CONVERGENCE OF DISCONTINUOUS GALERKIN SCHEMES FOR FRONT PROPAGATION WITH OBSTACLES

OLIVIER BOKANOWSKI, YINGDA CHENG, AND CHI-WANG SHU

ABSTRACT. We study semi-Lagrangian discontinuous Galerkin (SLDG) and Runge-Kutta discontinuous Galerkin (RKDG) schemes for some front propagation problems in the presence of an obstacle term, modeled by a nonlinear Hamilton-Jacobi equation of the form $\min(u_t+cu_x, u-g(x)) = 0$, in one space dimension. New convergence results and error bounds are obtained for Lipschitz regular data. These "low regularity" assumptions are the natural ones for the solutions of the studied equations.

1. INTRODUCTION

In this paper, we establish convergence of a class of discontinuous Galerkin (DG) methods for the one-dimensional Hamilton-Jacobi (HJ) equation below, hereafter also called the "obstacle" equation,

$$\min(u_t + c\,u_x,\ u - g(x)) = 0, \quad x \in I = (0,1), \quad t > 0, \tag{1a}$$

$$u(0,x) = u_0(x), \quad x \in (0,1), \tag{1b}$$

with periodic boundary conditions on I and a constant $c \in \mathbb{R}$. In (1), the function g is called the "obstacle" function.

It is well known that taking constraints in optimal control problems is not an obvious task. Within the viscosity theory, it is possible to devise schemes for obstacle equations such as (1). However a monotonicity condition is needed in general for proving the convergence of the scheme. The monotonicity condition can yield a convergence proof of one-half order in the mesh grid [11] (see also [4] for more specific partial differential equations (PDEs) with an obstacle term and related error estimates). However, a serious restriction of such monotonicity condition is that the schemes become at most first order accurate for smooth solutions or in smooth regions, thus making the schemes highly inefficient for practical computation. On

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the other hand, it is very difficult to show convergence of formally higher order schemes, such as the ones studied in this paper, when the solution is not regular enough.

In our previous work [3], we proposed a class of Runge-Kutta DG (RKDG) methods adapted to front propagation problems with obstacles. The DG methods under consideration were originally devised to solve conservation laws [10]. As for the DG-HJ solvers, in [13, 15], the first efforts relied on solving the conservation law system satisfied by the derivative of the solution. In [7], a DG method for directly solving the Hamilton-Jacobi equation was developed and was later generalized to solve front propagation problems [2] and obstacle problems [3]. Other direct DG solvers include the central DG scheme [16] and the local DG scheme [20]. The schemes proposed in [3] feature a simple treatment of the obstacle functions. Stability analysis is performed with forward Euler, a Heun scheme and a TVD Runge-Kutta third order (TVD-RK3) time discretization using the techniques developed in Zhang and Shu [21].

On the other hand, the semi-Lagrangian DG (SLDG) methods were proposed in [18, 19, 17] to compute incompressible flow and Vlasov equations, as well as in [6] for some general linear first and second order PDEs. The advantage of SLDG is its ability to take large time steps without a CFL restriction. However, it is difficult to design SLDG methods for nonlinear problems (some SL schemes for Hamilton-Jacobi-Bellman equations were proposed in [5], but without convergence proof).

Beyond the scope of the present paper, yet of great interest, we also mention the second order PDE with obstacle terms, such as $\min(u_t - cu_{xx}, u - g(x)) = 0$, with $c > 0$. This is the case of the so-called "American options" in mathematical finance [1]. Explicit schemes were proposed and proved to converge within the viscosity theory (see for instance [14]) yet with a reduced rate of convergence. Variational methods for nonlinear obstacle equations can be also devised [12] but will lead in general to nonlinear implicit schemes which are computationally more demanding.

The scope of the present paper is to study convergence of the SLDG and RKDG schemes for the obstacle problem (1). The main challenges include the low regularity of the solution and the nonlinear treatment needed to obtain the obstacle solution. Due to the fully discrete nature of the method, traditional semi-discrete error estimates of DG methods for hyperbolic problems cannot directly apply here. Therefore, fully discrete analysis is necessary. Fully discrete analysis of RKDG methods for conservation laws have been performed in the literature. In [21], error estimates for RKDG methods for scalar conservation laws were provided for smooth solutions. In [9, 22], discontinuous solutions with RK2 and linear polynomials and RK3 time discretizations with general polynomials were studied for linear conservation laws. However, to our best knowledge, convergence results for second and higher order DG schemes solving nonlinear hyperbolic equations with irregular solutions are not available.

As a consequence of our results, assuming that h is a space step and Δt a time step, we shall show error bounds of the order of $O(h^{9/10})$ for the SLDG schemes (under time stepping $\Delta t \equiv Ch^{3/5}$, larger time step can be taken with a lower convergence rate), and of order $O(h^{1/2})$ for the RKDG schemes (under time stepping $\Delta t \equiv Ch$). We will need a natural "no shattering" regularity assumption on the exact solution that will be made precise in the sequel, otherwise we will typically assume the exact solution to be Lipschitz regular and piecewise C^q regular for some $q \geq 1$ for SLDG (resp. $q \geq 2$ for RKDG).

The main idea of our proof is based on the dynamic programming principles as illustrated below. The viscosity solution of (1) also corresponds to the following optimal control problem:

$$u(t,x) = \max\left(u_0(x - ct),\ \max_{\theta\in[0,t]} g(x - c\theta)\right). \tag{2}$$

The function u is also the solution of the Bellman's dynamic programming principle (DPP): for any $\Delta t > 0$,

$$u(t + \Delta t, x) = \max\left(u(t, x - c\Delta t), \max_{\theta\in[0,\Delta t]} g(x - c\theta)\right), \quad \forall\, t \geq 0. \tag{3}$$

Notice conversely that the DPP (3), together with $u(0,x) = u_0(x)$, implies (2).

If we denote

$$u^n(x) := u(t_n, x)$$

and

$$g_t(x) := \max_{\theta\in[0,t]} g(x - c\theta),$$

then the DPP implies in particular for any x and $n \geq 0$:

$$u^{n+1}(x) = \max(u^n(x - c\Delta t), g_{\Delta t}(x)). \tag{4}$$

Using formula (2), we can see that when u_0 and g are Lipschitz regular, then u is also Lipschitz regular in space and time, and in general no more regularity can be assumed (the maximum of two regular functions is in general no more than Lipschitz regular). When u_0 is a discontinuous function (otherwise piecewise regular), formula (2) implies also some regularity on the solution u.

The rest of the paper is organized as follows. In Section 2, we describe the DG methods for the obstacle equations. In Section 3, we collect some lemmas that will be used in our convergence proofs. Section 4 and Section 5 are devoted to the convergence of SLDG and RKDG methods, respectively. In both sections, we will first establish error estimates for SLDG and RKDG schemes for linear transport equations without obstacles. Then we will use DPP illustrated above to prove convergence of the numerical solution in the presence of obstacles. We conclude with a few remarks in Section 6.

2. DG schemes for the obstacle equation

In this section, we will introduce the SLDG and RKDG methods for the obstacle equation. For simplicity of discussion, in the rest of the paper, we will assume c to be a positive constant.

Let $I_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, $j = 1, \ldots, N$ be a set of intervals forming a partition of $I = (0, 1)$. We denote $h := \max_j h_j$ where $h_j = |I_j|$ is the length of the interval I_j. For a given integer $k \geq 0$, let V_h be the DG space of piecewise polynomial of degree at most k on each interval I_j:

$$V_h := \{v : I \to \mathbb{R}, \quad v_{|I_j} \in P^k, \ \forall j\}. \tag{5}$$

To introduce the methods for the obstacle problems, we follow two steps. Firstly, we describe the DG solvers for the linear advection equation $v_t + cv_x = 0$.

To advance the numerical solution in one step from $v_h^n \in V_h$ to $v_h^{n+1} \in V_h$, we consider one of the two DG methods described below.

<u>SLDG Scheme</u>:

$$v_h^{n+1} := \Pi_h\big(v_h^n(\cdot - c\Delta t)\big), \tag{6}$$

where Π_h is the L^2 projection onto the space V_h. We shall denote this SLDG solver by $v_h^{n+1} = G_{\Delta t}^{SL}(v_h^n)$.

<u>RKDG Scheme</u> (by TVD-RK3 time stepping): find $v_h^{n,1}$, $v_h^{n,2}$, $v_h^{n+1} \in V_h$, such that

$$(v_h^{n,1} - v_h^n, \varphi_h) = \Delta t \mathcal{H}(v_h^n, \varphi_h), \quad \forall \varphi_h \in V_h \tag{7a}$$

$$(v_h^{n,2} - \frac{3}{4}v_h^n - \frac{1}{4}v_h^{n,1}, \varphi_h) = \frac{\Delta t}{4}\mathcal{H}(v_h^{n,1}, \varphi_h), \quad \forall \varphi_h \in V_h \tag{7b}$$

$$(v_h^{n+1} - \frac{1}{3}v_h^n - \frac{2}{3}v_h^{n,2}, \varphi_h) = \frac{2\Delta t}{3}\mathcal{H}(v_h^{n,2}, \varphi_h), \quad \forall \varphi_h \in V_h. \tag{7c}$$

where

$$(\phi, \varphi) = \int_I \phi \, \varphi \, dx$$

and

$$\mathcal{H}_j(\phi_h, \varphi_h) = \int_{I_j} c\phi_h \, (\varphi_h)_x dx - c((\phi_h)^-_{j+\frac{1}{2}}(\varphi_h)^-_{j+\frac{1}{2}} - (\phi_h)^-_{j-\frac{1}{2}}(\varphi_h)^+_{j-\frac{1}{2}}),$$

$$\mathcal{H}(\phi_h, \varphi_h) = \sum_j \mathcal{H}_j(\phi_h, \varphi_h).$$

We shall denote this RKDG solver by $v_h^{n+1} = G_{\Delta t}^{RK}(v_h^n)$.

After introducing DG schemes for the linear transport problem, for the obstacle equation (1), we shall consider two approaches: one by using the L^2 projection

$$u_h^{n+1} := \Pi_h\bigg(\max(G_{\Delta t}(u_h^n),\ \tilde{g})\bigg), \tag{8}$$

where $G_{\Delta t} = G_{\Delta t}^{SL}$ or $G_{\Delta t}^{RK}$ and

$$\tilde{g} \equiv g_{\Delta t} \quad \text{or} \quad \tilde{g} \equiv \max(g(x), g(x - c\Delta t)). \tag{9}$$

The idea of the formulation above is to try to follow the relation (4), and the projection step is to project the function into the piecewise polynomial space V_h.

Unfortunately, the scheme (8) is difficult to implement, because we need to compute the maximum of two functions, which requires locating the roots of $G_{\Delta t}(u_h^n) - \tilde{g}$. Another more practical approach is to define u_h^{n+1} as the unique polynomial in V_h such that

$$u_h^{n+1}(x_\alpha^j) := \max\big(G_{\Delta t}(u_h^n)(x_\alpha^j),\ \tilde{g}(x_\alpha^j)\big), \quad \forall\, j = 1, \ldots, N,\ \alpha = 0, \ldots, k, \tag{10}$$

where $(x_\alpha^j)_{\alpha=0,\ldots,k}$ are the $k+1$ Gauss-Legendre quadrature points on the interval I_j, and w_α^j are the corresponding quadrature weights. Those schemes were studied in details and stability was established in [3]. We shall see in later sections that definitions (8) or (10) lead to similar error estimates, although the second approach is much easier to implement.

