x) dx \leq -\frac{4}{a} \int_{I_j} e_q \, d_j(x)(x - x_{j+1/2}) \, dx.$$

Define piecewise polynomials $s(x), d(x), \phi_1(x), \phi_2(x)$, such that $s(x) = s_j(x), d(x) = d_j(x), \phi_1(x) = x - x_{j-1/2}, \phi_2(x) = x - x_{j+1/2}$ on I_j, then

$$||s||_{L^2}^2 \leq 4||(e_u)_t||_{L^2}||s||_{L^2}||\phi_1||_{L^\infty}, \qquad ||d||_{L^2}^2 \leq \frac{4}{a}||e_q||_{L^2}||d||_{L^2}||\phi_2||_{L^\infty}.$$

However, $||\phi_1||_{L^\infty} = ||\phi_2||_{L^\infty} = h$, hence we conclude

$$||s||_{L^2} \leq Ch||(e_u)_t||_{L^2}, \qquad ||d||_{L^2} \leq Ch||e_q||_{L^2}. \tag{3.33}$$

Moreover, from (3.10), at $t = 0$, $\bar{e}_z = 0$. Hence, by (3.27), at $t = 0$, $\int_{I_j} (e_u)_t v_h \, dx = 0$, for any $v \in V_h^k$. This implies $||(\bar{e}_u)_t|| \leq Ch^{k+1}$ at $t = 0$.

The bound for $||(e_u)_t||_{L^2}$ and $||e_q||_{L^2}$ at time t can be obtained similar to Lemma 3.3.

Lemma 3.4. Under the same condition as in Theorem 3.2, we have

$$||e_u||_{L^2} \leq Ce^{C_1 t} h^{k+1}, \tag{3.34}$$

$$||(e_u)_t||_{L^2} \le Ce^{C_1 t}h^{k+1}, \tag{3.35}$$

$$||e_q||_{L^2} \le Ce^{C_1 t}h^{k+1}, \tag{3.36}$$

with $C = C(||u||_{k+5}, a)$, $C_1 = C_1(a) > 0$.

The proof of this lemma is given in the Appendix. By Lemma 3.4, (3.33) leads to

$$||s||_{L^2} \le Ce^{C_1 t}h^{k+2}, \qquad ||d||_{L^2} \le Ce^{C_1 t}h^{k+2}$$

with $C = C(||u||_{k+5}, a)$, $C_1 = C_1(a) > 0$.

We are now ready for the final step of our proof. In (3.32),

$$\int_I (\varepsilon_u)_t\, \bar{e}_u dx = \sum_j \int_{I_j} (\varepsilon_u)_t\, (r_j + d_j(x)(x-x_j)/h_j) dx = \sum_j \int_{I_j} (\varepsilon_u)_t\, d_j(x)(x-x_j)/h_j dx,$$

$$\int_I \varepsilon_z\, \bar{e}_z dx = \sum_j \int_{I_j} \varepsilon_z\, (b_j + s_j(x)(x-x_j)/h_j) dx = \sum_j \int_{I_j} \varepsilon_z\, s_j(x)(x-x_j)/h_j dx,$$

$$\int_I \varepsilon_u\, \bar{e}_z dx = \sum_j \int_{I_j} \varepsilon_u\, (b_j + s_j(x)(x-x_j)/h_j) dx = \sum_j \int_{I_j} \varepsilon_u\, s_j(x)(x-x_j)/h_j dx.$$

Recall $\phi_3(x) = (x-x_j)/h_j$, and $||\phi_3(x)||_{L^\infty} = \frac{1}{2}$, thus

$$\begin{aligned}
&\frac{d}{dt}||\bar{e}_u||^2_{L^2} + \frac{1}{a^2}||\bar{e}_z||^2_{L^2} \\
&\le ||\phi_3||_{L^\infty}\left(2||(\varepsilon_u)_t||_{L^2}\, ||d||_{L^2} + \frac{2}{a^2}||\varepsilon_z||_{L^2}\, ||s||_{L^2} + \frac{2}{a^2}||\varepsilon_u||_{L^2}\, ||s||_{L^2}\right) + \frac{1}{a^2}||\bar{e}_u||_{L^2} \\
&\le Ce^{C_1 t}h^{2k+3} + C_2||\bar{e}_u||_{L^2}
\end{aligned}$$

and $C = C(||u||_{k+5}, a)$, $C_1 = C_1(a) > 0, C_2 = C_2(a) > 0$. Using the fact that $||\bar{e}_u||_{L^2} \le C(||u||_{k+3}, \lambda)h^{k+3/2}$ at $t = 0$, which is due to the special choice of the initial condition and Lemma 3.1, we have proved

$$||\bar{e}_u(\cdot, t)||^2_{L^2} + \int_0^t ||\bar{e}_z(\cdot, s)||^2_{L^2}\, ds \le Ce^{2C_1 t}h^{2k+3},$$

and in particular

$$||\bar{e}_u(\cdot, t)||_{L^2} \le Ce^{C_1 t}h^{k+3/2},$$

where $C = C(||u||_{k+5}, a, \lambda)$, $C_1 = C_1(a) > 0$. ■

Theorem 3.2 can be generalized to equations in the case of $c \leq 0$. The case of $c < 0$ is symmetric to that of $c > 0$ and is omitted. For $c = 0$, we have the following theorem.

