J. Math. Anal. Appl. 344 (2008) 981–998

Contents lists available at ScienceDirect

J. Math. Anal. Appl.

www.elsevier.com/locate/jmaa

Orbital stability of standing waves for semilinear wave equations with indefinite energy

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A R T I C L E I N F O

Article history:
Received 25 October 2007
Available online 29 March 2008
Submitted by P.J. McKenna

Keywords:
Nonlinear Klein–Gordon equation
Indefinite energy
Standing waves
Orbital stability

A B S T R A C T

The orbital stability of standing waves for semilinear wave equations is studied in the case that the energy is indefinite and the underlying space domain is bounded or a compact manifold or whole $\mathbf{R}^n$ with $n \geqslant 2$. The stability is determined by the convexity on ω of the lowest energy $d(\omega)$ of standing waves with frequency ω. The arguments rely on the conservation of energy and charge and the construction of suitable invariant manifolds of solution flows.

1. Introduction

In this paper, we consider the nonlinear Klein–Gordon equation (NLKG)

$$\begin{cases} u_{tt} - \Delta u + m(x)u + f(x,u) = 0 \quad \text{in } \mathbf{R}^+ \times \Omega, \\ u = 0 \quad \text{on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset, \\ u(0,x) = U_1(x), \qquad u_t(0,x) = U_2(x), \end{cases} \tag{1.1}$$

where m is a real function representing the potential, f is the nonlinear interaction force and is assumed to satisfy $f(x,u) = g(x,|u|)u$ for some real function g, and Ω denotes $\mathbf{R}^n$ or a bounded domain or a compact manifold.

Due to the gauge invariance $f(x, e^{i\theta}u) = e^{i\theta} f(x,u)$, we can look for the so-called standing wave solutions of (NLKG) of the form $u(t,x) = e^{i\omega t}\phi(x)$ with appropriate initial conditions in (1.1), where ω is a real number called frequency.

The search for the standing waves of (NLKG) equation (1.1) leads to the following nonlinear elliptic equation

$$\begin{cases} -\Delta\phi + \big(m(x) - \omega^2\big)\phi + f(x,\phi) = 0, \\ \phi = 0 \quad \text{on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset. \end{cases} \tag{1.2}$$

It is easy to see that every solution ϕ of Eq. (1.2) is a critical point of energy functional

$$J_\omega(\phi) = \frac{1}{2}\int \big(|\nabla\phi|^2 + \big(m(x) - \omega^2\big)|\phi|^2\big) + \int F\big(x,|\phi|\big) \tag{1.3}$$

and satisfies the functional identity

$$K_\omega(\phi) \equiv \int \big(|\nabla\phi|^2 + \big(m(x) - \omega^2\big)|\phi|^2\big) + \int |\phi| f\big(x,|\phi|\big) = 0, \tag{1.4}$$

where $F(x,s) = \int_0^s f(x,\tau)\,d\tau$ for all $x \in \Omega$ and all $s \geqslant 0$.

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doi:10.1016/j.jmaa.2008.03.056

A natural attempt to find nontrivial solutions to (1.2) is to solve the minimization problem

$$d(\omega) = \inf_{\phi \in M_\omega} J_\omega(\phi), \tag{1.5}$$

where

$$M_\omega = \{\phi \in H \mid K_\omega(\phi) = 0,\ \phi \neq 0\}, \tag{1.6}$$

and H is a suitable Hilbert space to be specified later.

In general, problem (1.5) may not have a solution and even when it has a solution the solution may not solve the semilinear elliptic equation (1.2). Therefore, we need to introduce some conditions on the nonlinearity $f(x,u)$ and the potential $m(x)$ that ensure the existence of minimizers of problem (1.5) which are also solutions to the elliptic equation (1.2). Throughout this paper, we assume the following conditions:

(m) $m(x)$ is a bounded real smooth function on Ω and the greatest lower bound λ_1 of the spectrum of the operator $-\Delta + m$ is positive;
(f_1) $f \in C^1(\Omega \times \mathbf{R})$ and $f'_u(x,s)$ and $f(x,s) \to 0$ uniformly as $s \to 0$;
(f_2) there exist constants $2 < l < 2n/(n-2)$ and C such that $|f'_u(x,s)| \leqslant Cs^{l-2}$ for large $s > 0$ and for all $x \in \Omega$;
(f_3) $f'_u(x,s) < 0$ and $sf'_u(x,s) \leqslant \theta f(x,s)$ for all $x \in \Omega$ and all $s > 0$, where $\theta > 1$ is a constant.

Remark 1.1. From the assumptions (f_1)–(f_3) the following statements are true:

(1) $F(x,s) = \int_0^s f(x,\tau)\,d\tau < 0$ for all $x \in \Omega$ and $s > 0$;
(2) $sf(x,s) < 0$ for all $x \in \Omega$ and $s > 0$;
(3) for any given $x \in \Omega$, $F(x,s) - \frac{1}{\theta+1} sf(x,s)$ is a nondecreasing function of s on $(0,\infty)$;
(4) the simplest example of such functions f is $f(x,u) = -|u|^{l-2}u$.

Indeed, under these conditions, we shall show in Section 2 that for every $\omega^2 < \lambda_1$, $d(\omega)$ is achieved at some nontrivial ϕ and all minimizers of (1.5) are also solutions to (1.2), which will be called the least energy solutions or ground states of Eq. (1.2). However, uniqueness of the ground states is a much different and difficult problem that will not be discussed in the present paper; see e.g. [3,4,10,12] and [13].

After establishing the existence and compactness of the ground state standing waves (in Theorem 2.5 and Corollary 2.6), we study their stabilities. It should be pointed out that a strong stability (see e.g. [5,9,11,17,18,20,21]) cannot be expected in the sense that $\|U_1 - \phi_{\omega_0}\|_H + \|U_2 - i\omega_0\phi_{\omega_0}\|_2 < \delta$ implies that

$$\big\|u(.,t) - e^{i\omega_0 t}\phi_{\omega_0}\big\|_H + \big\|u_t(.,t) - i\omega_0 e^{i\omega_0 t}\phi_{\omega_0}\big\|_2 < \epsilon$$

for all $t \in [0,\infty)$. In fact, for ω close to ω_0, we see that ϕ_ω and ϕ_{ω_0} will be close to each other. Furthermore, for $e^{i\omega t} = -e^{-i\omega_0 t}$

$$\big\|e^{i\omega t}\phi_\omega - e^{i\omega_0 t}\phi_{\omega_0}\big\|_H = \|\phi_\omega + \phi_{\omega_0}\|_H \geqslant \frac{3}{2}\|\phi_{\omega_0}\|_H.$$

The best stability we can hope for of these solutions is the so-called orbital stability in this case. Let us first give a definition of such a stability that will be used throughout this paper (see e.g. [1,2,8,9,14,15]).

Definition 1.1. Let S be a set in $H \times L^2$. We say that S is orbitally stable under the solution flow of (1.1) if for every given $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for any solution $u(t)$ of (1.1)

$$\inf_{(\phi_1,\phi_2)\in S}\big(\|U_1 - \phi_1\|_H + \|U_2 - \phi_2\|_2\big) \leqslant \delta$$

implies

$$\inf_{(\phi_1,\phi_2)\in S}\big(\big\|u(t) - \phi_1\big\|_H + \big\|u_t(t) - \phi_2\big\|_2\big) \leqslant \epsilon \quad \text{for all } t \geqslant 0.$$

In this paper we are concerned with the orbital stability and instability of the set

$$S_\omega = \{(\phi, i\omega\phi) \in H \times L^2 \mid \phi \in G_\omega\},$$

where $G_\omega = \{\phi \in H \mid J_\omega(\phi) = d(\omega),\ \phi \in M_\omega\}$ is usually called the set of ground states with frequency ω. The set S_ω is usually called the ground state orbit with frequency ω.

The main results of this paper are the following two theorems.

Theorem 1.1. *Suppose there exists a C^1-curve $\omega \mapsto \phi_\omega$ in G_ω near $\omega = \omega_0$ and suppose $0 < |\omega_0|^2 < \lambda_1$. If $d(\omega)$ is strictly convex at $\omega = \omega_0$, then S_{ω_0} is orbitally stable.*

Remark 1.2. If S_{ω_0} consists only of finitely many orbits, then each individual orbit $S_{\omega_0}(\phi) = \{(e^{i\theta}\phi, i\omega_0 e^{i\theta_0}\phi) \mid \theta \in [0, 2\pi)\}$ generated by a $\phi \in G_{\omega_0}$ is orbitally stable.