Finally, we remark that in [3], forward Euler and TVD-RK2 temporal discretizations are also considered. However, the stability restriction for the time step for the forward Euler method is rather severe as $\Delta t \le Ch^2$, and the stability proof of TVD-RK2 only works for piecewise linear polynomials. Therefore, in this paper, we will only consider TVD-RK3 time discretizations.

3. Preliminaries

In this section, we collect some lemmas which will be used in our convergence proof, and discuss properties of the obstacle solutions. Here and below, we use C (possibly with subscripts) to denote a positive constant depending solely on the exact solution, which may have a different value in each occurrence.

Let us introduce, for $\ell \ge 0$, the following function sets:

$$\begin{aligned} \mathcal{C}^{\ell+1}_{p,L,c_0}(0,1) := \Big\{ & v : (0,1) \to \mathbb{R},\ v \text{ Lipschitz continuous with } \|v'\|_{L^\infty} \le L, \\ & v \text{ piecewise } C^{\ell+1} \text{ with } \|v^{(\ell+1)}\|_{L^\infty} \le c_0, \\ & \text{and } v \text{ admits at most } p \ge 0 \text{ non regular points.} \Big\} \end{aligned}$$

where $v^{(\ell+1)}$ denotes the $(\ell+1)$-th derivative almost everywhere, and

$$\Delta_q^2(0,1) := \Big\{ g : (0,1) \to \mathbb{R},\ g \text{ has at most } q \text{ local maxima points}$$
$$\text{and } g \text{ is twice differentiable at each local maxima.} \Big\}$$

The following ℓ^2 pseudo-norm definition will also be used:

$$\|f\|_{\ell^2} := \left(\sum_{i,\alpha} w_\alpha^i |f(x_\alpha^i)|^2 h_i \right)^{1/2}. \tag{11}$$

In particular, using the Gaussian quadrature rule, for any $f \in V_h$ we have $\|f\|_{\ell^2} = \|f\|_{L^2}$. From this point on, we will use $\|f\|$ to denote $\|f\|_{L^2}$, and $\|f\|_D$ to denote $\|f\|_{L^2(D)}$ for a given domain D.

3.1. **Properties of projections and the obstacle function.** For the RKDG method, it is necessary to consider the following Gauss-Radau projection $\mathbb{P}_h$. For any function φ, $\mathbb{P}_h\varphi \in V_h$, and for any element I_j, it holds that

$$(\mathbb{P}_h\varphi)^-_{j+1/2} = \varphi^-_{j+1/2}, \qquad \int_{I_j} (\mathbb{P}_h\varphi - \varphi)\, \psi_h dx = 0, \quad \forall\, \psi_h \in P^{k-1}(I_j).$$

In the lemma below, we will first establish the projection properties for functions in the space $C^{\ell+1}_{p,L,c_0}$.

Lemma 3.1. *Let $\ell \geq 0$ and let φ be in $C^{\ell+1}_{p,L,c_0}$: φ is Lipschitz continuous, piecewise $C^{\ell+1}$, with at most $p \geq 0$ non regular points. Then there exists a constant $C \geq 0$, depending only on ℓ, p, L, c_0 such that*

$$\|\varphi - P_h\varphi\| \leq C h^{q_{k,\ell}},$$

where the projection P_h can be either $\mathbb{P}_h$ or Π_h, and $q_{k,\ell}$ is defined as

$$q_{k,\ell} := \min\left(\min(k,\ell)+1, \frac{3}{2}\right) \equiv \begin{cases} 3/2 & \text{if } k,\ell \geq 1 \\ 1 & \text{if } k=0 \text{ or } \ell = 0 \end{cases} \tag{12}$$

Proof. Using the property of the projections [8], on each regular cell I_j, we have

$$\|\varphi - P_h\varphi\|_{I_j} \leq C h^{\min(\ell,k)+1} \|\varphi\|_{H^{\ell+1}(I_j)} \leq C h^{\min(\ell,k)+1}$$

since $\varphi \in C^{l+1}$ on I_j. Therefore,

$$\|\varphi - P_h\varphi\|_{L^2(\cup I_j,\ \varphi_{|I_j} \text{regular})} = \left(\int_{\cup I_j,\ \varphi_{|I_j} \text{regular}} |\varphi - P_h\varphi|^2 dx \right)^{1/2} \leq C h^{\min(\ell,k)+1}.$$

On the other hand, on the intervals I_j where φ is not regular, we have

$$\|\varphi - P_h\varphi\|_{I_j} \leq C h^{\min(0,k)+1} \|\varphi\|_{H^1(I_j)} \leq C h \|\varphi\|_{H^1(I_j)} \leq C h^{3/2}$$

because φ is Lipschitz continuous. Hence summing up on the "bad" intervals I_j (with at most p such intervals),

$$\|\varphi - P_h\varphi\|_{L^2(\cup_{bad} I)} \leq Cp^{1/2}h^{3/2}.$$

Summing up the bounds with bad intervals and good ones, we prove the desired result. □

We now state a similar estimate for the ℓ^2 norm:

Lemma 3.2. *Assume that φ belongs to $C^{\ell+1}_{p,L,c_0}$, $\ell \geq 0$, then we have*

$$\|\varphi - P_h\varphi\|_{\ell^2} = \left(\sum_{j,\alpha} w^j_\alpha \left|v(x^j_\alpha) - (P_h\varphi)(x^j_\alpha)\right|^2 h_j\right)^{1/2} \leq Ch^{q_{k,\ell}} \tag{13}$$

where $P_h = \mathbb{P}_h$ or Π_h, $q_{k,\ell}$ is defined as in (12), and the constant C depends only on p, L and c_0.

Proof. On each regular cell I_j, we have [8],

$$\|\varphi - P_h\varphi\|_{L^\infty(I_j)} \leq Ch^{\min(\ell,k)+1/2}\|\varphi\|_{H^{\ell+1}(I_j)} \leq Ch^{\min(\ell,k)+1}.$$

Then using the fact that $w^j_\alpha \geq 0$, $\sum_\alpha w^j_\alpha = 1$ we obtain that

$$\left(\sum_{I_j,\ \varphi_{|I_j} \text{ regular}} \sum_\alpha w^j_\alpha \left|\varphi^n(x^j_\alpha) - (P_h\varphi^n)(x^j_\alpha)\right|^2 h_j\right)^{1/2} \leq Ch^{\min(\ell,k)+1}.$$

On the other hand, when the interval I_j is such that φ contains a non-regular point, we can write

$$\|\varphi - P_h\varphi\|_{L^\infty(I_j)} \leq Ch^{\min(0,k)+1/2}\|\varphi\|_{H^1(I_j)} \leq Ch.$$

Therefore

$$\begin{aligned}
&\left(\sum_{I_j,\ \varphi_{|I_j} \text{ not regular}} \sum_\alpha w^j_\alpha \left|\varphi^n(x^j_\alpha) - (P_h\varphi^n)(x^j_\alpha)\right|^2 h_j\right)^{1/2} \\
\leq\ &\left(\sum_{I_j,\ \varphi_{|I_j} \text{ not regular}} h(Ch)^2\right)^{1/2} \\
\leq\ &Cp^{1/2}h^{3/2}.
\end{aligned}$$

This concludes our proof. □

We now turn to some estimates relating to the obstacle function $g_{\Delta t}$.

Lemma 3.3. *Assume that* $g \in \Delta_q^2(0,1)$ *for some integer* $q \geq 1$. *Let* $\tilde{g}(x) := \max(g(x), g(x - c\Delta t))$. *There exists a constant* $C \geq 0$ *such that*

$$\|\tilde{g} - g_{\Delta t}\| \leq C\sqrt{q}\Delta t^{5/2}. \tag{14}$$

and

$$\|\tilde{g} - g_{\Delta t}\|_{\ell^2} \leq C\sqrt{q}\ \sqrt{\Delta t + h}\,\Delta t^2. \tag{15}$$

Proof. Denote $\mathcal{M}_g$ the set of local maximum points of g, if $[x - c\Delta t, x] \cap \mathcal{M}_g = \emptyset$, then we see that $\tilde{g}(x) - g_{\Delta t}(x) = 0$. Furthermore

$$\int_{\{x,\ g_{\Delta t}(x) \neq \tilde{g}(x)\}} dx \ \leq\ \int_{\{x,\ [x - c\Delta t, x] \cap \mathcal{M}_g \neq \emptyset\}} dx \ \leq\ q\, c\Delta t \tag{16}$$

since there are at most q local maxima.

Consider the case when $[x - c\Delta t, x]$ contains at least a local maxima of g. Assuming that Δt is small enough we can assume that x^* is the only local maximum of g on the interval $[x - c\Delta t, x]$, so that $g_{\Delta t}(x) = g(x^*)$. Then,

$$|g(x) - g_{\Delta t}(x)| = |g(x) - g(x^*)| \leq C|x - x^*|^2 \leq C\Delta t^2, \tag{17}$$

since g is twice differentiable at x^*.

Combining (17) and (16) we obtain the bound (14).

For the second estimate (15), we make use of a minimal covering $\cup I$ of the set

$$\Big\{x,\ [x - c\Delta t, x] \cap \mathcal{M}_g \neq \emptyset\Big\},$$

using mesh intervals. The length of this covering is bounded by $c\Delta t + 2h$ for each maximum point, since in order to cover any interval $[a, b]$ we may need two more mesh intervals I, of length $\leq 2h$ than the minimum required length $b - a$. Overall the length of the total covering is bounded by $q(c\Delta t + 2h)$, hence we obtain the desired result.

$\square$

3.2. **Properties of the obstacle solutions.** We shall impose some restrictions on the regularity of the obstacle solutions as described below.

Definition 3.1. **[“no shattering” property]** *We will say that the exact solution* u *of the problem* (1) *is “not shattering” if there exists some* $\ell \geq 0$ *and constants* p, L, c_0 *such that the exact solution satisfies, for any* $n \geq 0$, $u^n = u(t_n, .) \in C^{\ell+1}_{p,L,c_0}$.

Recall that the exact solution satisfies $u(t, x) = \max(u_0(x - ct), g_t(x)) = u_0(x - ct) + \max(0, g_t(x) - u_0(x - ct))$. Therefore if u is not shattering, it implies that $u_0(x - ct) - g_t(x)$ has bounded number of zeros (since otherwise $x \to u(t, x)$ would have an unbounded number of singularities).

A typical example where shattering occurs (therefore not satisfying definition 3.1), can be constructed as follows. Suppose the domain Ω contains the interval $(-2, 2)$, let $u_0(x) \equiv (x - 1) + (x - 1)^3 \sin(1/(x - 1))$ and $g(x) \equiv x$ on $[-2, 2]$, together with velocity constant $c = 1$. The function u_0 is of class C^2 on the interval $(-2, 2)$. Notice then, $g_t(x) = \max_{\theta\in[0,t]} g(x-\theta) = g(x)$ for $t \in [0, 1]$ and $x \in [-1, 1]$, and the exact solution is therefore $u(t, x) = \max(u_0(x-t), g(x))$ for $t \in [0, 1]$ and $x \in [-1, 1]$. So at time $t = 1$, $u(1, x) = \max(u_0(x-1), g(x)) \equiv x+\max(x^3 \sin(1/x), 0)$ (for $x \in [-1, 1]$). This function has an infinite number of non regular points in the interval $[-1, 1]$, and therefore does not satisfy the "no shattering" property at time $t = 1$.