Theorem 3.5. If $k \geq 1$, let u, $q = au_x$ be the exact solution of the diffusion equation (3.1) when $c = 0$, and u_h, q_h be the LDG solution of (3.3). If the fluxes (3.4) are used, we define $P_h u = P_h^- u$, $P_h q = P_h^+ q$, and we choose the initial condition as $u_h(\cdot, 0) = P_h^1 u_0$. We have the following error estimate:

$$||\bar{e}_u(\cdot, t)||_{L^2}^2 + \int_0^t ||\bar{e}_q(\cdot, s)||_{L^2}^2 \, ds \leq Ch^{2k+3}(t+1)^2,$$

and in particular

$$||\bar{e}_u(\cdot, t)||_{L^2} \leq Ch^{k+3/2}(t+1).$$

The situation with the fluxes (3.5) is symmetric to that with the fluxes (3.4).

The proof of this theorem is similar to that for the case of $c > 0$ given above and is therefore omitted.

4 Conclusion and future work

In this paper, we have studied the superconvergence property of the DG and LDG methods applied to one-dimensional linear conservation laws and convection-diffusion equations. We improve the proof in [8, 9] to arbitrary regular mesh and any order $k \geq 1$.

Future work includes the study of superconvergence of DG and LDG for two-dimensional problems and for nonlinear equations.

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A Appendix: Proofs of some of the lemmas

In this appendix, we collect the proofs of some of the technical lemmas.

A.1 The proof of Lemma 2.1

We will only prove (2.7). The proof for (2.8) follows similar lines and is omitted.

$$\begin{aligned}
\mathcal{B}_j^-(M) &= \int_{I_j} M(x)\frac{x-x_{j-1/2}}{h_j}\left(M'(x)\frac{x-x_j}{h_j}+M(x)\frac{1}{h_j}\right)dx \\
&= \int_{I_j}\left(M(x)M'(x)\frac{(x-x_j)(x-x_{j-1/2})}{h_j^2}+M^2(x)\frac{x-x_{j-1/2}}{h_j^2}\right)dx \\
&= \int_{I_j}\frac{d}{dx}\left(\frac{M^2(x)}{2}\right)\frac{(x-x_j)(x-x_{j-1/2})}{h_j^2}dx+\int_{I_j}M^2(x)\frac{x-x_{j-1/2}}{h_j^2}dx \\
&= \frac{M^2(x_{j+1/2})}{4}-\int_{I_j}\frac{M^2(x)}{2}\frac{2x-x_j-x_{j-1/2}}{h_j^2}dx+\int_{I_j}M^2(x)\frac{x-x_{j-1/2}}{h_j^2}dx \\
&= \frac{1}{4h_j}\int_{I_j}M^2(x)dx+\frac{M^2(x_{j+1/2})}{4}.
\end{aligned}$$

This finishes the proof of Lemma 2.1.

A.2 The proof of Lemma 2.3

From the projection results [10], $||\varepsilon_t||_{L^2}\le C'||u_t||_{k+1}h^{k+1}\le C'||u||_{k+2}h^{k+1}$, and $||\varepsilon_{tt}||_{L^2}\le C'||u_{tt}||_{k+1}h^{k+1}\le C'||u||_{k+3}h^{k+1}$, where the constant C' only depends on k.

Since $e=\varepsilon+\bar{e}$, we will only need to prove that $||\bar{e}_t||_{L^2}\le Ch^{k+1}(t+1)$. Starting from the error equation (2.12) and taking the derivative with respect to t, we get

$$\int_{I_j} e_{tt}v_h dx-\int_{I_j}\bar{e}_t(v_h)_x dx+\bar{e}_t^- v_h^-|_{j+\frac{1}{2}}-\bar{e}_t^- v_h^+|_{j-\frac{1}{2}}=0$$

which holds for any $v_h\in V_h^k$. By taking $v_h=\bar{e}_t$, and summing up over j, we have

$$\int_I(\bar{e})_{tt}\,\bar{e}_t dx+\frac{1}{2}\sum_{j=1}^N[\bar{e}_t]^2_{j+\frac{1}{2}}+\int_I\varepsilon_{tt}\,\bar{e}_t dx=0$$

for periodic boundary conditions, and

$$\int_I (\bar{e})_{tt}\, \bar{e}_t dx + \frac{1}{2}\sum_{j=1}^{N-1} [\bar{e}_t]^2_{j+\frac{1}{2}} + \frac{1}{2}(\bar{e}_t^+)^2_{\frac{1}{2}} + \frac{1}{2}(\bar{e}_t^-)^2_{N+\frac{1}{2}} + \int_I \varepsilon_{tt}\, \bar{e}_t dx = 0$$

for initial-boundary value problems. In both cases,

$$\frac{1}{2}\frac{d}{dt}\int_I \bar{e}_t^2 dx \le ||\varepsilon_{tt}||_{L^2} \cdot ||\bar{e}_t||_{L^2} = C_1 h^{k+1} ||\bar{e}_t||_{L^2}$$

where $C_1 = C'||u||_{k+3}$. Therefore,

$$\frac{d}{dt}||\bar{e}_t||_{L^2} \le C_1 h^{k+1}.$$

This gives us

$$||\bar{e}_t||_{L^2} \le C_1 h^{k+1} t + ||\bar{e}_t(\cdot, 0)||_{L^2}.$$

To bound $||\bar{e}_t(\cdot, 0)||_{L^2}$, we take $t = 0$ in the error equation (2.13). At $t = 0$, $\bar{e} = 0$, thus $\int_{I_j} e_t v_h dx = 0$. This means