Theorem 1.2. *Let there exist a C^2-curve $\omega \mapsto \phi_\omega$ in G_ω near $\omega = \omega_0$ with $0 < |\omega_0|^2 < \lambda_1$. If ϕ_{ω_0} is an isolated point in G_{ω_0} and $d''(\omega_0) < 0$, then $S_{\omega_0}(\phi_{\omega_0})$ is not orbitally stable under the regular solution flow of* (1.1).

As applications of our main results, we obtain the existence of orbitally stable standing waves for some special (NLKG) equations in Section 6. For example, we prove that under the conditions (m) and (f_1)–(f_3) given above the (NLKG) equation (1.1) has orbitally stable standing waves for some frequency ω with $0 < \omega^2 < \lambda_1$ provided that $\lambda_1 > 0$ (defined in condition (m)) is also an eigenvalue; see Theorem 6.1.

Our second example deals with the special (NLKG) equation

$$u_{tt} - \Delta u + u - |u|^{p-1}u = 0 \quad \text{in } \mathbf{R}^n \times \mathbf{R}.$$

We prove that this equation always has orbitally stable ground state standing waves if $1 < p < 1 + 4/n$ (Theorem 6.2).

It should be pointed out that although our main results are similar to those in [15] and [16], our results apply to both $\Omega = \mathbf{R}^n$ and $\Omega =$ compact manifold or bounded domain, and our method allows nonlinearity f to depend on space variable x as well as on u. In the case of bounded domains, the dilation arguments in [15] and [16] cannot be used here. In addition, our results in this paper hold for all dimensions $\geqslant 2$, while those in [15] and [16] are restricted to dimensions $\geqslant 3$.

2. Existence of the least energy solutions

In this section, we shall prove the existence of a positive least energy solution to (1.2). First of all, we define Hilbert spaces H and L^2. Let

$$H = \begin{cases} H_r^1(\mathbf{R}^n) & \text{if } \Omega = \mathbf{R}^n, \\ H_0^1(\Omega) & \text{if } \Omega \text{ is a bounded domain}, \\ H^1(\Omega) & \text{if } \Omega \text{ is a compact manifold}, \end{cases}$$

$$L^2 = \begin{cases} L_r^2(\mathbf{R}^n) & \text{if } \Omega = \mathbf{R}^n, \\ L^2(\Omega) & \text{if } \Omega \text{ is a bounded domain or a compact manifold}, \end{cases}$$

where subscript r indicates that the space consists of only radially symmetric functions. When $\Omega = \mathbf{R}^n$, we shall also assume that $m(x) = m(|x|)$ and $f(x,u) = f(|x|, u)$.

Besides functionals J_ω and K_ω introduced above, we also consider the functional

$$I_\omega(\phi) = J_\omega(\phi) - \frac{1}{\theta+1}K_\omega(\phi) = \frac{\theta-1}{2(\theta+1)}\int \big(|\nabla\phi|^2 + \big(m(x) - \omega^2\big)|\phi|^2\big) + \int \bigg(F\big(x, |\phi|\big) - \frac{1}{\theta+1}|\phi| f\big(x, |\phi|\big)\bigg),$$

and set

$$M_\omega^- = \big\{\phi \in H \;\big|\; K_\omega(\phi) \leqslant 0,\ \phi \neq 0\big\}.$$

Next we establish several lemmas to lay foundation for existence of the ground states. The first lemma is about the equivalence between H-norm and quadratic part of I_ω.

Lemma 2.1. *Let $\mu < \lambda_1$, define*

$$B(\mu) = \inf\bigg\{\int \big(|\nabla v|^2 + \big(m(x) - \mu\big)|v|^2\big) \;\Big|\; \|v\|_H = 1\bigg\},$$

then $B(\mu)$ is a positive decreasing function of μ.

Proof. $B(\mu)$ is a decreasing function since the integral is a decreasing function of μ.

We prove the positivity of $B(\mu)$ by contradiction. For $\mu < \lambda_1$, suppose that there exists a sequence $\{v_k\}$ such that

$$\|v_k\|_H^2 = \int \big(|\nabla v_k|^2 + |v_k|^2\big) = 1, \tag{2.1}$$

$$(\lambda_1 - \mu)\int |v_k|^2 \leqslant \int \big(|\nabla v_k|^2 + \big(m(x) - \mu\big)|v_k|^2\big) \to 0 \tag{2.2}$$

as $k \to \infty$. From (2.2), $\|v_k\|_2 \to 0$ as $k \to \infty$. By boundedness of m and the second part of (2.2), we obtain

$$\int |\nabla v_k|^2 \to 0 \quad \text{as } k \to \infty. \tag{2.3}$$

Therefore, the combination of (2.3) and $\|v_k\|_2 \to 0$ yields a contradiction to (2.1). $\square$

From the hypotheses (f_1)–(f_3), we have the following growth estimate.

Lemma 2.2. *For any fixed $\phi \in H$, $\phi \geqslant 0$ and $\phi \neq 0$, let*

$$G(s) = \frac{1}{s} \int_\Omega \phi(x) f\big(x, s\phi(x)\big)\, dx.$$

Then

$$G'(s) \leqslant \frac{\theta - 1}{s} G(s), \quad s > 0; \qquad G(s) \leqslant G(1)s^{\theta-1}, \quad s \geqslant 1.$$

Proof. Note that by Remark 1.1 $G(s) < 0$ for $s > 0$ and

$$G'(s) = \frac{1}{s^2} \int_\Omega \big[s\phi^2 f'_u(x, s\phi) - \phi f(x, s\phi)\big]\, dx \leqslant \frac{1}{s^2} \int (\theta - 1)\phi f(x, s\phi) = \frac{\theta - 1}{s} G(s).$$

Therefore we have, using $G(s) < 0$ for $s > 0$,

$$\frac{G'(s)}{G(s)} \geqslant \frac{\theta - 1}{s},$$

from which it follows that $G(s) \leqslant G(1)s^{\theta-1}$ for $s \geqslant 1$. □

It is useful later to relate the minimization problem (1.5) to an equivalent problem.

Proposition 2.3. *For any $\omega^2 < \lambda_1$, M_ω and M_ω^- are nonempty, and*

$$d(\omega) = \inf_{\phi \in M_\omega^-} I_\omega(\phi).$$

Furthermore, $I_\omega(\phi) > d(\omega)$ if $K_\omega(\phi) < 0$.

Proof. First let us show that M_ω^- is nonempty. Choose any $\phi \in H$ with $\phi \not\equiv 0$ and consider

$$g(s) = K_\omega(s\phi) = \frac{s^2}{2} \int \big(|\nabla\phi|^2 + \big(m - \omega^2\big)|\phi|^2\big)\, dx + s^2 G(s),$$

where $G(s)$ is as defined in Lemma 2.2. Thus we see that $g(s) < 0$ for sufficiently large $s > 1$ and hence $s\phi \in M_\omega^-$.

Next, to prove that M_ω is nonempty, we choose $v \in H$ such that $K_\omega(v) < 0$ and consider $K_\omega(sv)$. Now for $s = 1$, $K_\omega(v) < 0$; for $s > 0$ close to zero, $K_\omega(sv) > 0$ from the expression for K. Therefore there exists an $\alpha_0 \in (0, 1)$ such that $K_\omega(\alpha_0 v) = 0$, i.e., $\alpha_0 v \in M_\omega$.

Finally, by Remark 1.1, $I_\omega(sv)$ is an increasing function of $s \in (0, \infty)$, which yields

$$d(\omega) \leqslant I_\omega(\alpha_0 v) < I_\omega(v).$$

Hence

$$d(\omega) \leqslant \inf_{\phi \in M_\omega^-} I_\omega(\phi).$$

But by definition

$$d(\omega) = \inf_{\phi \in M_\omega} J_\omega(\phi) = \inf_{\phi \in M_\omega} I_\omega(\phi) \geqslant \inf_{\phi \in M_\omega^-} I_\omega(\phi),$$

which concludes our proof. □

Lemma 2.4. *For $\omega^2 < \lambda_1$, M_ω is a C^1-hypersurface in H, and both M_ω and M_ω^- are bounded away from zero.*

Proof. Hypotheses (f_1)–(f_3) imply that K_ω is a C^1-functional in H, which in turn implies that M_ω is C^1 hypersurface.