It is not easy to state precise conditions on the initial data u_0 and g to ensure that the no shattering property will be satisfied. Mainly, $\forall t$, $x \to g_t(x)-u_0(x-ct)$ should have only a finitely bounded number of zeros, as is detailed below. However, we hope the readers agree that it is clear that shattering will not occur for generic data u_0 and g. This definition still allows for a finite (bounded) number of singularities in $u(t_n, .)$, as is generally the case when taking the maximum of two regular functions.

Lemma 3.4. *Assume that*

$$g \in C^{\ell+1}_{p_g,L_g,c_g}(0, 1), \tag{18a}$$

$$u_0 \in C^{\ell+1}_{p_{u_0},L_{u_0},c_{u_0}}(0, 1), \tag{18b}$$

and that, $\forall\, t \in [0, T]$,

$$x \to g_t(x) - u_0(x - ct) \quad \textit{has a finitely bounded number of zeros in } (0, 1) \tag{18c}$$

(the bound being independent of t*).*

Then there exists constants p, L, c_0 *such that,* $\forall t_n \le T$, $u^n(x) \equiv u(t_n, x)$ *belongs to* $C^{\ell+1}_{p,L,c_0}$ *(with* $L = \max(L_g, L_{u_0})$, $c_0 = \max(c_g, c_{u_0})$, *and* p *that is independent of* n*.)*

Proof. We first establish that for $g \in C^{\ell+1}_{p_g,L_g,c_g}(0, 1)$, denoting M (resp. m) the number of local maxima (resp. minima) of g, then we have also $g_{\Delta t} \in C^{\ell+1}_{p_g+2M+m,L_g,c_g}(0, 1)$. This is because the Lipschitz constant of $g_{\Delta t}$ is bounded by L_g by elementary verifications. And each local maxima of g may develop two singularities in $g_{\Delta t}$, and each local minima may develop one singularity in $g_t(\cdot)$, as well as each singular point of g. Hence the total number of non-regular points in $g_t(\cdot)$ will be bounded by $2M + m + p_g$. The bound of the $(\ell + 1)$ derivative can also be obtained easily.

Because the exact solution is given by $u(t, x) = \max(u_0(x - ct), g_t(x)) = u_0(x - ct) + \max(0, g_t(x) - u_0(x - ct))$, the Lipschitz constant of $u(t, .)$ is therefore bounded by $\max(L_{u_0(\cdot-ct)}, L_g) = \max(L_{u_0}, L_g)$.

On the other hand, the number of singular points of $\max(0, g_t(x) - u_0(x - ct))$ is bounded by the sum of the number of singular points of the function $x \to g_t(x) - u_0(x - ct))$, plus the number of zeros of the same function, which is assumed to be bounded independently of $t \ge 0$. Hence the the number of singular points of $x \to u(t, x)$ is bounded independently of $t \ge 0$.

Finally the bound on the x-partial derivative $\|u^{(\ell+1)}(t, \cdot)\|_{L^\infty}$, in the regular region of u, is easily obtained, because u is then locally one of the two functions g_t or $u_0(\cdot - ct)$. $\square$

4. Convergence of the SLDG schemes

In this section, we will provide the convergence proof of the SLDG scheme. In particular, we will proceed in three steps. First, we will establish error estimates of the SLDG methods for the linear transport equation

$$v_t + c\,v_x = 0, \qquad v(0,x) = v_0(x).$$

We will then generalize the results to scheme (8), and finally to scheme (10) for the obstacle problem.

4.1. **Convergence of the SLDG scheme for the linear advection equation.** We first consider the linear equation $v_t + cv_x = 0$, for which

$$v(t+\Delta t) = v(t, x - c\Delta t).$$

We denote $v^n(\cdot) = v(t^n, \cdot)$, and we define the numerical solution of the SLDG method at t^n to be v_h^n. In particular, the scheme writes: initialize with $v_h^0 := \Pi_h v_0$, and $v_h^{n+1} = G^{SL}_{\Delta t}(v_h^n) = \Pi_h(v_h^n(\cdot - c\Delta t))$ for $n \geq 0$.

Theorem 4.1. *We consider $v_t + cv_x = 0$, $v(0,x) = v_0(x)$. If $v_0 \in \mathcal{C}^{\ell+1}_{p,L,c_0}(0,1)$, then we have for all n such that $n\Delta t \leq T$,*

$$\|v_h^n - v^n\| \leq CT\,\frac{h^{q_{k,\ell}}}{\Delta t}$$

for some constant $C \geq 0$ independent of n, and $q_{k,\ell}$ is defined in (12).

Proof. If $v_0 \in \mathcal{C}^{\ell+1}_{p,L,c_0}(0,1)$, then $v^n \in \mathcal{C}^{\ell+1}_{p,L,c_0}(0,1)$, and due to Lemma 3.1, we have:

$$\|v^n(\cdot - c\Delta t) - \Pi_h(v^n(\cdot - c\Delta t))\| \leq Ch^{q_{k,\ell}},$$

for some constant C independent of n. Hence

$$\begin{aligned}
&\|v_h^{n+1} - v^{n+1}\| = \|\Pi_h(v_h^n(\cdot - c\Delta t) - v^n(\cdot - c\Delta t)\| \\
&\leq \|\Pi_h(v_h^n(\cdot - c\Delta t) - \Pi_h(v^n(\cdot - c\Delta t)\| + \|\Pi_h(v^n(\cdot - c\Delta t) - v^n(\cdot - c\Delta t)\| \\
&\leq \|\Pi_h(v_h^n(\cdot - c\Delta t) - \Pi_h(v^n(\cdot - c\Delta t)\| + Ch^{q_{k,\ell}} \\
&\leq \|v_h^n(\cdot - c\Delta t) - v^n(\cdot - c\Delta t)\| + Ch^{q_{k,\ell}} \\
&\leq \|v_h^n - v^n\| + Ch^{q_{k,\ell}},
\end{aligned}$$

where we have used the fact $\|\Pi_h u\| \leq \|u\|$ in the fourth row, and the periodic boundary conditions in the last row. As for the initial condition, we have

$$\|v_h^0 - v_0\| = \|\Pi_h v_0 - v_0\| \leq Ch^{q_{k,\ell}}.$$

Finally we obtain for any given $T \geq 0$ the existence of a constant $C \geq 0$ (independent of T and of n), such that

$$\|v_h^n - v^n\| \leq C(n+1)\,h^{q_{k,\ell}} \leq CT\,\frac{h^{q_{k,\ell}}}{\Delta t},$$

for all n such that $n\Delta t \leq T$. $\square$

Therefore, if $\min(k,\ell)=0$, then $\|v_h^n - v^n\| \le CT\frac{h}{\Delta t}$, and we need $h = o(\Delta t)$ for the convergence. If otherwise $\min(k,\ell)\ge 1$, it holds $\|v_h^n - v^n\| \le CT\frac{h^{3/2}}{\Delta t}$ and we would only need $h = o(\Delta t^{2/3})$ for the convergence.

4.2. **Convergence of the first SLDG scheme in the obstacle case.** Now we turn to scheme (8) for the nonlinear equation (1). In particular, the scheme writes: initialize with $u_h^0 := \Pi_h u_0$, and $u_h^{n+1} = \Pi_h(\max(G^{SL}_{\Delta t}(u_h^n), \tilde g))$ for $n \ge 0$.

Theorem 4.2. *Assume that the exact solution u is not shattering in the sense of Definition 3.1, for some integer $\ell \ge 0$. Then the following error bound holds:*

$$\|u_h^n - u^n\| \le CT\ \frac{h^{q_{k,\ell}}}{\Delta t} + CT\ \frac{\|\tilde g - g_{\Delta t}\|}{\Delta t}$$

for some constant $C \ge 0$ independent of n. In particular,

(i) *If $\tilde g = g_{\Delta t}$, then $\forall\, t_n \le T$:*

$$\|u_h^n - u^n\| \le CT\ \frac{h^{q_{k,\ell}}}{\Delta t}$$

(ii) *If $\tilde g(x) := \max\big(g(x), g(x - c\Delta t)\big)$, and if $g \in \Delta_q^2(0,1)$, then $\forall\, t_n \le T$:*

$$\|u_h^n - u^n\| \le CT\ \frac{h^{q_{k,\ell}}}{\Delta t} + CT\,\Delta t^{3/2}.$$

Proof. By Lemma 3.1 and the "no shattering" assumption,

$$\|\Pi_h u^n - u^n\| \le Ch^{q_{k,\ell}}. \tag{19}$$

Hence, from the definitions,

$$\begin{aligned}
&\|u_h^{n+1} - u^{n+1}\| \le \|u_h^{n+1} - \Pi_h u^{n+1}\| + \|\Pi_h u^{n+1} - u^{n+1}\|\\
&\le \|\Pi_h(\max(G^{SL}_{\Delta t}(u_h^n), \tilde g)) - \Pi_h(\max(u^n(\cdot - c\Delta t), g_{\Delta t}))\| + Ch^{q_{k,\ell}}\\
&\le \|\max(G^{SL}_{\Delta t}(u_h^n), g_{\Delta t}) - \max(u^n(\cdot - c\Delta t), \tilde g)\| + Ch^{q_{k,\ell}}.
\end{aligned}$$

Using the fact that

$$|\max(a_1,b_1) - \max(a_2,b_2)| \le |a_1 - a_2| + |b_1 - b_2|,$$

we obtain

$$\begin{aligned}
\|u_h^{n+1} - u^{n+1}\| \ &\le\ \|G^{SL}_{\Delta t}(u_h^n) - u^n(\cdot - c\Delta t)\| + \|\tilde g - g_{\Delta t}\| + Ch^{q_{k,\ell}}\\
&\le \|\Pi_h(u_h^n(\cdot - c\Delta t)) - \Pi_h(u^n(\cdot - c\Delta t))\| + \|\tilde g - g_{\Delta t}\| + Ch^{q_{k,\ell}}\\
&\le \|u_h^n - u^n\| + \|\tilde g - g_{\Delta t}\| + Ch^{q_{k,\ell}} \qquad (20)
\end{aligned}$$

where we have used again (19) and the periodic boundary condition. Using Lemma 3.3 and by induction on n, we are done. ∎

Remark 4.1. *Defining $\tilde g$ as in (ii), and assuming* $\min(\ell,k)\ge 1$, *the error is bounded by $O(\frac{h^{3/2}}{\Delta t}) + O(\Delta t^{3/2})$. Therefore the optimal estimate is obtained when $h^{3/2} \equiv \Delta t^{5/2}$, or $\Delta t \equiv h^{3/5}$, and the error is of order $O(h^{9/10})$.*

4.3. **Convergence of the SLDG scheme defined with Gauss points.** Now we turn to scheme (10) for the nonlinear equation (1). In particular, the scheme writes: initialize with $u_h^0 := \Pi_h u_0$, and u_h^{n+1} is defined as the unique polynomial in V_h such that:

$$u_h^{n+1}(x_\alpha^j) := \max(G_{\Delta t}^{SL}(u_h^n)(x_\alpha^j),\ \tilde{g}(x_\alpha^j)), \quad \forall j,\ \alpha. \tag{21}$$