$$\int_{I_j} \bar{e}_t v_h dx = -\int_{I_j} \varepsilon_t v_h dx$$

for any $v_h \in V_h^k$ at $t = 0$. Let $v_h = \bar{e}_t(x, 0)$, then

$$||\bar{e}_t(\cdot, 0)||^2_{L^2} \le ||\varepsilon_t(\cdot, 0)||_{L^2}\ ||\bar{e}_t(\cdot, 0)||_{L^2} \le C_2 h^{k+1} ||\bar{e}_t(\cdot, 0)||_{L^2},$$

where $C_2 = C'||u||_{k+2}$, thus

$$||\bar{e}_t(\cdot, 0)||_{L^2} \le C_2 h^{k+1} \tag{A.1}$$

and we have proved Lemma 2.3 by taking $C = \max(C_1, C_2)$.

A.3 The proof of Lemma 3.1

We will first prove the existence and uniqueness of $P_h^1 u$. Since $q - au_x = 0$, we have

$$\int_{I_j} q w_h dx + \int_{I_j} au(w_h)_x dx - au^- w_h^-|_{j+\frac{1}{2}} + au^- w_h^+|_{j-\frac{1}{2}} = 0$$

for any $w_h \in V_h^k$. Combined with (3.6), we get an error equation,

$$\int_{I_j} (q - q_h) w_h dx + \int_{I_j} a(u - P_h^1 u)(w_h)_x dx - a(u - P_h^1 u)^- w_h^-|_{j+\frac{1}{2}} + a(u - P_h^1 u)^- w_h^+|_{j-\frac{1}{2}} = 0,$$

which, by the property of the projection P_h^-, is equivalent to

$$\int_{I_j}(q-q_h)w_h dx+\int_{I_j} a(P_h^- u-P_h^1 u)(w_h)_x dx-a(P_h^- u-P_h^1 u)^- w_h^-|_{j+\frac{1}{2}}+a(P_h^- u-P_h^1 u)^- w_h^+|_{j-\frac{1}{2}}=0.$$

We introduce the notation for the two errors $E_u = P_h^- u - P_h^1 u$, $E_q = P_h^+ q - q_h$, and use the notation $\varepsilon_q = q - P_h^+ q$. The above equality can then be rewritten as

$$\int_{I_j}(\varepsilon_q + E_q)w_h dx + \int_{I_j} aE_u(w_h)_x dx - aE_u^- w_h^-|_{j+\frac{1}{2}} + aE_u^- w_h^+|_{j-\frac{1}{2}} = 0, \tag{A.2}$$

and the conditions (3.7), (3.8) are now

$$\int_{I_j}(E_u - aE_q)\, v_h\, dx = 0 \tag{A.3}$$

for any $v_h \in P^{k-1}$ on I_j and

$$E_u^- = aE_q^+ \qquad \text{at } x_{j-1/2}. \tag{A.4}$$

(A.2) is equivalent to the original definition of q_h (3.6). Thus, we only need to prove there is a unique solution E_u to the equations (A.2), (A.3), (A.4), then $P_h^1 u = P_h^- u - E_u$ will exist and will be unique.

Combining (A.2), (A.3), (A.4), we arrive at the following equation

$$\int_{I_j}(\varepsilon_q + E_q)w_h dx + \int_{I_j} a^2 E_q(w_h)_x dx - a^2 E_q^+ w_h^-|_{j+\frac{1}{2}} + a^2 E_q^+ w_h^+|_{j-\frac{1}{2}} = 0, \tag{A.5}$$

or

$$\int_{I_j} E_q w_h dx + \int_{I_j} a^2 E_q(w_h)_x dx - a^2 E_q^+ w_h^-|_{j+\frac{1}{2}} + a^2 E_q^+ w_h^+|_{j-\frac{1}{2}} = -\int_{I_j}\varepsilon_q w_h dx, \tag{A.6}$$

or equivalently,

$$\int_{I_j}(\varepsilon_q + E_q)w_h dx - a^2\int_{I_j}(E_q)_x w_h dx - a^2[E_q]w_h^-|_{j+\frac{1}{2}} = 0, \tag{A.7}$$

for any $w_h \in V_h^k$. For any given u, ε_q is uniquely defined, thus the equation (A.6) is a $n(k+1)\times n(k+1)$ linear system for $E_q \in V_h^k$ with a known right hand side. Hence, if we can prove uniqueness for E_q, then existence will follow.

The solution E_q to (A.6) is unique. Otherwise, suppose there are two solutions E_q^1 and E_q^2. Define $g = E_q^1 - E_q^2$, then

$$\int_{I_j} g w_h dx + \int_{I_j} a^2 g (w_h)_x dx - a^2 g^+ w_h^-|_{j+\frac{1}{2}} + a^2 g^+ w_h^+|_{j-\frac{1}{2}} = 0$$

for any $w_h \in V_h^k$. Let $w_h = g$, and sum over all j, we obtain

$$\int_I g^2 \, dx + \frac{a^2}{2} \sum_j [g]_{j+\frac{1}{2}}^2 = 0.$$

Thus, $g = 0$. We have proved E_q exists and is unique. Similarly, given E_q, (A.3) and (A.4) is a $n(k+1) \times n(k+1)$ linear system for E_u. The solution is unique, because

$$\int_{I_j} g \, v_h \, dx = 0$$

for any $v_h \in P^{k-1}$ on I_j and

$$g^- = 0 \qquad \text{at } x_{j-1/2}.$$

will imply $g = 0$. That proves the existence and uniqueness of E_u, thus $P_h^1 u$.