For any small $\epsilon > 0$, from (f_1) and (f_2) there exists a $C(\epsilon) > 0$ such that

$$|\phi| f\big(x, |\phi|\big) \geqslant -\epsilon|\phi|^2 - C(\epsilon)|\phi|^l. \tag{2.4}$$

Using (2.4), Lemma 2.1 and the Sobolev embedding theorem, we have that for $\epsilon < \lambda_1 - \omega^2$

$$\begin{aligned}K_\omega(\phi) &\geqslant \frac{1}{2}\int \big(|\nabla\phi|^2 + \big(m-\omega^2-\epsilon\big)|\phi|^2\big) - C(\epsilon)\int|\phi|^l \\ &\geqslant \frac{B(\omega^2+\epsilon)}{2}\|\phi\|_H^2 - C(\epsilon)\int|\phi|^l \\ &\geqslant C_1\|\phi\|_H^2 - C_2\|\phi\|_H^l,\end{aligned}$$

which implies that M_ω and M_ω^- are bounded away from zero. □

Remark 2.1. For $\Omega = \mathbf{R}^n$, if the potential m and nonlinear interaction force f are independent of space variable x, then any nontrivial solution $v \in H$ to (1.2) also lies on another C^1 hypersurface

$$\tilde{M}_\omega = \big\{\phi \in H \;\big|\; \tilde{K}_\omega(\phi) = 0,\ \phi \neq 0\big\},$$

where

$$\tilde{K}_\omega(u) = \frac{n-2}{2}\int|\nabla u|^2 + n\int\bigg[\frac{1}{2}\big(m-\omega^2\big)|u|^2 + F(u)\bigg].$$

Proof. To see this, we use the scaling property of functions in $H_r^1(\mathbf{R}^n)$. Let $u \in H_r^1(\mathbf{R}^n)$ be a solution of (1.2). Put $u_\mu(x) = u(x/\mu)$, then

$$J_\omega(u_\mu) = \frac{1}{2}\int|\nabla u_\mu|^2 + \frac{1}{2}\int\big(m-\omega^2\big)|u_\mu|^2 + \int F(u_\mu) = \frac{\mu^{n-2}}{2}\int|\nabla u|^2 + \frac{\mu^n}{2}\int\big(m-\omega^2\big)|u|^2 + \mu^n\int F(u).$$

Since u is a solution, $d(J_\omega(u_\mu))/d\mu = 0$ at $\mu = 1$. An easy computation shows that

$$\frac{d(J_\omega(u_\mu))}{d\mu}\bigg|_{\mu=1} = \frac{n-2}{2}\int|\nabla u|^2 + n\int\bigg[\frac{1}{2}\big(m-\omega^2\big)|u|^2 + F(u)\bigg].$$

Note that for $n = 1$ and $n = 2$, $\tilde{M}_\omega$ is not bounded away from zero, and the minimization problem

$$\min\{J_\omega \mid \phi \in \tilde{M}_\omega\}$$

has no solution. This is why the dimension must be restricted to $n \geqslant 3$ in [15] and [16] by using $\tilde{K}_\omega$ instead of K_ω. □

Now we are ready to present our existence theorem for ground states.

Theorem 2.5. *Let $\omega^2 \in (0, \lambda_1)$. Then*

(1) *$d(\omega)$ is positive.*
(2) *Every minimizing sequence of problem* (1.5) *possesses a convergent subsequence. In particular, $d(\omega)$ is attained at some $\phi_\omega \in M_\omega$.*
(3) *This minimizer ϕ_ω can be chosen positive.*
(4) *Every minimizer of problem* (1.5) *is a solution of Eq.* (1.2).

Proof. Let $\{\phi_k\}$ be any minimizing sequence in M_ω for problem (1.5). Remark 1.1 and Lemma 2.1 imply that there exists a constant $C(\omega, \theta) > 0$ such that for all $\phi \in M_\omega$

$$C(\omega,\theta)\|\phi\|_H^2 \leqslant \frac{\theta-1}{2(\theta+1)}\int|\nabla\phi|^2 + \big(m-\omega^2\big)|\phi|^2 \leqslant I_\omega(\phi) = J_\omega(\phi), \tag{2.5}$$

which implies that ϕ_k is bounded in H. Thus by the Sobolev embedding theorem (if $\Omega = \mathbf{R}^n$ we need corresponding embedding theorem developed in [19]), there exist a $\phi_0 \in H$ and a subsequence, still denoted by $\{\phi_k\}$, such that

$$\begin{aligned}&\phi_k \rightharpoonup \phi_0 \quad \text{weakly in } H,\\ &\phi_k \to \phi_0 \quad \text{strongly in } L^p(\Omega),\\ &\phi_k \to \phi_0 \quad \text{a.e. on } \Omega,\end{aligned}$$

where $2 < p < 2n/(n-2)$ when $\Omega = \mathbf{R}^n$, and $1 < p < 2n/(n-2)$ otherwise.

Next we want to get a stronger convergence of sequence $\{\phi_k\}$. To that end, let $0 < \sigma = \frac{1}{2}(\lambda_1 - \omega^2)$ and rewrite $K_\omega(\phi)$ as

$$K_\omega(\phi) = S(\phi) + P(\phi)$$

with

$$S(\phi) = \int|\nabla\phi|^2 + \int\big(m-\omega^2-\sigma\big)|\phi|^2,$$

and

$$P(\phi)=\sigma\int|\phi|^2+\int|\phi|f\big(x,|\phi|\big).$$

By Lemma 2.1, $\sqrt{S(\phi)}$ is an equivalent norm to $\|\phi\|_H$. We see that $\phi_k\rightharpoonup\phi_0$ weakly under the new norm $\sqrt{S(\phi)}$. By weak lower semicontinuity of the norm $\sqrt{S(\cdot)}$, we have

$$\liminf_{k\to\infty}\int|\nabla\phi_k|^2+\int\big(m-\omega^2-\sigma\big)|\phi_k|^2\geqslant\int|\nabla\phi_0|^2+\int\big(m-\omega^2-\sigma\big)|\phi_0|^2. \tag{2.6}$$

Without loss of generality, we may assume the existences of $\lim\int|\phi_k|^2$ and $\lim P(\phi_k)$. We choose $0<\epsilon<\sigma$ in (2.4) and obtain that for some positive constants C_1 and C_2

$$\sigma|\phi|^2+|\phi|f\big(x,|\phi|\big)\geqslant C_1|\phi|^2-C_2|\phi|^l,$$

which implies that by Fatou's lemma

$$\lim_{k\to\infty}\left[P(\phi_k)+C_2\int|\phi_k|^l\right]\geqslant P(\phi_0)+C_2\int|\phi_0|^l.$$

Since $\phi_k\to\phi_0$ strongly in L^p for $2<p<2n/(n-2)$, we obtain

$$\lim_{k\to\infty}P(\phi_k)\geqslant P(\phi_0). \tag{2.7}$$

This combined with (2.6) and Remark 1.1 yields that

$$I_\omega(\phi_0)\leqslant\liminf_{k\to\infty}I_\omega(\phi_k)=d(\omega), \tag{2.8}$$

$$K_\omega(\phi_0)\leqslant\liminf_{k\to\infty}K_\omega(\phi_k)=0. \tag{2.9}$$

A strict inequality in (2.6) would imply strict inequalities in both (2.8) and (2.9), which in turn would imply $\phi_0\neq0$ and by Proposition 2.3 a strict inequality in (2.8) would give us a contradiction

$$d(\omega)<I_\omega(\phi_0)<d(\omega).$$

Therefore (2.6) must be an equality, which implies that $\phi_k\to\phi_0$ under the equivalent norm $\sqrt{S(\phi)}$. Hence $\phi_k\to\phi_0$ in H and $\phi_0\neq0$.

Lemma 2.4 and (2.5) show that $\phi_0\neq0$ and $d(\omega)>0$. Let $\phi_\omega=|\phi_0|\in H$. Then $\phi_w\geqslant0$, $K_\omega(\phi_\omega)=K_\omega(\phi_0)=0$ and $J_\omega(\phi_\omega)=J_\omega(\phi_0)=d(\omega)$.