Theorem 4.3. *Let $\ell \geq 0$ and assume that the exact solution is not shattering in the sense of Definition 3.1. The following error bound holds:*

$$\|u_h^n - u^n\| \leq CT\ \frac{h^{q_{k,\ell}}}{\Delta t} + CT\ \frac{\|\tilde{g} - g_{\Delta t}\|}{\Delta t},$$

for some constant $C \geq 0$ independent of n. In particular,

(i) If $\tilde{g} = g_{\Delta t}$, then $\forall t_n \leq T$:

$$\|u_h^n - u^n\| \leq CT\ \frac{h^{q_{k,\ell}}}{\Delta t}.$$

(ii) If $\tilde{g}(x) := \max\big(g(x), g(x - c\Delta t)\big)$ and $g \in \Delta_q^2(0,1)$, then $\forall t_n \leq T$:

$$\|u_h^n - u^n\| \leq CT\ \frac{h^{q_{k,\ell}}}{\Delta t} + CT\,\Delta t\sqrt{\Delta t + h}.$$

Proof. In view of Lemma 3.1 and the "no shattering" property, we have

$$\|u^{n+1} - \Pi_h u^{n+1}\| \leq Ch^{q_{k,\ell}}.$$

Now turn back to the error estimate, and consider the case of $\tilde{g} = g_{\Delta t}$:

$$\begin{aligned}
&\|u_h^{n+1} - u^{n+1}\| \leq \|u_h^{n+1} - \Pi_h u^{n+1}\| + Ch^{q_{k,\ell}} \\
&= \|u_h^{n+1} - \Pi_h u^{n+1}\|_{\ell^2} + Ch^{q_{k,\ell}} \\
&\leq \|u_h^{n+1} - u^{n+1}\|_{\ell^2} + Ch^{q_{k,\ell}} \\
&\leq \left(\sum_{j,\alpha} w_\alpha^j \left|u_h^{n+1}(x_\alpha^j) - u^{n+1}(x_a^j)\right|^2 h_j\right)^{1/2} + Ch^{q_{k,\ell}}
\end{aligned}$$

where in the third line we have used (13) for u^{n+1}. Because of the DPP, for all points x, the exact solution satisfies:

$$u^{n+1}(x) = \max(u^n(x - c\Delta t),\ g_{\Delta t}(x)).$$

Therefore, for $\forall j,\ \alpha$:

$$\left|(u_h^{n+1} - u^{n+1})(x_\alpha^j)\right| \leq \left|G_{\Delta t}^{SL}(u_h^n)(x_\alpha^j) - u^n(x_\alpha^j - c\Delta t)\right| + \left|\tilde{g}(x_\alpha^j) - g_{\Delta t}(x_\alpha^j)\right|$$

and

$$\begin{aligned}
\|u_h^{n+1} - u^{n+1}\| &\le \left(\sum_{j,\alpha} w_\alpha^j \left| G_{\Delta t}^{SL}(u_h^n)(x_\alpha^j) - u^n(x_\alpha^j - c\Delta t) \right|^2 h_j \right)^{1/2} \\
&\qquad + \left(\sum_{j,\alpha} w_\alpha^j \left| \tilde{g}(x_\alpha^j) - g_{\Delta t}(x_\alpha^j) \right|^2 h_j \right)^{1/2} + Ch^{q_{k,\ell}} \\
&\le \|G_{\Delta t}^{SL}(u_h^n) - u^n(\cdot - c\Delta t)\|_{\ell^2} + \|\tilde{g} - g_{\Delta t}\|_{\ell^2} + Ch^{q_{k,\ell}} \\
&\le \|G_{\Delta t}^{SL}(u_h^n) - \Pi_h u^n(\cdot - c\Delta t)\|_{\ell^2} + \|\tilde{g} - g_{\Delta t}\|_{\ell^2} + Ch^{q_{k,\ell}} \\
&= \|\Pi_h u_h^n(\cdot - c\Delta t) - \Pi_h u^n(\cdot - c\Delta t)\|_{\ell^2} + \|\tilde{g} - g_{\Delta t}\|_{\ell^2} + Ch^{q_{k,\ell}} \\
&= \|\Pi_h u_h^n(\cdot - c\Delta t) - \Pi_h u^n(\cdot - c\Delta t)\| + \|\tilde{g} - g_{\Delta t}\|_{\ell^2} + Ch^{q_{k,\ell}} \\
&\le \|u_h^n(\cdot - c\Delta t) - u^n(\cdot - c\Delta t)\| + \|\tilde{g} - g_{\Delta t}\|_{l^2} + Ch^{q_{k,\ell}} \\
&\le \|u_h^n - u^n\| + \|\tilde{g} - g_{\Delta t}\|_{\ell^2} + Ch^{q_{k,\ell}}
\end{aligned}$$

where in the fourth line we have used Lemma 3.2 for the function $u^n(\cdot - c\Delta t)$. Using Lemma 3.3 and by induction on n, we are done. □

Remark 4.2. *Defining $\tilde{g}$ as in (ii), and assuming $\min(\ell, k) \ge 1$, the error is bounded by $O(\frac{h^{3/2}}{\Delta t}) + O(\Delta t\sqrt{\Delta t + h})$. Therefore the optimal estimate is obtained when $h^{3/2} \equiv \Delta t^2\sqrt{\Delta t + h}$. So $\frac{h}{\Delta t} \to 0$, $h^{3/2} \equiv \Delta t^{5/2}$, or $\Delta t \equiv h^{3/5}$ (as in Remark 4.1), and the error of the scheme defined with gauss points is again of order $O(h^{9/10})$ for this particular time stepping.*

5. Convergence of RKDG schemes

In this section, we will prove convergence for the RKDG schemes. We will proceed in three steps similar to the previous section.

Firstly, let us recall the following properties for the bilinear operator $\mathcal{H}$.

Lemma 5.1. [21] *For any $\phi_h, \varphi_h \in V_h$, we have*

$$\mathcal{H}(\phi_h, \varphi_h) + \mathcal{H}(\varphi_h, \phi_h) = -\sum_j c[\phi_h]_{j+1/2} \cdot [\varphi_h]_{j+1/2}$$

$$\mathcal{H}(\phi_h, \phi_h) = -\frac{1}{2}\sum_j c[\phi_h]_{j+1/2}^2.$$

We also recall inverse inequalities [8] for the finite element space V_h. In particular, there exists a constant C (independent of h), such that, for any $\varphi_h \in V_h$,

$$\|(\varphi_h)_x\| \le Ch^{-1}\|\varphi_h\|, \qquad \|\varphi_h\|_{L^\infty} \le Ch^{-1/2}\|\varphi_h\|.$$

5.1. **Convergence of the RKDG scheme for the linear advection equation.** We first consider the linear equation $v_t + cv_x = 0$, for which

$$v(t + \Delta t) = v(t, x - c\Delta t).$$

We still denote $v^n(\cdot) = v(t^n, \cdot)$. In particular, the scheme writes: initialize with $v_h^0 := \Pi_h v_0$, and $v_h^{n+1} = G_{\Delta t}^{RK}(v_h^n)$ for $n \geq 0$.

This convergence proof closely follows the work in [21] for smooth solution, but additional difficulties are encountered because we consider solutions with less regularity. The main technique is to introduce piece-wisely defined intermediate stage functions and the careful treatment of intervals containing irregular points.

Theorem 5.1. *We consider $v_t + cv_x = 0$, $v(0, x) = v_0(x)$. Let v_0 be in $\mathcal{C}^{\ell+1}_{p,L,c_0}(0, 1)$, $\ell \geq 2$, $k \geq 1$, and assume the CFL condition*

$$\Delta t \leq C_0 h$$

for C_0 small enough (the usual CFL condition for stability of the RKDG scheme). The following bound holds:

$$\|v_h^n - v^n\| \leq C_1 (h + \frac{h^3}{\Delta t^2})^{1/2},$$

for some constant $C_1 \geq 0$ independent of $h, \Delta t, v_h$.

In particular, if $\Delta t/h$ is bounded from below ($\frac{\Delta t}{h} \geq \bar{C}_0$ for some constant $\bar{C}_0 > 0$), then

$$\|v_h^n - v^n\| \leq C_1 h^{1/2}.$$

Proof. We need to introduce some intermediate stages of the exact solution. Firstly we define

$$v^{(1)} = v^n - c\Delta t\ (v^n)_x$$

where the spatial derivative $(v^n)_x$ should be understood in the weak sense. We notice that $v^{(1)}$ may become discontinuous at the irregular points of v^n.

To define the second intermediate stage $v^{(2)}$, we need to distinguish the "good" and "bad" intervals. Since $v^n \in \mathcal{C}^{\ell+1}_{p,L,c_0}(0, 1)$, when the mesh is fine enough, there are at most p irregular intervals. Because of the CFL condition (which we assume implies in particular that $c\triangle t \leq h$), each irregular point at t^n may influence at most three intervals at time t^{n+1}. Now we introduce sets $\mathcal{B}^n$ and $\mathcal{I}^n$ such that

$$\mathcal{B}^n = \bigcup_j I_j, \text{s.t.}\ I_j \text{ or its immediate neighbors contain an irregular point of}\ v^n$$

and the corresponding set of indices:

$$\mathcal{I}^n = \bigcup_j j, \text{s.t.}\ I_j \text{ or its immediate neighbors contain an irregular point of}\ v^n.$$

Therefore $meas(\mathcal{B}^n) \leq 3ph$ and $Card(\mathcal{I}^n) \leq 3p$. (In the case of the irregular points located exactly at the cell interface $x_{j+1/2}$, we include the point's neighboring cells I_j, I_{j+1} in $\mathcal{B}^n$, and j, $j+1$ in $\mathcal{I}^n$.)