To prove the estimate (3.11), we start with (A.7). Similar to the proof of Theorem 2.2, we define $E_q = b_j + s_j(x)(x - x_j)/h_j$ on I_j, where b_j is a constant and $s_j(x) \in P^{k-1}$, then let $w_h = s_j(x)(x - x_{j+1/2})/h_j$ on I_j. Since $w_h(x_{j+\frac{1}{2}}^-) = 0$, from (A.7) we get

$$\int_{I_j} (\varepsilon_q + E_q) \, s_j(x)(x - x_{j+1/2})/h_j \, dx - a^2 \mathcal{B}_j^+(s_j) = 0,$$

which by Lemma 2.1 is,

$$\int_{I_j} (\varepsilon_q + E_q) \, s_j(x)(x - x_{j+1/2})/h_j \, dx + a^2 \left(\frac{1}{4h_j} \int_{I_j} s_j^2(x) dx + \frac{s_j^2(x_{j-1/2})}{4} \right) = 0.$$

Thus,

$$\int_{I_j} s_j(x)^2 dx \leq -\frac{4}{a^2} \int_{I_j} (\varepsilon_q + E_q) \, s_j(x)(x - x_{j+1/2}) \, dx.$$

Define piecewise polynomials $s(x)$ and $\phi_2(x)$, such that $s(x) = s_j(x)$, $\phi_2(x) = x - x_{j+1/2}$ on I_j, then

$$||s||_{L^2}^2 \leq \frac{4}{a^2} ||\varepsilon_q + E_q||_{L^2} ||s||_{L^2} ||\phi_2||_{L^\infty}.$$

However, $||\phi_2||_{L^\infty} = h$, hence

$$||s||_{L^2} \leq Ch||\varepsilon_q + E_q||_{L^2} \leq Ch(||\varepsilon_q||_{L^2} + ||E_q||_{L^2}) \leq Ch^{k+2} + Ch||E_q||_{L^2}, \tag{A.8}$$

where $C = C(a, ||u||_{k+2})$. Now, plugging $w_h = E_q$ in (A.6) and summing over all j, we get

$$\int_I E_q^2\,dx + \frac{a^2}{2}\sum_j [E_q]^2_{j+\frac{1}{2}} = -\int_I \varepsilon_q E_q\,dx,$$

thus

$$||E_q||^2_{L^2} \leq \left|\int_I \varepsilon_q E_q\,dx\right|. \tag{A.9}$$

Since ε_q is orthogonal to any constant, we have

$$\left|\int_I \varepsilon_q E_q\,dx\right| = \left|\sum_j \int_{I_j} \varepsilon_q\, s_j(x)(x - x_j)/h_j dx\right|.$$

Define a new function $\phi(x) = (x - x_j)/h_j$ on I_j, then $||\phi(x)||_{L^\infty} = \frac{1}{2}$, and

$$\left|\int_I \varepsilon_q\, E_q dx\right| \leq ||\varepsilon_q||_{L^2}\,||\phi||_{L^\infty}\,||s||_{L^2} \leq Ch^{k+1}||s||.$$

Therefore, (A.9) and (A.8) imply

$$||E_q||^2_{L^2} \leq Ch^{k+1}||s|| \leq Ch^{2k+3} + Ch^{k+2}||E_q||_{L^2}.$$

Thus, we have proved a bound for E_q,

$$||E_q||_{L^2} \leq Ch^{k+3/2},$$

where $C = C(||u||_{k+2})$. Using the relations (A.3) and (A.4), we will be able to prove a bound for E_u. Suppose

$$E_u = \sum_{m=0}^{k} a^j_m P_m\left(\frac{2(x - x_j)}{h_j}\right)$$

and

$$E_q = \sum_{m=0}^{k} b^j_m P_m\left(\frac{2(x - x_j)}{h_j}\right)$$

on I_j, with $P_m(\cdot)$ denoting the $m-$th order Legendre polynomial. From the orthogonality, (A.3) means

$$a^j_m = a\,b^j_m, \qquad \text{for} \quad m = 0, 1, \ldots k-1 \quad \text{and any } j.$$

Now (A.4) implies

$$\sum_{m=0}^{k} a_m^j = a \sum_{m=0}^{k} b_m^{j+1}(-1)^m,$$

thus

$$a_k^j = a \left(\sum_{m=0}^{k} b_m^{j+1}(-1)^m - \sum_{m=0}^{k-1} b_m^j \right).$$