Finally we show that ϕ_ω is a positive solution of Eq. (1.2). It is known from the Lagrange multiplier that $\delta J_\omega(\phi_\omega)=\lambda\delta K_\omega(\phi_\omega)$, or

$$-\Delta\phi_\omega+\big(m(x)-\omega^2\big)\phi_\omega+f(x,\phi_\omega)=\lambda\big[-2\Delta\phi_\omega+2\big(m-\omega^2\big)\phi_\omega+f(x,\phi_\omega)+\phi_\omega f_u'\big(x,|\phi_\omega|\big)\big].$$

Taking inner product with ϕ_ω on both sides and using $K_\omega(\phi_\omega)=0$ lead to

$$0=\lambda\int\big[\phi_\omega^2f_u'(x,\phi_\omega)-\phi_\omega f(x,\phi_\omega)\big]\,dx\leqslant\lambda(\theta-1)\int\phi_\omega f(x,\phi_\omega)\,dx,$$

which implies that $\lambda=0$ since $\int\phi_\omega f(x,\phi_\omega)\,dx<0$, i.e., ϕ_ω is a solution of (1.2). The positivity of ϕ_ω follows from the strong maximum principle. □

From the proof of Theorem 2.5, we see that

Corollary 2.6. *Every minimizing sequence of the minimization problem*

$$\inf_{\phi\in M_\omega^-}I_\omega(\phi) \tag{2.10}$$

has a subsequence converging to a $\phi_\omega\in M_\omega$. In particular, ϕ_ω is also a minimizer of (1.5).

3. Standing wave as a function of frequency

In this section, we prove that standing waves are smooth functions of frequency if some additional conditions are assumed.

Lemma 3.1. *$d(\omega)$ and $\|\phi_\omega\|_H$ are uniformly bounded for ω^2 on compact subsets of* $(0,\lambda_1)$.

Proof. The uniform boundedness of $d(\omega)$ follows from the fact that given $\omega_0 \in (0, \lambda_1)$, there exists $\phi_0 \in H$ such that $K_{\omega_0}(\phi_0) < 0$ (see the proof of Proposition 2.3); hence there exists an $\epsilon > 0$ such that

$$K_\omega(\phi_0) < 0 \quad \text{for } \omega \in (\omega_0 - \epsilon, \omega_0 + \epsilon),$$

from which and Proposition 2.3 it follows that

$$d(\omega) \leqslant I_\omega(\phi_0) \leqslant C \quad \text{for } \omega \in (\omega_0 - \epsilon, \omega_0 + \epsilon).$$

By Remark 1.1 and Lemma 2.1, we have

$$d(\omega) = J_\omega(\phi_\omega) = I_\omega(\phi_\omega) \geqslant \frac{\theta - 1}{2(\theta + 1)} \int \left[|\nabla \phi_\omega|^2 + \left(m - \omega^2\right)|\phi_\omega|^2 \right] \geqslant \frac{B(\omega^2)(\theta - 1)}{2(\theta + 1)} \|\phi_\omega\|_H^2, \tag{3.1}$$

which implies the uniform boundedness of $\|\phi_\omega\|_H$. □

Lemma 3.2. *$d(\omega)$ is a decreasing and continuous function of ω for $\omega \in (0, \sqrt{\lambda_1})$.*

Proof. Let $0 < \omega_1 < \omega_2 < \sqrt{\lambda_1}$ and $d(\omega_1) = J_{\omega_1}(\phi_{\omega_1})$, then

$$K_{\omega_2}(\phi_{\omega_1}) = K_{\omega_1}(\phi_{\omega_1}) - \frac{1}{2}\left(\omega_2^2 - \omega_1^2\right) \int |\phi_{\omega_1}|^2 < 0.$$

Therefore by Proposition 2.3 we have

$$d(\omega_2) \leqslant I_{\omega_2}(\phi_{\omega_1}) < I_{\omega_1}(\phi_{\omega_1}) = d(\omega_1).$$

This concludes the proof for monotonicity of d. To show the continuity of $d(\omega)$ at any $\omega_0 \in (0, \sqrt{\lambda_1})$, we show d is left and right continuous at ω_0.

For the left continuity, let $0 < \omega < \omega_0$ and $A(\omega, s) \equiv \frac{1}{s^2} K_\omega(s\phi_{\omega_0})$. Then $A(\omega, s)$ is a smooth function of $s \in (0, \infty)$ and ω. Moreover, we have $A(\omega_0, 1) = 0$ and $A'_s(\omega_0, 1) < 0$ from Lemma 2.2. Therefore, by implicit function theorem, there exist a neighborhood of ω_0 and a C^1 function $\alpha = \alpha(\omega)$ in this neighborhood such that $\alpha(\omega_0) = 1$ and $A(\omega, \alpha(\omega)) = 0$. Hence $\alpha(\omega)\phi_{\omega_0} \in M_\omega$ and we thus have

$$d(\omega_0) \leqslant d(\omega) \leqslant I_\omega\big(\alpha(\omega)\phi_{\omega_0}\big) = \alpha^2(\omega) \frac{\theta - 1}{2(\theta + 1)} \left(\omega_0^2 - \omega^2\right) \int |\phi_{\omega_0}|^2 + I_{\omega_0}\big(\alpha(\omega)\phi_{\omega_0}\big).$$

Let $\omega \to \omega_0^-$, then $\alpha(\omega) \to 1$ and $I_{\omega_0}(\alpha(\omega)\phi_{\omega_0}) \to d(\omega_0)$. Hence $d(\omega) \to d(\omega_0)$ as $\omega \to \omega_0^-$, which concludes the proof of left continuity.

To show that $\lim_{\omega \to \omega_0^+} d(\omega) = d(\omega_0)$, it suffices to find a function $\alpha(\omega) \geqslant 1$ such that

$$K_{\omega_0}\big(\alpha(\omega)\phi_\omega\big) = 0 \quad \text{and } \alpha(\omega) \to 1 \text{ as } \omega \to \omega_0^+. \tag{3.2}$$

In fact, for such a $\alpha(\omega)$, from the definition of $d(\omega)$,

$$\begin{aligned} d(\omega_0) = J_{\omega_0}(\phi_{\omega_0}) &\leqslant J_{\omega_0}\big(\alpha(\omega)\phi_\omega\big) = J_\omega\big(\alpha(\omega)\phi_\omega\big) + \frac{\alpha^2(\omega)}{2}\left(\omega^2 - \omega_0^2\right) \int \phi_\omega^2 \\ &= d(\omega) + \left[J_\omega\big(\alpha(\omega)\phi_\omega\big) - J_\omega(\phi_\omega)\right] + \frac{\alpha^2(\omega)}{2}\left(\omega^2 - \omega_0^2\right) \int \phi_\omega^2. \end{aligned}$$

Note also that $J_\omega(t\phi) - J_\omega(\phi) \to 0$ as $t \to 1^+$ uniformly for ω in a compact set and $\phi \geqslant 0$ in a bounded set of H. Hence $d(\omega) \to d(\omega_0)$ as $\omega \to \omega_0^+$ if $\alpha(\omega) \to 1$.

We now come back to find $\alpha(\omega)$ satisfying (3.2). Choose an arbitrary $\omega_2 \in [\omega_0, \sqrt{\lambda_1})$ and consider function

$$B(\omega, s) = \frac{K_{\omega_0}(s\phi_\omega)}{s^2} = \int |\nabla \phi_\omega|^2 + \left(m - \omega_0^2\right)\phi_\omega^2 + G(\omega, s),$$

where $\omega \in [\omega_0, \omega_2]$ and

$$G(\omega, s) = \frac{1}{s} \int \phi_\omega f(x, s\phi_\omega)\, dx$$

is the function studied in Lemma 2.2. It is easy to see that $B(\omega, 1) = (\omega^2 - \omega_0^2) \int \phi_\omega^2 > 0$ if $\omega > \omega_0$. Lemma 2.2 tells us that $G(\omega, s)$ is monotonically decreasing as a function of s and $G(\omega, s) \leqslant G(\omega, 1)s^{\theta - 1} \to -\infty$ as $s \to \infty$. Therefore there exists a unique $\alpha(\omega) \geqslant 1$ such that

$$\big(\alpha(\omega)\big)^2 B\big(\omega, \alpha(\omega)\big) = K_{\omega_0}\big(\alpha(\omega)\phi_\omega\big) = 0.$$

Furthermore, note that

$$G(\omega,1)=\int \phi_\omega f(\cdot,\phi_\omega)=-\int \big[|\nabla\phi_\omega|^2+\big(m-\omega^2\big)\phi_\omega^2\big]\leqslant -2d(\omega)\leqslant -2d(\omega_2),$$

hence we have

$$0=B\big(\omega,\alpha(\omega)\big)=B\big(\omega,\alpha(\omega)\big)-B(\omega,1)+B(\omega,1)\leqslant G(\omega,1)\big[\big(\alpha(\omega)\big)^{\theta-1}-1\big]+B(\omega,1)$$
$$\leqslant -2d(\omega_2)\big[\big(\alpha(\omega)\big)^{\theta-1}-1\big]+B(\omega,1),$$

which implies that

$$1\leqslant \big(\alpha(\omega)\big)^{\theta-1}\leqslant 1+\frac{B(\omega,1)}{2d(\omega_2)}\to 1,$$

hence $\alpha(\omega)\to 1$ as $\omega\to\omega_0^+$, as desired. □

The following result gives the derivative of $d(\omega)$ in the case the curve $\omega\mapsto\phi_\omega$ is smooth.