Now, we define

$$\tilde{v}^{(2)} = \begin{cases} \frac{3}{4}v^n + \frac{1}{4}v^{(1)} - c\frac{\Delta t}{4}\ (v^{(1)})_x, & \text{if } x \notin \mathcal{B}^n, \\ \frac{3}{4}v^n + \frac{1}{4}v^{(1)} - c\frac{\Delta t}{4}\ (v^n)_x \equiv v^n - c\frac{\Delta t}{2}\ (v^n)_x, & \text{if } x \in \mathcal{B}^n, \end{cases} \tag{22}$$

For points not located in $\mathcal{B}^n$, the definition coincides with [21]. For points in $\mathcal{B}^n$, v^n is used instead of $v^{(1)}$ to avoid discontinuity at the irregular points. Notice that this causes $\tilde{v}^{(2)}$ to be discontinuous at $\partial\mathcal{B}^n$. For example, if $x_a \in \partial\mathcal{B}^n$, then the jump of $\tilde{v}^{(2)}$ at x_a is of magnitude $c\frac{\Delta t}{4}\ (v^{(1)} - v^n)_x(x_a)$, and this is bounded by $CL\Delta t^2$. By these arguments, we could add a linear interpolating function defined by $\frac{1}{4}L_a(x)$ which is nonzero only on $\mathcal{B}^n$ to enforce continuity at $\partial\mathcal{B}^n$, and $||L_a||_\infty < C\Delta t^2$, i.e. we introduce

$$v^{(2)} = \begin{cases} \frac{3}{4}v^n + \frac{1}{4}v^{(1)} - c\frac{\Delta t}{4}\ (v^{(1)})_x, & \text{if } x \notin \mathcal{B}^n, \\ v^n - c\frac{\Delta t}{2}\ (v^n)_x + \frac{1}{4}L_a(x), & \text{if } x \in \mathcal{B}^n, \end{cases} \tag{23}$$

and L_a is chosen to be a linear polynomial so that $v^{(2)}$ is continuous at $\partial\mathcal{B}^n$. We can now define

$$v^{(3)} = \begin{cases} \frac{1}{3}v^n + \frac{2}{3}v^{(2)} - c\frac{2\Delta t}{3}\ (v^{(2)})_x, & \text{if } x \notin \mathcal{B}^n, \\ \frac{1}{3}v^n + \frac{2}{3}v^{(2)} - c\frac{2\Delta t}{3}\ (v^n)_x \equiv v^n - c\Delta t\ (v^n)_x + \frac{1}{6}L_a(x), & \text{if } x \in \mathcal{B}^n. \end{cases} \tag{24}$$

Notice that for $x \notin \mathcal{B}^n$, the definition is still consistent with [21] for smooth solutions, and it is well defined because $\ell \geq 2$. However, for irregular intervals, the definition is modified due to the lower regularity of the solution.

Now we are ready to define the errors

$$\begin{aligned} &e^{(1)} = v^{(1)} - v_h^{n,1}, \quad \xi^{(1)} = \mathbb{P}_h v^{(1)} - v_h^{n,1}, \quad \eta^{(1)} = \mathbb{P}_h v^{(1)} - v^{(1)}, \\ &e^{(2)} = v^{(2)} - v_h^{n,2}, \quad \xi^{(2)} = \mathbb{P}_h v^{(2)} - v_h^{n,2}, \quad \eta^{(2)} = \mathbb{P}_h v^{(2)} - v^{(2)}, \\ &e^n = v^n - v_h^n, \quad \xi^n = \mathbb{P}_h v^n - v_h^n, \quad \eta^n = \mathbb{P}_h v^n - v^n. \end{aligned} \tag{25}$$

Clearly,

$$e^{(1)} = \xi^{(1)} - \eta^{(1)}, \quad e^{(2)} = \xi^{(2)} - \eta^{(2)}, \quad e^n = \xi^n - \eta^n.$$

Our next step is to establish the error equations. First, let us recall that the numerical solution satisfies:

$$\begin{aligned}
\int_{I_j} v_h^{n,1}\varphi_h dx &= \int_{I_j} v_h^n\varphi_h dx + \Delta t\mathcal{H}_j(v_h^n,\varphi_h), \quad \forall\varphi_h\in V_h\\
\int_{I_j} v_h^{n,2}\varphi_h dx &= \frac{3}{4}\int_{I_j} v_h^n\varphi_h dx + \frac{1}{4}\int_{I_j} v_h^{n,1}\varphi_h dx + \frac{\Delta t}{4}\mathcal{H}_j(v_h^{n,1},\varphi_h), \quad \forall\varphi_h\in V_h\\
\int_{I_j} v_h^{n+1}\varphi_h dx &= \frac{1}{3}\int_{I_j} v_h^n\varphi_h dx + \frac{2}{3}\int_{I_j} v_h^{n,2}\varphi_h dx + \frac{2\Delta t}{3}\mathcal{H}_j(v_h^{n,2},\varphi_h). \quad \forall\varphi_h\in V_h.
\end{aligned}$$

From the definitions of $v^{(1)}, v^{(2)}, v^{(3)}$, we can verify

$$\begin{aligned}
\int_{I_j} v^{(1)}\varphi_h dx &= \int_{I_j} v^n\varphi_h dx + \Delta t\mathcal{H}_j(v^n,\varphi_h), \quad \forall\varphi_h\in V_h\\
\int_{I_j} v^{(2)}\varphi_h dx &= \frac{3}{4}\int_{I_j} v^n\varphi_h dx + \frac{1}{4}\int_{I_j} v^{(1)}\varphi_h dx + \frac{\Delta t}{4}\mathcal{H}_j(v^{(\star 1)},\varphi_h), \quad \forall\varphi_h\in V_h\\
&\quad + \begin{cases} 0, & j\notin\mathcal{I}^n\\ \frac{1}{4}(L_a,\varphi_h), & j\in\mathcal{I}^n\end{cases}\\
\int_{I_j} v^{(3)}\varphi_h dx &= \frac{1}{3}\int_{I_j} v^n\varphi_h dx + \frac{2}{3}\int_{I_j} v^{(2)}\varphi_h dx + \frac{2\Delta t}{3}\mathcal{H}_j(v^{(\star 2)},\varphi_h), \quad \forall\varphi_h\in V_h
\end{aligned}$$

where for $j\in\mathcal{I}^n, (\star 1)=n, (\star 2)=n$; otherwise, $(\star 1)=(1), (\star 2)=(2)$. Notice that the formulations above are correct because we have enforced continuity of the first function appeared in operator $\mathcal{H}_j$ in all cases. In particular, the procedure to enforce continuity of $v^{(2)}$ at $\partial\mathcal{B}^n$ turns out to be necessary here. Combining the previous two relations, we derive the error equations

$$\begin{aligned}
\int_{I_j} e^{(1)}\varphi_h dx &= \int_{I_j} e^n\varphi_h dx + \Delta t\mathcal{H}_j(e^n,\varphi_h), \quad \forall\varphi_h\in V_h\\
\int_{I_j} e^{(2)}\varphi_h dx &= \frac{3}{4}\int_{I_j} e^n\varphi_h dx + \frac{1}{4}\int_{I_j} e^{(1)}\varphi_h dx + \frac{\Delta t}{4}\mathcal{H}_j(e^{(1)},\varphi_h), \quad \forall\varphi_h\in V_h\\
&\quad + \begin{cases} 0, & j\notin\mathcal{I}^n\\ \frac{\Delta t}{4}\mathcal{H}_j(v^n-v^{(1)},\varphi_h)+\frac{1}{4}(L_a,\varphi_h), & j\in\mathcal{I}^n\end{cases}\\
\int_{I_j} e^{n+1}\varphi_h dx &= \int_{I_j}\Upsilon\varphi_h dx + \frac{1}{3}\int_{I_j} e^n\varphi_h dx + \frac{2}{3}\int_{I_j} e^{(2)}\varphi_h dx, \quad \forall\varphi_h\in V_h\\
&\quad + \frac{2\Delta t}{3}\mathcal{H}_j(e^{(2)},\varphi_h) + \begin{cases} 0, & j\notin\mathcal{I}^n\\ \frac{2\Delta t}{3}\mathcal{H}_j(v^n-v^{(2)},\varphi_h), & j\in\mathcal{I}^n\end{cases}
\end{aligned}$$

where $\Upsilon = v^{n+1} - v^{(3)}$. Using the decomposition of errors (25), we get

$$\int_{I_j} \xi^{(1)} \varphi_h dx = \int_{I_j} \xi^n \varphi_h dx + \Delta t \mathcal{J}_j(\varphi_h), \quad \forall \varphi_h \in V_h \tag{26a}$$

$$\int_{I_j} \xi^{(2)} \varphi_h dx = \frac{3}{4} \int_{I_j} \xi^n \varphi_h dx + \frac{1}{4} \int_{I_j} \xi^{(1)} \varphi_h dx + \frac{\Delta t}{4} \mathcal{K}_j(\varphi_h), \quad \forall \varphi_h \in V_h \tag{26b}$$

$$\int_{I_j} \xi^{n+1} \varphi_h dx = \frac{1}{3} \int_{I_j} \xi^n \varphi_h dx + \frac{2}{3} \int_{I_j} \xi^{(2)} \varphi_h dx + \frac{2\Delta t}{3} \mathcal{L}_j(\varphi_h), \quad \forall \varphi_h \in V_h \tag{26c}$$

where

$$\mathcal{J}_j(\varphi_h) = \int_{I_j} \frac{1}{\Delta t}(\eta^{(1)} - \eta^n)\varphi_h dx + \mathcal{H}_j(e^n, \varphi_h), \quad \forall \varphi_h \in V_h \tag{27a}$$

$$\begin{aligned} \mathcal{K}_j(\varphi_h) = & \int_{I_j} \frac{1}{\Delta t}(4\eta^{(2)} - 3\eta^n - \eta^{(1)})\varphi_h dx + \mathcal{H}_j(e^{(1)}, \varphi_h), \quad \forall \varphi_h \in V_h \\ & + \begin{cases} 0, & j \notin \mathcal{I}^n \\ \mathcal{H}_j(v^n - v^{(1)}, \varphi_h) + \frac{1}{\Delta t}(L_a, \varphi_h), & j \in \mathcal{I}^n \end{cases} \end{aligned} \tag{27b}$$

$$\begin{aligned} \mathcal{L}_j(\varphi_h) = & \int_{I_j} \frac{1}{2\Delta t}(3\eta^{n+1} - \eta^n - 2\eta^{(2)} + 3\Upsilon)\varphi_h dx + \mathcal{H}_j(e^{(2)}, \varphi_h), \quad \forall \varphi_h \in V_h \\ & + \begin{cases} 0, & j \notin \mathcal{I}^n \\ \mathcal{H}_j(v^n - v^{(2)}, \varphi_h), & j \in \mathcal{I}^n \end{cases} \end{aligned} \tag{27c}$$

We further denote $\mathcal{J}(\varphi_h) = \sum_j \mathcal{J}_j(\varphi_h)$, $\mathcal{K}(\varphi_h) = \sum_j \mathcal{K}_j(\varphi_h)$, $\mathcal{L}(\varphi_h) = \sum_j \mathcal{L}_j(\varphi_h)$. By letting $\varphi_h = \xi^n, 4\xi^{(1)}, 6\xi^{(2)}$ in (27a), (27b), (27c), respectively, we get the following energy equation for ξ^n, [21]

$$\begin{aligned} 3\|\xi^{n+1}\|^2 - 3\|\xi^n\|^2 = & \Delta t[\mathcal{J}(\xi^n) + \mathcal{K}(\xi^{(1)}) + \mathcal{L}(\xi^{(2)})] \\ & + \|2\xi^{(2)} - \xi^{(1)} - \xi^n\|^2 + 3(\xi^{n+1} - \xi^n, \xi^{n+1} - 2\xi^{(2)} + \xi^n) \end{aligned} \tag{28}$$

Now we define $\Pi_1 = \Delta t[\mathcal{J}(\xi^n) + \mathcal{K}(\xi^{(1)}) + \mathcal{L}(\xi^{(2)})]$, $\Pi_2 = \|2\xi^{(2)} - \xi^{(1)} - \xi^n\|^2 + 3(\xi^{n+1} - \xi^n, \xi^{n+1} - 2\xi^{(2)} + \xi^n)$. We will estimate those two terms separately.