Now,

$$\int_{I_j} E_u^2 dx = \sum_{m=0}^{k} (a_m^j)^2 \int_{I_j} \left(P_m \left(\frac{2(x-x_j)}{h_j} \right) \right)^2 dx = \sum_{m=0}^{k} (a_m^j)^2 \frac{h_j}{2m+1}$$
$$= a^2 \sum_{m=0}^{k-1} (b_m^j)^2 \frac{h_j}{2m+1} + (a_k^j)^2 \frac{h_j}{2k+1} \leq a^2 \sum_{m=0}^{k} (b_m^j)^2 \frac{h_j}{2m+1} + (a_k^j)^2 \frac{h_j}{2k+1}$$
$$= a^2 \int_{I_j} E_q^2 dx + (a_k^j)^2 \frac{h_j}{2k+1},$$

and

$$(a_k^j)^2 \frac{h_j}{2k+1} = a^2 \frac{h_j}{2k+1} \left(\sum_{m=0}^{k} b_m^{j+1}(-1)^m - \sum_{m=0}^{k-1} b_m^j \right)^2$$
$$\leq a^2 \frac{2h_j}{2k+1} \left(\left(\sum_{m=0}^{k} b_m^{j+1}(-1)^m \right)^2 + \left(\sum_{m=0}^{k-1} b_m^j \right)^2 \right)$$
$$\leq a^2 \frac{2(k+1)h_j}{2k+1} \left(\sum_{m=0}^{k} (b_m^{j+1})^2 + \sum_{m=0}^{k-1} (b_m^j)^2 \right)$$
$$\leq a^2 (2(k+1)) \left(\sum_{m=0}^{k} (b_m^{j+1})^2 \frac{h_j}{2m+1} + \sum_{m=0}^{k-1} (b_m^j)^2 \frac{h_j}{2m+1} \right)$$
$$\leq a^2 (2(k+1)) \left(\frac{1}{\lambda} \sum_{m=0}^{k} (b_m^{j+1})^2 \frac{h_{j+1}}{2m+1} + \sum_{m=0}^{k} (b_m^j)^2 \frac{h_j}{2m+1} \right)$$
$$= a^2 (2(k+1)) \left(\frac{1}{\lambda} \int_{I_{j+1}} E_q^2 dx + \int_{I_j} E_q^2 dx \right).$$

Thus,

$$\int_{I_j} E_u^2 dx \leq a^2 \left((1 + 2(k+1)) \int_{I_j} E_q^2 dx + \frac{2(k+1)}{\lambda} \int_{I_{j+1}} E_q^2 dx \right).$$

Summing over all j, we obtain

$$||E_u||_{L^2}^2 \leq a^2 \left(1 + 2(k+1) + \frac{2(k+1)}{\lambda} \right) ||E_q||_{L^2}^2.$$

We now have

$$||E_u||_{L^2} \le C(\lambda)||E_q||_{L^2} \le C(\lambda, ||u||_{k+2})h^{k+3/2},$$

hence we have proved (3.11).

We now proceed to prove the existence and uniqueness of $P_h^2 u$. Since $q - au_x = 0$, we have

$$\int_{I_j} qw_h dx + \int_{I_j} au(w_h)_x dx - au^+ w_h^-|_{j+\frac{1}{2}} + au^+ w_h^+|_{j-\frac{1}{2}} = 0$$

for any $w_h \in V_h^k$. Combined with (3.9), we get an error equation,

$$\int_{I_j} (q - q_h)w_h dx + \int_{I_j} a(u - P_h^2 u)(w_h)_x dx - a(u - P_h^2 u)^+ w_h^-|_{j+\frac{1}{2}} + a(u - P_h^2 u)^+ w_h^+|_{j-\frac{1}{2}} = 0,$$

which, by the property of the projection P_h^+, is equivalent to

$$\int_{I_j} (q-q_h)w_h dx + \int_{I_j} a(P_h^+ u - P_h^2 u)(w_h)_x dx - a(P_h^+ u - P_h^2 u)^+ w_h^-|_{j+\frac{1}{2}} + a(P_h^+ u - P_h^2 u)^+ w_h^+|_{j-\frac{1}{2}} = 0$$

for any $w_h \in V_h^k$. We use the notation $z_h = cP_h^2 u - aq_h$, $E_u = P_h^+ u - P_h^2 u$ and $\varepsilon_u = u - P_h^+ u$, $\varepsilon_z = z - P_h^- z$. The above equality can be rewritten as

$$\int_{I_j} (q - q_h)w_h dx + \int_{I_j} aE_u(w_h)_x dx - aE_u^+ w_h^-|_{j+\frac{1}{2}} + aE_u^+ w_h^+|_{j-\frac{1}{2}} = 0, \tag{A.10}$$

or

$$\int_{I_j} (q - q_h)w_h dx - a\int_{I_j} (E_u)_x w_h dx - a[E_u]w_h^-|_{j+\frac{1}{2}} = 0 \tag{A.11}$$

for any $w_h \in V_h^k$. Since

$$q - q_h = \frac{cu - z}{a} - \frac{cP_h^2 u - z_h}{a} = \frac{-(z - z_h) + c(u - P_h^2 u)}{a},$$

using (3.10) here, we have

$$q - q_h = \frac{-(z - P_h^- z) + c(u - P_h^2 u)}{a} = \frac{-\varepsilon_z + c(\varepsilon_u + E_u)}{a}. \tag{A.12}$$

Thus (A.11) becomes

$$\int_{I_j} \frac{cE_u}{a} w_h dx + \int_{I_j} aE_u(w_h)_x dx - aE_u^+ w_h^-|_{j+\frac{1}{2}} + aE_u^+ w_h^+|_{j-\frac{1}{2}} = \int_{I_j} \frac{\varepsilon_z - c\varepsilon_u}{a} w_h dx \tag{A.13}$$

for any $w_h \in V_h^k$. This is a $n(k+1) \times n(k+1)$ linear system for E_u with a known right hand side. The solution is unique. Otherwise, suppose there are two solutions E_u^1 and E_u^2. Define $g = E_u^1 - E_u^2$, then

$$\int_{I_j} \frac{cg}{a} w_h dx + \int_{I_j} ag(w_h)_x dx - ag^+ w_h^- |_{j+\frac{1}{2}} + ag^+ w_h^+ |_{j-\frac{1}{2}} = 0$$

for any $w_h \in V_h^k$. Let $w_h = g$, and sum over all j,

$$\int_I \frac{cg^2}{a} dx + \frac{a}{2} \sum_j [g]_{j+\frac{1}{2}}^2 = 0.$$

Thus, $g = 0$. Now E_u exists and is unique. By the equivalence of (3.9) and (A.10), $P_h^2 u$ exists and is unique.