Proposition 3.3. *Assume that $\omega\mapsto\phi_\omega$ is a C^1 curve in H. Then we have*

$$d'(\omega)=-\omega\int |\phi_\omega|^2.$$

Proof. From

$$d(\omega)=J_\omega(\phi_\omega)=\frac{1}{2}\int \big(|\nabla\phi_\omega|^2+\big(m-\omega^2\big)|\phi_\omega|^2\big)+\int F(x,\phi_\omega),$$

we have

$$d'(\omega)=\int \big(-\Delta\phi_\omega+\big(m-\omega^2\big)\phi_\omega+f(x,\phi_\omega)\big)\frac{\partial\phi_\omega}{\partial\omega}-\omega\int |\phi_\omega|^2. \tag{3.3}$$

The first integral in (3.3) is zero since ϕ_ω is a solution of Eq. (1.2). □

We now give a sufficient condition for the smooth dependence of ground states ϕ_ω on ω. In addition to the previous structural conditions, we also assume the following condition.

Assumption 3.4. For ω near ω_0, assume that ϕ_ω is the unique positive solution of problem (1.5). Also assume that zero is not in the spectrum of the linearized operator $\mathcal{L}_0=-\Delta+m-\omega_0^2+f_u'(\cdot,\phi_{\omega_0})$ at ϕ_{ω_0} acting on L^2 (real valued).

We now establish following two lemmas using the same procedures as in [16, Theorem 18].

Lemma 3.5. *Suppose that Assumption* 3.4 *holds. Then $\omega\mapsto\phi_\omega$ is continuous with values in H.*

Proof. From Lemmas 3.1 and 3.2, $d(\omega)=I_\omega(\phi_\omega)$ is continuous in ω and $\|\phi_\omega\|_H$ is a bounded function of ω. Let $\{\omega_k\}$ be a sequence tending to ω_0. Then $\{\phi_{\omega_k}\}$ is bounded in H. A subsequence may be chosen converging weakly in H to some v. Note $v\geqslant 0$ since each ϕ_{ω_k} is positive and $\phi_{\omega_k}\to v$ a.e. on Ω. From $0=K_\omega(\phi_\omega)$ and lower semicontinuity of the norm we have

$$K_{\omega_0}(v)\leqslant 0,\qquad I_{\omega_0}(v)\leqslant \liminf_{k\to\infty} I_{\omega_k}(\phi_{\omega_k})=d(\omega_0).$$

By similar arguments in the proof of Theorem 2.5, we have $K_{\omega_0}(v)=0$, $I_{\omega_0}(v)=d(\omega_0)$ and $\phi_{\omega_k}\to v$ strongly in H. Then by uniqueness, $v=\phi_{\omega_0}$, which completes the proof. □

Lemma 3.6. *Let f be C^1. Suppose that Assumption* 3.4 *holds. Then $\omega\mapsto\phi_\omega$ is C^1 in H near $\omega=\omega_0$.*

Proof. We write (1.2) as

$$-\Delta\phi+m(x)\phi+\tau\phi+f(x,\phi)=0,$$

where $\tau=-\omega^2$. Let $\tau_0=-\omega_0^2$, $\phi_0=\phi_{\omega_0}$ and let

$$\mathcal{L}(\tau,v)=v+(m-\Delta+\tau)^{-1}f(\cdot,v),\quad \tau>-\lambda_1,\ v\in H.$$

Then $\mathcal{L}(\tau, v) \in H$ since $v \in H \subset L^{2n/(n-2)}$, $f(\cdot, v) \in L^{2n/(n+2)}$ by (f_1) and (f_2), and therefore $(\tau + m - \Delta)^{-1} f(\cdot, v) \in H$ by elliptic theory. In fact, $\mathcal{L}(\tau, v)$ is a C^1 operator from $(-\lambda_1, \infty) \times H$ into H. Note that $\mathcal{L}(\tau_0, \phi_0) = 0$. Now the operator $\mathcal{L}_0 = -\Delta + m - \omega_0^2 + f_u'(\cdot, \phi_{\omega_0})$ is invertible by assumption. It follows that the compact operator $(\tau_0 + m - \Delta)^{-\frac{1}{2}} f_u'(\cdot, \phi_0)(\tau_0 + m - \Delta)^{-\frac{1}{2}}$ on L^2 does not have -1 in its spectrum. Hence

$$\frac{\partial \mathcal{L}}{\partial v}(\tau_0, \phi_0) = I + (\tau_0 + m - \Delta)^{-1} f_u'(\cdot, \phi_0),$$

acting from H to H, is invertible. By implicit function theorem, the solutions of $\mathcal{L}(\tau, v) = 0$ in a neighborhood of (τ_0, ϕ_0) form a C^1 curve in $(-\lambda_1, \infty) \times H$. Thus by uniqueness assumption, $\omega \mapsto \phi_\omega$ is a C^1 curve near $\omega = \omega_0$. □

Remark 3.1. If f is a C^2 function, under the same Assumption 3.4 one can also prove that the curve $\omega \mapsto \phi_\omega$ is a C^2 curve in H near $\omega = \omega_0$. This follows from a regularity argument of elliptic equations; see [16].

4. Stability of standing waves

We consider (NLKG)

$$\begin{cases} u_{tt} - \Delta u + m(x)u + f(x, u) = 0 \quad \text{in } \mathbf{R}^+ \times \Omega, \\ u = 0 \quad \text{on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset, \\ u(0, x) = U_1(x) \in H, \qquad u_t(0, x) = U_2(x) \in L^2. \end{cases} \tag{4.1}$$

For $\Omega = \mathbf{R}^n$ it is shown in [6,7] that strong solutions $u \in C([0, T), H)$, $u_t \in C([0, T), L^2)$ exist for nonlinear interaction $f(x, u)$ satisfying conditions (f_1)–(f_3). For other cases of Ω, it is shown in [19] that weak solutions exist and for these solutions energy inequality holds. In this section, we study stability for the weak solutions of (NLKG); the case for strong solutions is relatively easier.

Let $X = H \times L^2$ and consider the modulated energy functional on X

$$\mathcal{E}_\omega(v_1, v_2) = \frac{1}{2} \int |v_2|^2 + J_\omega(v_1).$$

Define

$$R_\omega = \{(v_1, v_2) \in H \times L^2 \mid \mathcal{E}_\omega(v_1, v_2) < d(\omega)\}.$$

Next we introduce two invariant sets which play an important role in establishing stability

$$R_\omega^1 = \{(v_1, v_2) \in R_\omega \mid K_\omega(v_1) > 0\} \cup \{(0, v_2) \in R_\omega\},$$
$$R_\omega^2 = \{(v_1, v_2) \in R_\omega \mid K_\omega(v_1) < 0\}.$$

It is easy to prove the following equivalent expressions

$$R_\omega^1 = \{(v_1, v_2) \in R_\omega \mid I_\omega(v_1) < d(\omega)\},$$
$$R_\omega^2 = \{(v_1, v_2) \in R_\omega \mid I_\omega(v_1) > d(\omega)\}.$$

Proposition 4.1. *R_ω^1 and R_ω^2 are invariant regions under the solution flow of the following modulated equation*

$$\begin{cases} v_{tt} + 2i\omega v_t - \Delta v + (m(x) - \omega^2)v + f(x, v) = 0 \quad \text{in } \mathbf{R}^+ \times \Omega, \\ v = 0 \quad \text{on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset, \\ v(0, x) = V_1(x) \in H, \qquad v_t(0, x) = V_2(x) \in L^2. \end{cases} \tag{4.2}$$

Proof. Let $(V_1, V_2) \in R_\omega^1$ and assume that there exists a τ such that $(v(\tau), v_t(\tau)) \notin R_\omega^1$. Then $v(\tau) \neq 0$ and $K_\omega(v(\tau)) \leqslant 0$, i.e., $v(\tau) \in M_\omega^-$. Let

$$s = \inf\{0 \leqslant t \leqslant \tau \mid (v(t), v_t(t)) \notin R_\omega^1\}, \tag{4.3}$$

then $K_\omega(u(t)) \geqslant 0$ for all $0 \leqslant t < s$. Let $\{s_k\}$ be the minimizing sequence for problem (4.3), then $v(s_k) \in M_\omega^-$ and, arguing similarly as in the proof of Theorem 2.5, we have

$$K_\omega(v(s)) \leqslant \liminf_{k\to\infty} K_\omega(v(s_k)) \leqslant 0 \quad \text{and} \quad v(s) \neq 0. \tag{4.4}$$

On the other hand

$$I_\omega(v(s)) = \liminf_{t\to s^-} I_\omega(v(t)) \leqslant \liminf_{t\to s^-}\left(I_\omega(v(t)) + \frac{1}{\theta+1} K_\omega(v(t))\right) \leqslant \liminf_{t\to s^-} \mathcal{E}_\omega(v(t), v_t(t)) < d(\omega),$$

which contradicts (4.4) from Proposition 2.3. Therefore R_ω^1 is invariant.