Estimate of Π_1

Firstly, we notice that

$$\begin{aligned}
\Delta t\mathcal{J}(\xi^n) &= (\eta^{(1)}-\eta^n,\xi^n)+\Delta t\mathcal{H}(e^n,\xi^n)\\
&= (\eta^{(1)}-\eta^n,\xi^n)+\Delta t\mathcal{H}(\xi^n,\xi^n)\\
&= (\eta^{(1)}-\eta^n,\xi^n)-\frac{\Delta t}{2}\sum_j c[\xi^n]^2_{j+1/2}\\
&\le \|\eta^{(1)}-\eta^n\|\cdot\|\xi^n\|-\frac{\Delta t}{2}\sum_j c[\xi^n]^2_{j+1/2}\\
&\le \frac{1}{4\Delta t\epsilon}\|\eta^{(1)}-\eta^n\|^2+\epsilon\Delta t\|\xi^n\|^2-\frac{\Delta t}{2}\sum_j c[\xi^n]^2_{j+1/2}
\end{aligned}$$

where in the second line we have used the property of the Gauss-Radau projection to get $\mathcal{H}(\eta^n,\varphi_h)=0$. In the formulas above, ϵ is a positive constant of order 1. Since

$$\eta^{(1)}-\eta^n=\mathbb{P}_h(v^{(1)}-v^n)-(v^{(1)}-v^n)=-\Delta t\,c\,\left(\mathbb{P}_h(v^n)_x-(v^n)_x\right),$$

similar to Lemma 3.1, we get

$$\begin{aligned}
\|\mathbb{P}_h(v^n)_x-(v^n)_x\| &\le \|\mathbb{P}_h(v^n)_x-(v^n)_x\|_{\mathcal{B}^n}+\|\mathbb{P}_h(v^n)_x-(v^n)_x\|_{I\setminus\mathcal{B}^n}\\
&\le Ch^{1/2}+Ch^{\min(\ell,k+1)}\le Ch^{1/2},
\end{aligned}$$

Therefore $\|\eta^{(1)}-\eta^n\|\le C\Delta t h^{1/2}$ and

$$\Delta t\mathcal{J}(\xi^n)\le C\Delta t h+\epsilon\Delta t\|\xi^n\|^2-\frac{\Delta t}{2}\sum_j c[\xi^n]^2_{j+1/2}.$$

Similarly,

$$\begin{aligned}
&\Delta t\mathcal{K}(\xi^{(1)})\\
&=(4\eta^{(2)}-\eta^{(1)}-3\eta^n+L_a,\xi^{(1)})+\Delta t\mathcal{H}(e^{(1)},\xi^{(1)})+\Delta t\sum_{j\in\mathcal{I}^n}\mathcal{H}_j(v^n-v^{(1)},\xi^{(1)})\\
&=(4\eta^{(2)}-\eta^{(1)}-3\eta^n+L_a,\xi^{(1)})-\frac{\Delta t}{2}\sum_j c[\xi^{(1)}]^2_{j+1/2}+\Delta t\sum_{j\in\mathcal{I}^n}\mathcal{H}_j(v^n-v^{(1)},\xi^{(1)})\\
&\le\frac{1}{4\Delta t\epsilon}\|4\eta^{(2)}-\eta^{(1)}-3\eta^n\|^2+\frac{1}{4\Delta t\epsilon}\|L_a\|^2+\frac{\epsilon}{2}\Delta t\|\xi^{(1)}\|^2-\frac{\Delta t}{2}\sum_j c[\xi^{(1)}]^2_{j+1/2}\\
&+\Delta t\sum_{j\in\mathcal{I}^n}\mathcal{H}_j(v^n-v^{(1)},\xi^{(1)})
\end{aligned}$$

Since $||L_a||_\infty\le C\Delta t^2$, and $L_a\ne 0$ only on $\mathcal{B}^n$, therefore $||L_a||\le C\Delta t^2h^{1/2}$. Next, we will estimate the term $\|4\eta^{(2)}-\eta^{(1)}-3\eta^n\|$ and $\Delta t\sum_{j\in\mathcal{I}}\mathcal{H}_j(v^n-v^{(1)},\xi^{(1)})$. We can derive that

$$4\eta^{(2)} - \eta^{(1)} - 3\eta^n = \begin{cases} -\Delta t c(\mathbb{P}_h v_x^{(1)} - v_x^{(1)}), & x \notin \mathcal{B}^n \\ -\Delta t c(\mathbb{P}_h v_x^n - v_x^n) + (\mathbb{P}_h L_a - L_a), & x \in \mathcal{B}^n \end{cases}$$

Because L_a is a linear polynomial and $k \geq 1$, $\mathbb{P}_h L_a - L_a = 0$, and similar to the previous argument, we get $\|4\eta^{(2)} - \eta^{(1)} - 3\eta^n\| \leq C\Delta t h^{1/2}$.

As for $\Delta t \sum_{j\in\mathcal{I}} \mathcal{H}_j(v^n - v^{(1)}, \xi^{(1)})$, we have

$$v^n - v^{(1)} = c\Delta t (v^n)_x,$$

therefore for any j

$$|(v^n - v^{(1)})_{j\pm1/2}| \leq C\Delta t \|v\|_{W^{1,\infty}}$$

and

$$\|v^n - v^{(1)}\|_{I_j} \leq C\Delta t h^{1/2} \|v\|_{W^{1,\infty}}.$$

Hence

$$\begin{aligned} &\mathcal{H}_j(v^n - v^{(1)}, \xi^{(1)}) \\ &= \int_{I_j} c(v^n - v^{(1)})\xi_x^{(1)} dx - c(v^n - v^{(1)})^-(\xi^{(1)})^-_{j+1/2} + c(v^n - v^{(1)})^-(\xi^{(1)})^+_{j-1/2} \\ &\leq C\Delta t h^{1/2}\|\xi_x^{(1)}\|_{I_j} + C\Delta t|(\xi^{(1)})^-_{j+1/2}| + C\Delta t|(\xi^{(1)})^+_{j-1/2}| \\ &\leq C\Delta t h^{-1/2}\|\xi^{(1)}\|_{I_j} \end{aligned}$$

by inverse inequalities, and

$$\begin{aligned} \Delta t \sum_{j\in\mathcal{I}^n} \mathcal{H}_j(v^n - v^{(1)}, \xi^{(1)}) &\leq C\Delta t^2 h^{-1/2} \|\xi^{(1)}\|_{\mathcal{B}^n} \\ &\leq C\Delta t^2 + \frac{\epsilon}{2}\Delta t \|\xi^{(1)}\|^2_{B^n} \\ &\leq C\Delta t^2 + \frac{\epsilon}{2}\Delta t\|\xi^{(1)}\|^2, \end{aligned}$$

where in the second line we have used the CFL condition $\Delta t \leq C_{cfl} h$. Putting everything together, and using the CFL condition again, we have

$$\Delta t \mathcal{K}(\xi^{(1)}) \leq C\Delta t h + \epsilon \Delta t \|\xi^{(1)}\|^2 - \frac{\Delta t}{2}\sum_j c[\xi^{(1)}]^2_{j+1/2}$$

Finally,

$$
\begin{aligned}
&\Delta t\mathcal{L}(\xi^{(2)})\\
&=\frac{1}{2}(3\eta^{n+1}-2\eta^{(2)}-\eta^n+3\Upsilon,\xi^{(2)})+\Delta t\mathcal{H}(e^{(2)},\xi^{(2)})+\Delta t\sum_{j\in\mathcal{I}^n}\mathcal{H}_j(v^n-v^{(2)},\xi^{(2)})\\
&=\frac{1}{2}(3\eta^{n+1}-2\eta^{(2)}-\eta^n+3\Upsilon,\xi^{(2)})-\frac{\Delta t}{2}\sum_j c[\xi^{(2)}]^2_{j+1/2}+\Delta t\sum_{j\in\mathcal{I}^n}\mathcal{H}_j(v^n-v^{(2)},\xi^{(2)})\\
&\le\frac{1}{8\Delta t\epsilon}\|3\eta^{n+1}-2\eta^{(2)}-\eta^n\|^2+\frac{9}{8\Delta t\epsilon}\|\Upsilon\|^2+\epsilon\Delta t\|\xi^{(2)}\|^2-\frac{\Delta t}{2}\sum_j c[\xi^{(2)}]^2_{j+1/2}\\
&\quad+\Delta t\sum_{j\in\mathcal{I}^n}\mathcal{H}_j(v^n-v^{(2)},\xi^{(2)}).
\end{aligned}
$$

Using the same argument as the previous terms, we have $\|3\eta^{n+1}-2\eta^{(2)}-\eta^n\|\le C\Delta t h^{1/2}$ and $\Delta t\sum_{j\in\mathcal{I}^n}\mathcal{H}_j(v^n-v^{(2)},\xi^{(2)})\le C\Delta t^2+\epsilon\Delta t\|\xi^{(2)}\|^2$. As for Υ, we have

$$
\begin{aligned}
\|\Upsilon\|^2 &= \int_{\mathcal{B}^n}\Upsilon^2dx+\int_{I\setminus\mathcal{B}^n}\Upsilon^2dx\\
&= \int_{\mathcal{B}^n}(v^{n+1}-v^{(3)})^2dx+\int_{I\setminus\mathcal{B}^n}\Upsilon^2dx\\
&\le \int_{\mathcal{B}^n}(v^{n+1}-v^n+c\Delta t(v^n)_x-\frac{1}{6}L_a)^2dx+(C\Delta t^4)^2\\
&= \int_{\mathcal{B}^n}(v^{n+1}-v^n+c\Delta t(v^n)_x)^2dx+C\Delta t^4h+(C\Delta t^4)^2\\
&\le C\Delta t^2h+C\Delta t^4h+(C\Delta t^4)^2\le C\Delta t^2h.
\end{aligned}
$$

Finally we obtain

$$\Delta t\mathcal{L}(\xi^{(2)})\le C\Delta t h+\epsilon\Delta t\|\xi^{(2)}\|^2-\frac{\Delta t}{2}\sum_j c[\xi^{(2)}]^2_{j+1/2}.$$

Putting everything together, we have

$$
\begin{aligned}
\Pi_1 \le\ & C\Delta t h+\epsilon\Delta t\|\xi^n\|^2+\epsilon\Delta t\|\xi^{(1)}\|^2+\epsilon\Delta t\|\xi^{(2)}\|^2\\
&-\frac{\Delta t}{2}\sum_j c[\xi^n]^2_{j+1/2}-\frac{\Delta t}{2}\sum_j c[\xi^{(1)}]^2_{j+1/2}-\frac{\Delta t}{2}\sum_j c[\xi^{(2)}]^2_{j+1/2}
\end{aligned}
$$

Estimate of Π_2

To estimate Π_2, we first introduce

$$
\begin{aligned}
&\mathbb{G}_1=\xi^{(1)}-\xi^n\\
&\mathbb{G}_2=2\xi^{(2)}-\xi^{(1)}-\xi^n\\
&\mathbb{G}_3=\xi^{n+1}-2\xi^{(2)}+\xi^n.
\end{aligned}
$$

From the error equation (26), we can deduce

$$\int_{I_j} \mathbb{G}_1\varphi_h dx = \Delta t \mathcal{J}_j(\varphi_h), \quad \forall \varphi_h \in V_h \tag{29a}$$

$$\int_{I_j} \mathbb{G}_2\varphi_h dx = \frac{\Delta t}{2}(\mathcal{K}_j(\varphi_h) - \mathcal{J}_j(\varphi_h)), \quad \forall \varphi_h \in V_h \tag{29b}$$

$$\int_{I_j} \mathbb{G}_3\varphi_h dx = \frac{\Delta t}{3}(2\mathcal{L}_j(\varphi_h) - \mathcal{K}_j(\varphi_h) - \mathcal{J}_j(\varphi_h)), \quad \forall \varphi_h \in V_h \tag{29c}$$