The final step is to prove the error bound (3.12). Similar to the proof of Theorem 2.2, we define $E_u = r_j + d_j(x)(x - x_j)/h_j$ on I_j, where r_j is a constant and $d_j(x) \in P^{k-1}$, then let $w_h = d_j(x)(x - x_{j+1/2})/h_j$ in (A.11) to get,

$$\int_{I_j} (q - q_h)\, d_j(x)(x - x_{j+1/2})/h_j\, dx - a\mathcal{B}_j^+(d_j) = 0,$$

or, by Lemma 2.1,

$$\int_{I_j} (q - q_h)\, d_j(x)(x - x_{j+1/2})/h_j\, dx + \frac{a}{4h_j} \int_{I_j} d_j^2(x) dx + \frac{a d_j^2(x_{j-1/2})}{4} = 0.$$

Therefore

$$\int_{I_j} d_j(x)^2 dx \leq -\frac{4}{a} \int_{I_j} (q - q_h)\, d_j(x)(x - x_{j+1/2})\, dx.$$

Define piecewise polynomials $d(x), \phi_2(x)$, such that $d(x) = d_j(x), \phi_2(x) = x - x_{j+1/2}$ on I_j, then

$$||d||_{L^2}^2 \leq \frac{4}{a} ||q - q_h||_{L^2} ||d||_{L^2} ||\phi_2||_{L^\infty}$$

Since $||\phi_2||_{L^\infty} = h$, we have

$$||d||_{L^2} \leq Ch||q - q_h||_{L^2}.$$

In (A.13), let $w_h = E_u$, and sum over all j,

$$\int_I \frac{cE_u^2}{a}\,dx + \frac{a}{2}\sum_j [E_u]^2_{j+\frac{1}{2}} = \int_I \frac{\varepsilon_z - c\varepsilon_u}{a} E_u dx,$$

hence, since ε_z and ε_u are orthogonal to piecewise constants,

$$\begin{aligned} ||E_u||^2_{L^2} &\le \frac{1}{c}\left|\int_I (\varepsilon_z - c\varepsilon_u) E_u dx\right| = \frac{1}{c}\left|\sum_j \int_{I_j} (\varepsilon_z - c\varepsilon_u) d_j(x)(x-x_j)/h_j dx\right| \\ &\le C(||\varepsilon_z||_{L^2} + ||\varepsilon_u||_{L^2})||d||_{L^2} \le Ch^{k+1}||d||_{L^2} \le Ch^{k+2}||q-q_h||_{L^2}. \end{aligned}$$

However, by (A.12),

$$||q-q_h||_{L^2} \le ||\frac{\varepsilon_z - c\varepsilon_u}{a}||_{L^2} + \frac{c||E_u||_{L^2}}{a} \le Ch^{k+1} + \frac{c||E_u||_{L^2}}{a},$$

which tells us

$$||E_u||^2_{L^2} \le Ch^{2k+3} + Ch^{k+2}||E_u||_{L^2},$$

and (3.12) follows. This finishes the proof for Lemma 3.1.

A.4 The proof of Lemma 3.3

From the projection results, $||(\varepsilon_u)_t||_{L^2} \le C'||u_t||_{k+1}h^{k+1} \le C'||u||_{k+3}h^{k+1}$ with C' as a constant.

Since $e_u = \varepsilon_u + \bar{e}_u$, to prove (3.23), we will only need to prove $||(\bar{e}_u)_t||_{L^2} \le Ch^{k+1}(t+1)$. Similar to the proof of Lemma 2.3, we take the derivative with respect to time for the error equations (3.14) and (3.15), and let $v_h = (\bar{e}_u)_t, w_h = (\bar{e}_q)_t$, sum them over all j, to obtain

$$\int_I (\bar{e}_u)_{tt}\,(\bar{e}_u)_t dx + \int_I ((\bar{e}_q)_t)^2 dx + \int_I (\varepsilon_u)_{tt}\,(\bar{e}_u)_t dx + \int_I (\varepsilon_q)_t\,(\bar{e}_q)_t dx + \frac{1}{2}\sum_j [(\bar{e}_u)_t]^2_{j+\frac{1}{2}} = 0.$$