To show the invariance of R_ω^2, we just need to switch the roles of I_ω and K_ω. Let $(V_1, V_2) \in R_\omega^2$ and assume that there exists a τ such that $(v(\tau), v_t(\tau)) \notin R_\omega^2$, i.e., $I_\omega(u(\tau)) \leqslant d(\omega)$. Let

$$s = \inf\{0 \leqslant t \leqslant \tau \mid (v(t), v_t(t)) \notin R_\omega^2\}, \tag{4.5}$$

then $I_\omega(v(s)) \leqslant d(\omega)$ and $I_\omega(v(t)) > d(\omega)$ for all $0 < t < s$. On the other hand,

$$\begin{aligned} K_\omega(v(s)) &= \liminf_{t\to s^-}(\theta+1)\big(J_\omega(v(t)) - I_\omega(v(t))\big) \\ &\leqslant \liminf_{t\to s^-}(\theta+1)\big(\mathcal{E}_\omega(v(t), v_t(t)) - d(\omega)\big) \\ &\leqslant (\theta+1)\big(\mathcal{E}_\omega(V_1, V_2) - d(\omega)\big) < 0, \end{aligned}$$

which contradicts $I_\omega(u(s)) \leqslant d(\omega)$ from Proposition 2.3. □

Lemma 4.2. *For all $\omega^\pm$ near ω_0 such that $\omega^- < \omega_0 < \omega^+$, one has*

$$d(\omega^+) < I_{\omega^\pm}(\phi_{\omega_0}) < d(\omega^-).$$

Proof. Set $a = \frac{\theta-1}{2(\theta+1)} < \frac{1}{2}$. It is obvious from the definition of $I_\omega(u)$ that

$$I_\omega(\phi_{\omega_0}) = d(\omega_0) + a(\omega_0^2 - \omega^2)\int |\phi_{\omega_0}|^2 \equiv h(\omega) + d(\omega). \tag{4.6}$$

Since $a > 0$, by (4.6) and Lemma 3.2,

$$I_{\omega^+}(\phi_{\omega_0}) < d(\omega_0) < d(\omega^-); \qquad I_{\omega^-}(\phi_{\omega_0}) > d(\omega_0) > d(\omega^+).$$

On the other hand, by definition of $h(\omega)$ in (4.6) we have

$$h(\omega_0) = 0, \qquad h'(\omega_0) = (2a-1)d'(\omega_0) = (1-2a)\omega_0\int |\phi_{\omega_0}|^2 > 0,$$

hence $h(\omega^+) > 0 > h(\omega^-)$, which implies

$$d(\omega^+) < I_{\omega^+}(\phi_{\omega_0}); \qquad d(\omega^-) > I_{\omega^-}(\phi_{\omega_0}).$$

We complete the proof. □

The following result gives an important connection between stability of ground state solutions and convexity of $d(\omega)$.

Proposition 4.3. *Assume that $d(\omega)$ is strictly convex near $\omega = \omega_0$. Then there exists an $M(\omega_0) \geqslant 0$ such that for every $M > M(\omega_0)$ there exists a $\delta = \delta(M)$ such that if $u(t)$ is a weak solution of (NLKG) equation (4.1) with initial data satisfying*

$$\|U_1 - \phi_{\omega_0}\|_H + \|U_2 - i\omega_0\phi_{\omega_0}\|_2 < \delta$$

for some $\phi_{\omega_0} \in G_{\omega_0}$, then

$$d(\omega_+) \leqslant I_{\omega_\pm}(u(t)) \leqslant d(\omega_-) \quad \forall t > 0, \tag{4.7}$$

and

$$\frac{1}{2}\int |u_t(t) - i\omega_\pm u(t)|^2 + J_\pm(u(t)) < d(\omega_\pm) \quad \forall t > 0, \tag{4.8}$$

where $\omega_\pm = \omega_0 \pm 1/M$.

Proof. Set $v^\pm(t) = e^{-i\omega_\pm t}u(t)$. Then $v^\pm$ satisfies the modulated equation

$$\begin{cases} v_{tt}^\pm + 2i\omega_\pm v_t^\pm - \Delta v^\pm + (m(x) - \omega_\pm^2)v^\pm + f(x, v^\pm) = 0 & \text{in } \mathbf{R}^+ \times \Omega, \\ v^\pm = 0 \quad \text{on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset, \\ v^\pm(0,x) = U_1(x) \equiv V_1, \qquad v_t^\pm(0,x) = U_2(x) - i\omega_\pm U_1 \equiv V_2. \end{cases} \tag{4.9}$$

Note that

$$J_{\omega_\pm}(u) = J_{\omega_\pm}(v^\pm), \qquad \int |v_t^\pm|^2 = \int |u_t(t) - i\omega_\pm u(t)|^2.$$

The energy inequality of modulated equation (4.9) yields

$$\frac{1}{2}\int |v_t^\pm(t)|^2 + J_{\omega_\pm}(u(t)) \leqslant \mathcal{E}_{\omega_\pm}(U_1, U_2 - i\omega_\pm U_1). \tag{4.10}$$

To show (4.7) and (4.8), by invariance of $R^1_{\omega_\pm}$ and $R^2_{\omega_\pm}$ under solution flow of modulated equation (4.9) (Proposition 4.1) and by energy inequality (4.10), it suffices to prove that

$$d(\omega_+) < I_{\omega_\pm}(U_1) < d(\omega_-), \tag{4.11}$$

and

$$\mathcal{E}_{\omega_\pm}(U_1, U_2 - i\omega_\pm U_1) < d(\omega_\pm). \tag{4.12}$$

Note that $I_{\omega_\pm}(U_1) = I_{\omega_\pm}(\phi_{\omega_0}) + O(\delta)$. Thus, by Lemma 4.2, δ can be chosen so that (4.11) holds.

Now we turn our attention to (4.12). It is easy to see that

$$J_{\omega_\pm}(U_1) = J_{\omega_\pm}(\phi_{\omega_0}) + O(\delta) = J_{\omega_0}(\phi_{\omega_0}) + \frac{\omega_0^2 - \omega_\pm^2}{2}\int |\phi_{\omega_0}|^2 + O(\delta), \tag{4.13}$$

and

$$\|U_2 - i\omega_\pm U_1\|_2 \leqslant \|U_2 - i\omega_0\phi_{\omega_0}\|_2 + \|\omega_0\phi_{\omega_0} - \omega_\pm\phi_{\omega_0}\|_2 + \|\omega_\pm\phi_{\omega_0} - \omega_\pm U_1\|_2 = |\omega_0 - \omega_\pm|\|\phi_{\omega_0}\|_2 + O(\delta). \tag{4.14}$$

Therefore, if δ is chosen sufficiently small,

$$\mathcal{E}_{\omega_\pm}(U_1, U_2 - i\omega_\pm U_1) = \frac{1}{2}\|U_2 - i\omega_\pm U_1\|_2^2 + J_{\omega_\pm}(U_1) \leqslant d(\omega_0) + (\omega_\pm - \omega_0)d'(\omega_0) + O(\delta) < d(\omega_\pm)$$

since $d(\omega)$ is strictly convex near ω_0. Thus (4.12) follows and the proof is completed. □

We are now ready to prove the stability theorem.