Now,

$$\Pi_2 = (\mathbb{G}_2, \mathbb{G}_2) + 3(\mathbb{G}_1, \mathbb{G}_3) + 3(\mathbb{G}_2, \mathbb{G}_3) + 3(\mathbb{G}_3, \mathbb{G}_3).$$

First, let us estimate $(\mathbb{G}_2, \mathbb{G}_2) + 3(\mathbb{G}_1, \mathbb{G}_3)$.

$$\begin{aligned}
&(\mathbb{G}_2, \mathbb{G}_2) + 3(\mathbb{G}_1, \mathbb{G}_3)\\
&= -\|\mathbb{G}_2\|^2 + 2(\mathbb{G}_2, \mathbb{G}_2) + 3(\mathbb{G}_1, \mathbb{G}_3)\\
&= -\|\mathbb{G}_2\|^2 + \Delta t\,[\mathcal{K}(\mathbb{G}_2) - \mathcal{J}(\mathbb{G}_2) + 2\mathcal{L}(\mathbb{G}_1) - \mathcal{K}(\mathbb{G}_1) - \mathcal{J}(\mathbb{G}_1)]
\end{aligned}$$

We have

$$\begin{aligned}
& \Delta t(\mathcal{K}(\mathbb{G}_2) - \mathcal{J}(\mathbb{G}_2))\\
= \;& (4\eta^{(2)} - 3\eta^n - \eta^{(1)} - (\eta^{(1)} - \eta^n) + L_a, \mathbb{G}_2) + \Delta t \mathcal{H}(e^{(1)} - e^n, \mathbb{G}_2)\\
& + \Delta t \sum_{j\in\mathcal{I}^n} \mathcal{H}_j(v^n - v^{(1)}, \mathbb{G}_2)\\
= \;& (4\eta^{(2)} - 3\eta^n - \eta^{(1)} - (\eta^{(1)} - \eta^n) + L_a, \mathbb{G}_2) + \Delta t \mathcal{H}(\mathbb{G}_1, \mathbb{G}_2)\\
& + \Delta t \sum_{j\in\mathcal{I}^n} \mathcal{H}_j(v^n - v^{(1)}, \mathbb{G}_2)\\
\leq \;& \|4\eta^{(2)} - 3\eta^n - \eta^{(1)} - (\eta^{(1)} - \eta^n) + L_a\|^2 + \frac{1}{4}\|\mathbb{G}_2\|^2\\
& + \Delta t \mathcal{H}(\mathbb{G}_1, \mathbb{G}_2) + \Delta t \sum_{j\in\mathcal{I}^n} \mathcal{H}_j(v^n - v^{(1)}, \mathbb{G}_2)\\
\leq \;& C\Delta t^2 h + \frac{1}{4}\|\mathbb{G}_2\|^2 + \Delta t \mathcal{H}(\mathbb{G}_1, \mathbb{G}_2) + \Delta t \sum_{j\in\mathcal{I}^n} \mathcal{H}_j(v^n - v^{(1)}, \mathbb{G}_2)\\
\leq \;& C\Delta t^2 h + \frac{1}{4}\|\mathbb{G}_2\|^2 + \Delta t \mathcal{H}(\mathbb{G}_1, \mathbb{G}_2) + C\Delta t^2 h^{-1/2}\|\mathbb{G}_2\|\\
\leq \;& C\Delta t^2 h + \frac{1}{4}\|\mathbb{G}_2\|^2 + \Delta t \mathcal{H}(\mathbb{G}_1, \mathbb{G}_2) + C\Delta t^4 h^{-1} + \frac{1}{4}\|\mathbb{G}_2\|^2\\
\leq \;& C\Delta t^2 h + \frac{1}{2}\|\mathbb{G}_2\|^2 + \Delta t \mathcal{H}(\mathbb{G}_1, \mathbb{G}_2).
\end{aligned}$$

For the other term, similarly we get

$$\begin{aligned}
& \Delta t(2\mathcal{L}(\mathbb{G}_1) - \mathcal{K}(\mathbb{G}_1) - \mathcal{J}(\mathbb{G}_1)) \\
= \; & (3\eta^{n+1} - 2\eta^{(2)} - \eta^n + 3\Upsilon - (4\eta^{(2)} - 3\eta^n - \eta^{(1)}) - (\eta^{(1)} - \eta^n) - L_a, \mathbb{G}_1) \\
& + \Delta t \mathcal{H}(2e^{(2)} - e^{(1)} - e^n, \mathbb{G}_1) + \Delta t \sum_{j \in \mathcal{I}^n} \mathcal{H}_j(2(v^n - v^{(2)}) - (v^n - v^{(1)}), \mathbb{G}_1) \\
\leq \; & C\Delta t h + \Delta t \mathcal{H}(\mathbb{G}_2, \mathbb{G}_1) + \frac{\epsilon}{2} \Delta t \|\mathbb{G}_1\|^2 \\
\leq \; & C\Delta t h + \Delta t \mathcal{H}(\mathbb{G}_2, \mathbb{G}_1) + \epsilon \Delta t \|\xi^n\|^2 + \epsilon \Delta t \|\xi^{(1)}\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& (\mathbb{G}_2, \mathbb{G}_2) + 3(\mathbb{G}_1, \mathbb{G}_3) \\
& \leq -\frac{1}{2}\|\mathbb{G}_2\|^2 + C\Delta t h + \Delta t \mathcal{H}(\mathbb{G}_2, \mathbb{G}_1) + \Delta t \mathcal{H}(\mathbb{G}_1, \mathbb{G}_2) + \epsilon \Delta t \|\xi^n\|^2 + \epsilon \Delta t \|\xi^{(1)}\|^2 \\
& \leq -\frac{1}{2}\|\mathbb{G}_2\|^2 + C\Delta t h - \Delta t \sum_j c[\mathbb{G}_1][\mathbb{G}_2] + \epsilon \Delta t \|\xi^n\|^2 + \epsilon \Delta t \|\xi^{(1)}\|^2 \\
& \leq -\frac{1}{2}\|\mathbb{G}_2\|^2 + C\Delta t h + \frac{\Delta t}{4} \sum_j c|[\mathbb{G}_1]|^2 + \Delta t \sum_j c|[\mathbb{G}_2]|^2 + \epsilon \Delta t \|\xi^n\|^2 + \epsilon \Delta t \|\xi^{(1)}\|^2.
\end{aligned}$$

Also

$$\begin{aligned}
& 3(\mathbb{G}_3, \mathbb{G}_2) \\
= \; & \Delta t(2\mathcal{L}(\mathbb{G}_2) - \mathcal{K}(\mathbb{G}_2) - \mathcal{J}(\mathbb{G}_2)) \\
= \; & (3\eta^{n+1} - 2\eta^{(2)} - \eta^n + 3\Upsilon - (4\eta^{(2)} - 3\eta^n - \eta^{(1)}) - (\eta^{(1)} - \eta^n) - L_a, \mathbb{G}_2) \\
& + \Delta t \mathcal{H}(2e^{(2)} - e^{(1)} - e^n, \mathbb{G}_2) + \Delta t \sum_{j \in \mathcal{I}^n} \mathcal{H}_j(2(v^n - v^{(2)}) - (v^n - v^{(1)}), \mathbb{G}_2) \\
\leq \; & C\Delta t h + \Delta t \mathcal{H}(\mathbb{G}_2, \mathbb{G}_2) + \epsilon \Delta t \|\mathbb{G}_2\|^2 \\
\leq \; & C\Delta t h - \frac{\Delta t}{2} \sum_j c[\mathbb{G}_2]^2_{j+1/2} + \epsilon \Delta t \|\mathbb{G}_2\|^2.
\end{aligned}$$

and

$$\begin{aligned}
& 3\|\mathbb{G}_3\|^2 = 3(\mathbb{G}_3, \mathbb{G}_3) \\
= \; & \Delta t(2\mathcal{L}(\mathbb{G}_3) - \mathcal{K}(\mathbb{G}_3) - \mathcal{J}(\mathbb{G}_3)) \\
= \; & (3\eta^{n+1} - 2\eta^{(2)} - \eta^n + 3\Upsilon - (4\eta^{(2)} - 3\eta^n - \eta^{(1)}) - (\eta^{(1)} - \eta^n) - L_a, \mathbb{G}_3) \\
& + \Delta t \mathcal{H}(2e^{(2)} - e^{(1)} - e^n, \mathbb{G}_3) + \Delta t \sum_{j \in \mathcal{I}^n} \mathcal{H}_j(2(v^n - v^{(2)}) - (v^n - v^{(1)}), \mathbb{G}_3) \\
\leq \; & C\Delta t h^{1/2} \|\mathbb{G}_3\| + C\frac{\Delta t}{h} \|\mathbb{G}_2\| \cdot \|\mathbb{G}_3\| + C\Delta t^2 h^{-1/2} \|\mathbb{G}_3\|
\end{aligned}$$

Therefore

$$\|\mathbb{G}_3\| \leq C\Delta t h^{1/2} + C\|\mathbb{G}_2\|$$

due to the CFL condition and

$$3\|\mathbb{G}_3\|^2 \leq C\Delta t^2 h + \frac{1}{4}\|\mathbb{G}_2\|^2.$$

Putting everything together, we have

$$\begin{aligned}
\Pi_2 &\leq (-\frac{1}{4}+\epsilon\Delta t)\|\mathbb{G}_2\|^2 + C\Delta t h + \frac{\Delta t}{4}\sum_j c[\mathbb{G}_1]^2_{j+1/2} \\
&\quad + \frac{\Delta t}{2}\sum_j c[\mathbb{G}_2]^2_{j+1/2} + \epsilon\Delta t\|\xi^n\|^2 + \epsilon\Delta t\|\xi^{(1)}\|^2 \\
&\leq (-\frac{1}{4}+\epsilon\Delta t)\|\mathbb{G}_2\|^2 + C\Delta t h + \frac{\Delta t}{2}\sum_j c([\xi^n]^2_{j+1/2} + [\xi^{(1)}]^2_{j+1/2}) \\
&\quad + C\frac{\Delta t}{h}\|\mathbb{G}_2\|^2 + \epsilon\Delta t\|\xi^n\|^2 + \epsilon\Delta t\|\xi^{(1)}\|^2 \\
&\leq (-\frac{1}{4}+\epsilon\Delta t + CC_{cfl})\|\mathbb{G}_2\|^2 + C\Delta t h + \frac{\Delta t}{2}\sum_j c([\xi^n]^2_{j+1/2} + [\xi^{(1)}]^2_{j+1/2}) \\
&\quad + \epsilon\Delta t\|\xi^n\|^2 + \epsilon\Delta t\|\xi^{(1)}\|^2
\end{aligned}$$