Thus,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||(\bar{e}_u)_t||^2_{L^2} + ||(\bar{e}_q)_t)||^2_{L^2} &\le \left|\int_I (\varepsilon_u)_{tt}\,(\bar{e}_u)_t dx\right| + \left|\int_I (\varepsilon_q)_t\,(\bar{e}_q)_t dx\right| \\ &\le ||(\varepsilon_u)_{tt}||_{L^2}\,||(\bar{e}_u)_t||_{L^2} + \frac{||(\varepsilon_q)_t||^2_{L^2}}{4} + ||(\bar{e}_q)_t||^2_{L^2}. \end{aligned}$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}||(\bar{e}_u)_t||^2_{L^2} \le ||(\varepsilon_u)_{tt}||_{L^2}\,||(\bar{e}_u)_t||_{L^2} + \frac{||(\varepsilon_q)_t||^2_{L^2}}{4}.$$

Again by the projection results, $||(\varepsilon_u)_{tt}||_{L^2} \le C'||u||_{k+5}h^{k+1}$ and $||(\varepsilon_q)_t||_{L^2} \le C'||q||_{k+3}h^{k+1}$, thus,

$$\frac{d}{dt}||(\bar{e}_u)_t||^2_{L^2} \le 2C'||u(t)||_{k+5}h^{k+1}\,||(\bar{e}_u)_t||_{L^2} + \frac{1}{2}(C'||q(t)||_{k+3}h^{k+1})^2.$$

Denote $E(t) = ||(\bar{e}_u)_t||_{L^2}$, $A = 2C'||u(t)||_{k+5}h^{k+1}$, $B = \frac{1}{2}(C'||q(t)||_{k+3}h^{k+1})^2$, then

$$\frac{d}{dt}E^2(t) \le AE(t) + B.$$

Notice that here, although A, B do not explicitly depend on t, they are functions of time through the dependence on the norm of $u(t)$ and $q(t)$. In our example, $||u(t)||$ and $||q(t)||$ are exponentially decaying with respect to time. However, we just assume that A, B are bounded, namely, for any t, $A \le \alpha$, $B \le \beta$, whereas, $\alpha = C_2h^{k+1}$, $\beta = C_2h^{2k+2}$. Thus,

$$\frac{d}{dt}E^2(t) \le \alpha E(t) + \beta. \tag{A.14}$$

Because of the way the initial condition of u_h is chosen, using (3.7) and (3.8), and plugging in (3.14), we have, at $t = 0$,

$$\int_{I_j}(e_u)_t v_h dx = 0 \tag{A.15}$$

for any $v_h \in V_h^k$. Let $v_h = \bar{e}_u$, then similar to the proof of (A.1) in Lemma 2.3, at $t = 0$, we have

$$||(\bar{e}_u)_t(\cdot, 0)||_{L^2} \le ||(\varepsilon_u)_t(\cdot, 0)||_{L^2} \le C'||u||_{k+3}h^{k+1}. \tag{A.16}$$

We have thus proved

$$E(0) \le C'||u(\cdot, 0)||_{k+3}h^{k+1} = C_1h^{k+1}.$$

Combined with (A.14), we will be able to obtain a bound on $E(t)$. Integrate (A.14) with respect to t,

$$E^2(t) \le E^2(0) + \beta t + \alpha\int_0^t E(s)ds.$$

If $t \le T$, then

$$E^2(t) \le E^2(0) + \beta T + \alpha\int_0^t E(s)ds.$$

Define $w(t) = E^2(0) + \beta T + \alpha \int_0^t E(s)ds$, then the above inequality reads $E(t) \leq \sqrt{w(t)}$. Moreover,

$$\frac{d}{dt} w(t) = \alpha E(t) \leq \alpha \sqrt{w(t)}.$$

It is now easy to derive

$$w(t) \leq 2w(0) + \frac{(\alpha t)^2}{2} = 2(E^2(0) + \beta T) + \frac{(\alpha t)^2}{2},$$

therefore,

$$E^2(t) \leq 2(E^2(0) + \beta T) + \frac{(\alpha t)^2}{2}$$

for any $t \leq T$. We can simply take $T = t$ now to obtain

$$E^2(t) \leq 2(E^2(0) + \beta t) + \frac{(\alpha t)^2}{2} \leq 2(C_1^2 h^{2k+2} + C_2 h^{2k+2} t) + \frac{C_2^2}{2} h^{2k+2} t^2.$$

Taking a square root, we obtain

$$||(e_u)_t||_{L^2} \leq C h^{k+1}(t+1).$$

Similarly to the above, it is easy to derive (3.24) if we work with (3.18) directly. From (3.18), (3.23) and (3.24), we have

$$\begin{aligned} ||\bar{e}_q||_{L^2}^2 &\leq \left| \int_I (\bar{e}_u)_t \, \bar{e}_u dx \right| + \left| \int_I (\varepsilon_u)_t \, \bar{e}_u dx \right| + \left| \int_I \varepsilon_q \, \bar{e}_q dx \right| \\ &\leq ||(\bar{e}_u)_t||_{L^2} \; ||\bar{e}_u||_{L^2} + ||(\varepsilon_u)_t||_{L^2} \; ||\bar{e}_u||_{L^2} + ||\varepsilon_q||_{L^2} \; ||\bar{e}_q||_{L^2} \\ &\leq C h^{2k+2}(t+1)^2 + C h^{k+1} ||\bar{e}_q||_{L^2}, \end{aligned}$$

thus (3.25) follows. This finishes the proof for Lemma 3.3.