Theorem 4.4. *If $d(\omega)$ is strictly convex near $\omega = \omega_0$, then S_{ω_0} is orbitally stable.*

Proof. Suppose S_{ω_0} is not orbitally stable. Then there exist $\{(U_1^k, U_2^k)\}$, $\{t_k\}$, weak solutions $\{u^k(t)\}$ and $\epsilon_0 > 0$ such that

$$\inf_{\phi\in G_{\omega_0}} \left(\left\|U_1^k - \phi\right\|_H + \left\|U_2^k - i\omega_0\phi\right\|_2\right) \to 0, \tag{4.15}$$

and

$$\inf_{\phi\in G_{\omega_0}} \left(\left\|u^k(t_k) - \phi\right\|_H + \left\|u_t^k(t_k) - i\omega_0\phi\right\|_2\right) \geqslant \epsilon_0. \tag{4.16}$$

Since G_{ω_0} is compact in H, without loss of generality, we may assume that

$$\left(U_1^k, U_2^k\right) \to (\phi, i\omega_0\phi) \quad \text{for some } \phi \in G_{\omega_0}.$$

From Proposition 4.3, we may also assume that

$$d(\omega_0 + 1/k) \leqslant I_{\omega_\pm}\left(u^k(t_k)\right) \leqslant d(\omega_0 - 1/k), \tag{4.17}$$

and

$$\left\|u_t^k(t_k) - i\omega_+ u^k(t_k)\right\|_2^2 + J_{\omega_+}\left(u^k(t_k)\right) < d(\omega_0 + 1/k), \tag{4.18}$$

where $\omega_\pm = \omega_0 \pm 1/k$. (4.17) and (2.5) imply that $u^k(t_k)$ is bounded in H, therefore, by continuity of $d(\omega)$ and (4.17),

$$I_{\omega_0}\left(u^k(t_k)\right) \to d(\omega_0). \tag{4.19}$$

It follows from (4.18) that

$$\liminf_{k\to\infty} J_{\omega_0}\left(u^k(t_k)\right) = d \leqslant d(\omega_0) \quad \text{for some } d. \tag{4.20}$$

Hence (4.20) and (4.19) yield

$$\liminf_{k\to\infty} K_{\omega_0}\left(u^k(t_k)\right) = (\theta + 1)\liminf_{k\to\infty}\left(J_{\omega_0}\left(u^k(t_k)\right) - I_{\omega_0}\left(u^k(t_k)\right)\right) \leqslant 0. \tag{4.21}$$

Next we show that there is a subsequence of $\{u^k(t_k)\}$, still denoted by $\{u^k(t_k)\}$ such that $\|u^k(t^k) - \phi_{\omega_0}\|_H \to 0$ for some $\phi_{\omega_0} \in G_{\omega_0}$. If this is done, we see from (4.18) that

$$u_t^k(t_k) \to i\omega_0\phi_{\omega_0} \quad \text{in } L^2,$$

which is a contradiction to (4.16), and thus we complete the proof. To simplify our notation, we let $w_k = u^k(t^k)$. Consider

$$g(s,k) = \frac{1}{s^2}K_{\omega_0}(sw_k) = \int \left(|\nabla w_k|^2 + \left(m - \omega_0^2\right)|w_k|^2\right) + G(s,k),$$

where

$$G(s,k)=\frac{1}{s}\int w_k f(x,sw_k)\,dx$$

is a function we studied in Lemma 2.2, thus $G(s,k)\leqslant G(1,k)s^{\theta-1}$ for all $s\geqslant 1$. Note that

$$G(1,k)=K_{\omega_0}(w_k)-\int\big(|\nabla w_k|^2+\big(m-\omega_0^2\big)|w_k|^2\big)\leqslant K_{\omega_0}(w_k)-M$$

for some positive M independent of k since $\|w_k\|_H\geqslant M$ from $I_{\omega_+}(w_k)\geqslant d(\omega_0)+\frac{1}{k}$. Let us choose s_k such that

$$s_k^{\theta-1}=1+\max\left\{0,\frac{K_{\omega_0}(w_k)}{M-K_{\omega_0}(w_k)}\right\}.$$

Then

$$s_k^{-2}K_{\omega_0}(s_kw_k)=g(s_k,k)=g(s_k,k)-g(1,k)+g(1,k)\leqslant K_{\omega_0}(w_k)+G(1,k)\big(s_k^{\theta-1}-1\big)\leqslant 0,$$

and

$$\lim_{k\to\infty}s_k=1;\qquad I_{\omega_0}(s_kw_k)\to d(\omega_0).$$

That is, $\{s_ku^k(t_k)\}$ is a minimizing sequence of problem (2.10). Therefore, by Corollary 2.6, there exists a subsequence of $\{s_ku^k(t_k)\}$ such that $s_ku^k(t_k)\to\phi_{\omega_0}$ as $k\to\infty$ for a function $\phi_{\omega_0}\in G_{\omega_0}$. Since $s_k\to 1$, we have $u^k(t_k)\to\phi_{\omega_0}$ in H, as desired. □

5. Instability of ground state orbits

In this section we give a condition which ensures the instability of the orbit $S_\omega(\phi_\omega)$ generated by a positive ground state $\phi_\omega\in G_\omega$ found in Section 2. Although ϕ_ω may not be unique, we shall assume in this section that the curve $\omega\mapsto\phi_\omega$ is C^2 in H near a given number $\omega_0\in(0,\sqrt{\lambda_1})$.

It will be helpful to write Eq. (1.1) as a Hamiltonian in a suitable Banach space. To this end, we first introduce some notation.

Let $X=\{\vec{u}=(u_1,u_2)\mid u_1\in H,\ u_2\in L^2\}$ be the usual Banach space with norm $\|\vec{u}\|=\|u_1\|_H+\|u_2\|_2$. Denote by X^* the real dual space of X and $\langle l,\vec{u}\rangle$ the pairing between $l\in X^*$ and $\vec{u}\in X$. Let $P:X\to X^*$ be defined by

$$\langle P\vec{u},\vec{v}\rangle=\mathrm{Re}\int(-u_2\bar{v}_1+u_1\bar{v}_2)\quad\forall\vec{u},\vec{v}\in X.\tag{5.1}$$

We now define some real valued functionals on X. For $\vec{u}=(u_1,u_2)\in X$, set

$$\mathcal{E}(\vec{u})=\frac{1}{2}\int\big(|u_2|^2+|\nabla u_1|^2+m(x)|u_1|^2\big)+\int F\big(x,|u_1|\big),\tag{5.2}$$

$$\mathcal{Q}(\vec{u})=\mathrm{Im}\int u_2\bar{u}_1.\tag{5.3}$$

We also consider the corresponding functionals on H. For $\omega\in\mathbf{R}$, define

$$E_\omega(\phi)=\mathcal{E}(\vec{\phi}_\omega),\quad Q_\omega(\phi)=\mathcal{Q}(\vec{\phi}_\omega),\quad \vec{\phi}_\omega=(\phi,i\omega\phi)\in X\ \forall\phi\in H.$$

Recall

$$d(\omega)=\inf_{K_\omega(\phi)=0,\,\phi\neq0}J_\omega(\phi)=J_\omega(\phi_\omega),\qquad d'(\omega)=-\omega\int|\phi_\omega|^2=-Q_\omega(\phi_\omega).$$

It is easy to see that

Lemma 5.1. *For all $\omega\in\mathbf{R}$ and $\vec{u}=(u_1,u_2)\in X$, one has $\mathcal{E}(\vec{u})\geqslant J_\omega(u_1)+\omega\mathcal{Q}(\vec{u})$.*

Lemma 5.2. *$\mathcal{Q}'(\vec{u})=P(i\vec{u})$ for all $\vec{u}\in X$. Furthermore, for regular solutions, Eq.* (1.1) *is equivalent to the Hamiltonian system*

$$\begin{cases}P\dfrac{d\vec{u}}{dt}=\mathcal{E}'(\vec{u}),\quad t>0,\\ \vec{u}(0)=\vec{u}_0.\end{cases}\tag{5.4}$$

The main result of this section is the following

Theorem 5.3. *Assume that $\omega \mapsto \phi_\omega$ is C^2 near $\omega_0 \in (0, \sqrt{\lambda_1})$, where ϕ_ω is a positive ground state with frequency ω, and $d''(\omega_0) < 0$, then the orbit $S_{\omega_0}(\phi_{\omega_0})$ is not stable under the flow* (5.4)*.*

We shall prove this theorem by constructing a bounded C^1-functional $\mathcal{A}$ defined on a neighborhood of $S_{\omega_0}(\phi_{\omega_0})$ such that a sequence $\vec{u}_0^k$ and a constant $\sigma_0 > 0$ can be chosen so that

$$\operatorname{dist}\big(\vec{u}_0^k; S_{\omega_0}(\phi_{\omega_0})\big) \to 0, \qquad \frac{d\mathcal{A}(\vec{u}^k(t))}{dt} \geqslant \sigma_0 > 0 \tag{5.5}$$

for all t as long as the solution $\vec{u}^k(t)$ to (5.4) with initial data $\vec{u}_0^k$ exists and stays inside that neighborhood. Since we want the orbital instability, we should require $\mathcal{A}(e^{i\theta}\vec{u}) = \mathcal{A}(\vec{u})$. If $\mathcal{A}'(\vec{\phi}_{\omega_0}) = P\vec{y}_0$ for some $\vec{y}_0 \in X$, we have

$$0 = \langle P\vec{y}_0, i\vec{\phi}_{\omega_0}\rangle = \frac{d\mathcal{A}(e^{i\theta}\vec{\phi}_{\omega_0})}{d\theta}\bigg|_{\theta=0}. \tag{5.6}$$