When ϵ and C_{cfl} are small enough, $-\frac{1}{4}+\epsilon\Delta t + CC_{cfl} \leq 0$, and

$$\Pi_2 \leq C\Delta t h + \frac{\Delta t}{2}\sum_j c([\xi^n]^2_{j+1/2} + [\xi^{(1)}]^2_{j+1/2}) + \epsilon\Delta t\|\xi^n\|^2 + \epsilon\Delta t\|\xi^{(1)}\|^2.$$

Finally

$$\Pi_1 + \Pi_2 \leq C\Delta t h + 2\epsilon\Delta t\|\xi^n\|^2 + 2\epsilon\Delta t\|\xi^{(1)}\|^2 + \epsilon\Delta t\|\xi^{(2)}\|^2.$$

At this point, we need to provide an estimate of $\|\xi^{(1)}\|$, $\|\xi^{(2)}\|$ to finish the proof. Plug in the error equation (26),

$$\begin{aligned}
&\|\xi^{(1)}\|^2 = (\xi^{(1)},\xi^{(1)}) = (\xi^n,\xi^{(1)}) + \Delta t\mathcal{J}(\xi^{(1)}) \\
&\leq \|\xi^n\|\cdot\|\xi^{(1)}\| + (\eta^{(1)}-\eta^n,\xi^{(1)}) + \Delta t\mathcal{H}(\xi^n-\eta^n,\xi^{(1)}) \\
&\leq \|\xi^n\|\cdot\|\xi^{(1)}\| + C\Delta t h^{1/2}\|\xi^{(1)}\| + C\frac{\Delta t}{h}(\|\xi^n\| + \|\eta^n\|)\|\xi^{(1)}\| \\
&\leq C\|\xi^n\|\cdot\|\xi^{(1)}\| + C\Delta t h^{1/2}\|\xi^{(1)}\|
\end{aligned}$$

Therefore,

$$\|\xi^{(1)}\| \leq C\|\xi^n\| + C\Delta t h^{1/2}.$$

Similarly,

$$\|\xi^{(2)}\| \leq C\|\xi^n\| + C\|\xi^{(1)}\| + C\Delta t h^{1/2} \leq C\|\xi^n\| + C\Delta t h^{1/2}.$$

Overall,

$$\Pi_1 + \Pi_2 \leq C\Delta t h + C\Delta t\|\xi^n\|^2,$$

and

$$3\|\xi^{n+1}\|^2 - 3\|\xi^n\|^2 \leq C\Delta t h + C\Delta t\|\xi^n\|^2$$

i.e.

$$\|\xi^{n+1}\|^2 \leq (1 + C\Delta t)\|\xi^n\|^2 + C\Delta t h \tag{30}$$

and by induction with the initial condition satisfying $||\xi^0|| \leq Ch^{q_{k,\ell}}$,

$$\|\xi^n\| \leq Ch^{1/2}$$

and we are done using the projection property $\|\eta^n\| \leq Ch^{q_{k,\ell}}$, since $q_{k,\ell} = \frac{3}{2}$ in this case.

The final bound is

$$\|u_h^n - u^n\| \leq C\,(h + \frac{h^3}{\Delta t^2})^{1/2}.$$

In the case $h/\Delta t$ is bounded from below, we obtain a bound of order $h^{1/2}$. □

5.2. **Convergence of RKDG scheme in the obstacle case.** Now we turn to scheme (8) for the obstacle equation (1). In particular, the scheme writes: initialize with $u_h^0 := \Pi_h u_0$, and $u_h^{n+1} = \Pi_h(\max(G_{\Delta t}^{RK}(u_h^n), \tilde{g}))$ for $n \geq 0$. The main idea follows closely the proof of Theorem 4.2, but utilizes the estimates in Theorem 5.1.

Theorem 5.2. *Assume that the exact solution is not shattering in the sense of Definition 3.1. Under the same assumption as in Theorem 5.1 (in particular assuming the CFL condition)with both $\tilde{g} = g_{\Delta t}$ and $\tilde{g} = \max\big(g(\cdot), g(\cdot - c\Delta t)\big)$, scheme (8) with RKDG solver $G_{\Delta t}^{RK}$ satisfies*

$$\|u_h^n - u^n\| \leq C\,(h + \frac{h^3}{\Delta t^2})^{1/2}$$

for some constant $C \geq 0$ independent of $h, \Delta t, u_h$.

In particular, if $\Delta t/h$ is bounded from below ($\frac{\Delta t}{h} \geq \bar{C}_0$ for some constant $\bar{C}_0 > 0$), we have

$$\|v_h^n - v^n\| \leq C_1 h^{1/2}.$$

Proof. Similar to the proof of Theorem 4.2, we can obtain

$$\|e^{n+1}\| = \|u_h^{n+1} - u^{n+1}\| \leq \|G_{\Delta t}^{RK}(u_h^n) - u^n(\cdot - c\Delta t)\| + \|\tilde{g} - g_{\Delta t}\| + Ch^{q_{k,\ell}}.$$

We decompose the error

$$u_h^n - u^n = \eta^n - \xi^n,$$

where $\xi^n = \mathbb{P}_h u^n - u_h^n, \quad \eta^n = \mathbb{P}_h u^n - u^n$, and

$$G_{\Delta t}^{RK}(u_h^n) - u^n(\cdot - c\Delta t) = \xi' - \eta'$$

where $\xi' = \mathbb{P}_h u^n(\cdot - c\Delta t) - G_{\Delta t}^{RK}(u_h^n), \quad \eta' = \mathbb{P}_h u^n(\cdot - c\Delta t) - u^n(\cdot - c\Delta t).$

Therefore

$$\begin{aligned} \|e^{n+1}\| &\leq \|\xi'\| + \|\eta'\| + \|\tilde{g} - g_{\Delta t}\| + Ch^{q_{k,\ell}} \\ &\leq \|\xi'\| + \|\tilde{g} - g_{\Delta t}\| + Ch^{q_{k,\ell}} \end{aligned} \tag{31}$$

by using the projection property again. Hence

$$\|e^{n+1}\|^2 \leq (1+\epsilon)\|\xi'\|^2 + (1+\frac{1}{\epsilon})\left(\|\tilde{g} - g_{\Delta t}\| + Ch^{q_{k,\ell}}\right)^2 \tag{32}$$

Using (30) in the proof of Theorem 5.1, we obtain

$$\|\xi'\|^2 \leq (1 + C\Delta t)\|\xi^n\|^2 + C\Delta t h$$

Hence

$$\|e^{n+1}\|^2 \leq (1+\epsilon)\left((1+C\Delta t)\|\xi^n\|^2 + C\Delta t h\right) + (1+\frac{1}{\epsilon})\left(\|\tilde{g} - g_{\Delta t}\| + Ch^{q_{k,\ell}}\right)^2$$

Now we take $\epsilon = C\Delta t$,

$$\begin{aligned} \|e^{n+1}\|^2 &\leq (1+C\Delta t)\|\xi^n\|^2 + C\Delta t h + C\Delta t^{-1}\left(\|\tilde{g} - g_{\Delta t}\| + Ch^{q_{k,\ell}}\right)^2 \\ &\leq (1+C\Delta t)\|\xi^n\|^2 + C\Delta t h + C\Delta t^{-1}h^3 \\ &\leq (1+\epsilon)(1+C\Delta t)\|e^n\|^2 + (1+\frac{1}{\epsilon})(1+C\Delta t)\|\eta^n\|^2 + C\Delta t h + C\Delta t^{-1}h^3 \\ &\leq (1+C\Delta t)\|e^n\|^2 + C\Delta t h + C\Delta t^{-1}h^3 \end{aligned}$$

where in the last line we have taken $\epsilon = C\Delta t$ again. By induction on n, we are done. $\square$

5.3. **Convergence of the RKDG scheme defined with Gauss points.** Now we turn to scheme (10) for the nonlinear equation (1). In particular, the scheme writes: initialize with $u_h^0 := \Pi_h u_0$, and u_h^{n+1} is defined as the unique polynomial in V_h such that :

$$u_h^{n+1}(x_\alpha^j) := \max(G_{\Delta t}^{RK}(u_h^n)(x_\alpha^j),\ \tilde{g}(x_\alpha^j)), \quad \forall j,\ \alpha. \tag{33}$$

Theorem 5.3. *Assume that the exact solution is not shattering in the sense of Definition 3.1. Under the same assumption as in Theorem 5.1 (in particular assuming the CFL condition) with both $\tilde{g} = g_{\Delta t}$ and $\tilde{g}(x) := \max\left(g(x), g(x - c\Delta t)\right)$, scheme* (10) *with RKDG solver $G_{\Delta t}^{RK}$ satisfies*

$$\|u_h^n - u^n\| \leq C\,(h + \frac{h^3}{\Delta t^2})^{1/2}$$

for some constant $C \geq 0$ independent of $h, \Delta t, u_h$.

In particular, if $\Delta t/h$ is bounded from below ($\frac{\Delta t}{h} \geq \bar{C}_0$ for some constant $\bar{C}_0 > 0$), we have

$$\|v_h^n - v^n\| \leq C_1 h^{1/2}.$$

Proof. Following the same lines as the proof for Theorem 4.3, we obtain

$$\begin{aligned}\|e^{n+1}\| = \|u_h^{n+1} - u^{n+1}\| &\leq \|G_{\Delta t}^{RK}(u_h^n) - \Pi_h u^n(\cdot - c\Delta t)\|_{\ell^2} + \|\tilde{g} - g_{\Delta t}\|_{\ell^2} + Ch^{q_{k,\ell}} \\ &= \|G_{\Delta t}^{RK}(u_h^n) - \Pi_h u^n(\cdot - c\Delta t)\| + \|\tilde{g} - g_{\Delta t}\|_{\ell^2} + Ch^{q_{k,\ell}} \\ &\leq \|G_{\Delta t}^{RK}(u_h^n) - u^n(\cdot - c\Delta t)\| + \|\tilde{g} - g_{\Delta t}\|_{\ell^2} + Ch^{q_{k,\ell}}.\end{aligned}$$

By the same argument as in Theorem 5.2

$$\begin{aligned}\|e^{n+1}\|^2 &\leq (1 + C\Delta t)\|\xi^n\|^2 + C\Delta t h + C\Delta t^{-1}\left(\|\tilde{g} - g_{\Delta t}\|_{\ell^2} + Ch^{q_{k,\ell}}\right)^2 \\ &\leq (1 + C\Delta t)\|\xi^n\|^2 + C\Delta t h + C\Delta t^{-1}h^3 \\ &\leq (1 + C\Delta t)\|e^n\|^2 + C\Delta t h + C\Delta t^{-1}h^3,\end{aligned}$$

and we are done. □

6. Conclusion

In this paper, we prove convergence of the SLDG and RKDG methods for the obstacle problem under the "no shattering" assumption of the exact solution. We utilize the DPP of the obstacle solutions. The proof of the SLDG methods relies on the property of the L^2 projection, while new techniques of devising piecewise intermediate stage functions are developed for the convergence of the RKDG methods.

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Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie 75252 Paris Cedex 05 France
UFR de Mathématiques, Site Chevaleret, Université Paris-Diderot, 75205 Paris Cedex France
E-mail address: boka@math.jussieu.fr

Department of Mathematics, Michigan State University, East Lansing, MI, 48824 USA
E-mail address: ycheng@math.msu.edu

Division of Applied Mathematics, Brown University, Providence, RI 02912, USA
E-mail address: shu@dam.brown.edu