A.5 The proof of Lemma 3.4

We can rewrite the error equation (3.31) into the following form,

$$\begin{aligned} &\frac{a}{2}\frac{d}{dt}||\bar{e}_u||_{L^2}^2 + \frac{1}{a}||\bar{e}_z||_{L^2}^2 \\ \leq\ & a\left| \int_I (\varepsilon_u)_t \, \bar{e}_u dx \right| + \frac{1}{a}\left| \int_I \varepsilon_z \, \bar{e}_z dx \right| + \frac{1}{a}\left| \int_I \varepsilon_u \, \bar{e}_z dx \right| + \frac{1}{a}\left| \int_I \bar{e}_u \, \bar{e}_z dx \right| \\ \leq\ & \frac{a}{2}(||(\varepsilon_u)_t||_{L^2}^2 + ||\bar{e}_u||_{L^2}^2) + \frac{1}{2a}(||\varepsilon_z||_{L^2}^2 + ||\bar{e}_z||_{L^2}^2) + \frac{1}{a}||\varepsilon_u||_{L^2}^2 + \frac{1}{4a}||\bar{e}_z||_{L^2}^2 + \frac{1}{a}||\bar{e}_u||_{L^2}^2 + \frac{1}{4a}||\bar{e}_z||_{L^2}^2, \end{aligned}$$

then

$$\frac{d}{dt}||\bar{e}_u||_{L^2}^2 \leq ||(\varepsilon_u)_t||_{L^2}^2 + \frac{1}{a^2}||\varepsilon_z||_{L^2}^2 + \frac{2}{a^2}||\varepsilon_u||_{L^2}^2 + \left(\frac{2}{a^2}+1\right)||\bar{e}_u||_{L^2}^2.$$

From the projection properties, $||\varepsilon_u||_{L^2} \leq C'||u||_{k+1}h^{k+1}$,$||\varepsilon_z||_{L^2} \leq C'||u||_{k+2}h^{k+1}$ and $||(\varepsilon_u)_t||_{L^2} \leq C'||u||_{k+3}h^{k+1}$ with C' as a constant. Therefore,

$$\frac{d}{dt}||\bar{e}_u||_{L^2}^2 \leq Ch^{2k+2} + C_1||\bar{e}_u||_{L^2}^2,$$

and $C = C(||u||_{k+3})$, $C_1 = C_1(a)$. Since $||\bar{e}_u|| \leq Ch^{k+1}$ initially,

$$||\bar{e}_u||_{L^2}^2 \leq \frac{C}{C_1}e^{C_1 t}h^{2k+2}.$$

Combined with $||\varepsilon_u||_{L^2} \leq C'||u||_{k+1}h^{k+1}$, we have

$$||e_u||_{L^2} \leq \sqrt{\frac{C}{C_1}}e^{C_1 t/2}h^{k+1}.$$

We can rewrite it as

$$||e_u||_{L^2} \leq Ce^{C_1 t}h^{k+1}, \tag{A.17}$$

with $C = C(||u||_{k+3}, a)$, $C_1 = C_1(a) > 0$. Notice that, although the bound in (A.17) shows an exponential growth, the constant C is compensating the growth, since our exact solution u is exponentially decaying, as well as all of its norms. This explains why in the numerical experiments, exponential decay of the error is observed, see [9].

To prove (3.35), we take the time derivative of the error equations (3.27) and (3.28), and let the test functions be $w_h = (\bar{e}_z)_t$, $v_h = (\bar{e}_u)_t$. At $t = 0$, we still have (A.15) because of the choice of initial condition, hence we have (A.16), or $||(\bar{e}_u)_t|| \leq Ch^{k+1}$ at $t = 0$. The remaining proof is then very similar to the above and is thus omitted. From the error equation (3.31) we have

$$\begin{aligned} & a\int_I (\bar{e}_u)_t\bar{e}_u dx + \frac{1}{a}||\bar{e}_z||_{L^2}^2 \\ \leq\ & \frac{a}{2}(||(\varepsilon_u)_t||_{L^2}^2 + ||\bar{e}_u||_{L^2}^2) + \frac{1}{a}||\varepsilon_z||_{L^2}^2 + \frac{1}{4a}||\bar{e}_z||_{L^2}^2 + \frac{1}{a}||\varepsilon_u||_{L^2}^2 + \frac{1}{4a}||\bar{e}_z||_{L^2}^2 + \frac{1}{a}||\bar{e}_u||_{L^2}^2 + \frac{1}{4a}||\bar{e}_z||_{L^2}^2, \end{aligned}$$

thus,

$$
\begin{aligned}
&||\bar{e}_z||^2_{L^2}\\
&\le -4a^2\int_I (\bar{e}_u)_t\bar{e}_u dx + 2a^2(||(\varepsilon_u)_t||^2_{L^2} + ||\bar{e}_u||^2_{L^2}) + 4||\varepsilon_z||^2_{L^2} + 4||\varepsilon_u||^2_{L^2} + 4||\bar{e}_u||^2_{L^2}\\
&\le 2a^2(||(\bar{e}_u)_t||^2_{L^2} + ||\bar{e}_u||^2_{L^2}) + 2a^2(||(\varepsilon_u)_t||^2_{L^2} + ||\bar{e}_u||^2_{L^2}) + 4||\varepsilon_z||^2_{L^2} + 4||\varepsilon_u||^2_{L^2} + 4||\bar{e}_u||^2_{L^2}.
\end{aligned}
$$

We already have bounds on every term on the right hand side of the above inequality, thus (3.36) follows by taking into account $e_q = (e_u - e_z)/a$. This finishes the proof of Lemma 3.4.