Note that $\mathcal{A}$ should be close to

$$\big\langle P\vec{y}_0, e^{-i\theta(\vec{u})}\vec{u}\big\rangle$$

in a small neighborhood of $S_{\omega_0}(\phi_{\omega_0})$, where $e^{-i\alpha}e^{-i\theta(\vec{u})} = e^{-i\theta(e^{i\alpha}\vec{u})}$ for any $\alpha \in [0, 2\pi)$ and $\theta(\vec{\phi}_{\omega_0}) = 0$. We can choose, without loss of generality, that

$$\mathcal{A}(\vec{u}) = \big\langle P\vec{y}_0, e^{-i\theta(\vec{u})}\vec{u}\big\rangle. \tag{5.7}$$

The problem now is how to choose $\vec{y}_0$, $\theta(\vec{u})$ and $\vec{u}_0^k$ such that (5.5) is satisfied. The choice of θ is relatively easy, and we will carry out in Lemma 5.4.

Next an easy computation shows that

$$\frac{d}{dt}\mathcal{A}\big(\vec{u}(t)\big) = \big\langle \mathcal{E}'\big(\vec{u}(t)\big), -P^{-1}\mathcal{A}'\big(\vec{u}(t)\big)\big\rangle.$$

Let us define

$$\mathcal{B}(\vec{u}) = \big\langle \mathcal{E}'(\vec{u}), -P^{-1}\mathcal{A}'(\vec{u})\big\rangle \tag{5.8}$$

for all $u \in X$. We would like that the set

$$\mathcal{S}^+ = \big\{\vec{u} \bigm| \mathcal{Q}(\vec{u}) \equiv Q(\vec{\phi}_{\omega_0}),\ \mathcal{E}(\vec{u}) < \mathcal{E}(\vec{\phi}_{\omega_0}),\ \mathcal{B}(\vec{u}) > 0\big\}$$

is an invariant set under the flow of the equation. The conditions given in $\mathcal{S}^+$ about $\mathcal{Q}$ and $\mathcal{E}$ are easy to verify from the conservation of charge and energy of the equation. To see that the condition on $\mathcal{B}$ can be preserved under the flow, we need to choose $\vec{y}_0$ carefully by using the condition on $d(\omega)$. We follow the idea in [8,16] to work out the choice of $\vec{y}_0$ in Proposition 5.5. Note that the arguments in [8] and [16] using spatial dilations do not work in the case when domain Ω is bounded. After we specify the choice of $\vec{y}_0$ and prove the invariance of $\mathcal{S}^+$, the proof of our main instability result (Theorem 5.3) follows from the standard argument and will be carried out in Proposition 5.12.

We now construct a neighborhood of $S_{\omega_0}(\phi_{\omega_0})$ and a function $\theta(\vec{u})$ on this set, which is motivated by [16, Lemma 9]. Let

$$\mathcal{N}(\vec{u}) = \int (\operatorname{Im} u_1 - \omega_0 \operatorname{Re} u_2)\phi_{\omega_0} \quad \forall \vec{u} = (u_1, u_2) \in X.$$

Note that by definition $\mathcal{N}(\vec{\phi}_{\omega_0}) = 0$ and

$$\mathcal{N}(i\vec{\phi}_{\omega_0}) = \big(1 + \omega_0^2\big)\int \phi_{\omega_0}^2 > 0, \qquad \mathcal{N}\big(e^{-i\theta}\vec{u}\big) = (\cos\theta)\mathcal{N}(\vec{u}) - (\sin\theta)\mathcal{N}(i\vec{u}). \tag{5.9}$$

Lemma 5.4. *Let $L_\delta = \{\vec{u} \in X \mid \mathcal{N}(\vec{u}) = 0,\ \|\vec{u} - \vec{\phi}_{\omega_0}\| < \delta\}$. If δ is small enough then*

(i) $(e^{i\theta}L_\delta) \cap L_\delta = \emptyset$ *for* $\theta \in (0, 2\pi)$;
(ii) $\vec{U}_\delta = \bigcup_{\theta \in [0,2\pi)} e^{i\theta}L_\delta$ *is an open neighborhood of* $S_{\omega_0}(\phi_{\omega_0})$ *in* X; *and*
(iii) *for any* $\vec{u} \in U_\delta$, *there is a unique* $\theta = \theta(\vec{u}) \in [0, 2\pi)$ *such that* $e^{-i\theta}u \in L_\delta$.

Proof. (i) Let $\vec{v} \in (e^{i\theta}L_\delta) \cap L_\delta$ for some $\theta \in (0, 2\pi)$. Then $\vec{v}$ and $e^{-i\theta}\vec{v}$ both belong to L_δ. By (5.9), $(\sin\theta)\mathcal{N}(i\vec{v}) = 0$. However, if $\|\vec{v} - \vec{\phi}_{\omega_0}\| < \delta$ is small enough, then $\mathcal{N}(i\vec{v}) \neq 0$ since $\mathcal{N}(i\vec{\phi}_{\omega_0}) \neq 0$, thus $\sin\theta = 0$ and $\theta = \pi$. This implies $-\vec{v} \in L_\delta$, thus $\delta > \|-\vec{v} - \vec{\phi}_{\omega_0}\|_X \geqslant 2\|\vec{\phi}_{\omega_0}\| - \delta$, which is impossible if $\delta > 0$ is small enough. This proves (i).

(ii) We only need to show that any point $\vec{u} \in L_\delta$ is an interior point of U_δ. From (i), there exists a $\delta_1 > 0$ such that $\|\vec{v} - \vec{u}\|_X < \delta_1$ implies that $\mathcal{N}(i\vec{v}) \neq 0$. For any such v, let

$$\alpha = \alpha(\vec{v}) = \tan^{-1} \frac{\mathcal{N}(\vec{v})}{\mathcal{N}(i\vec{v})}. \tag{5.10}$$

Then by (5.9) $\mathcal{N}(e^{-i\alpha}\vec{v}) = 0$. Hence $e^{-\alpha}\vec{v} \in L_\delta$, i.e., $\vec{v} \in \vec{U}_\delta$ by definition.

(iii) It follows immediately from (i). □

The characterization of $\vec{y}_0$ is given in the following lemma and the proof of which will be given at the end of this section for the clarity of the argument.

Proposition 5.5. *If $d''(\omega_0) < 0$, then there exists a $\vec{y}_0 = (y_1, y_2) \in X$ such that*

(i) $\langle P\vec{y}_0, i\vec{\phi}_{\omega_0}\rangle = \langle \mathcal{Q}'(\vec{\phi}_{\omega_0}), \vec{y}_0\rangle = 0$,
(ii) $\langle [\mathcal{E}''(\vec{\phi}_{\omega_0}) - \omega_0 \mathcal{Q}''(\vec{\phi}_{\omega_0})]\vec{y}_0, \vec{y}_0\rangle \leqslant d''(\omega_0) < 0$,
(iii) $\langle K'_{\omega_0}(\phi_{\omega_0}), y_1\rangle \neq 0$; *here the pairing is in H^* and H.*

Using this vector $\vec{y}_0 \in X$ and the angle $\theta(\vec{u})$ determined in (iii) of Lemma 5.4, we define $\mathcal{A}(\vec{u})$ on $\vec{U}_\delta$ by formula (5.7). Before we go to details of the proof of Theorem 5.3, we summarize some properties of $\mathcal{A}$ which are useful in our argument.

Proposition 5.6. *$\mathcal{A}$ is a C^1 functional on $\vec{U}_\delta$ and satisfies that*

(i) $\mathcal{A}(e^{i\theta}\vec{u}) = \mathcal{A}(\vec{u})$ *for all* $\vec{u} \in \vec{U}_\delta$,
(ii) $\mathcal{A}'(\vec{\phi}_{\omega_0}) = P\vec{y}_0$,
(iii) $\mathcal{A}'(\vec{u}) \in \mathrm{Range}(P) \subset X^*$ *for all* $\vec{u} \in \vec{U}_\delta$,
(iv) $\langle \mathcal{Q}'(\vec{u}), P^{-1}\mathcal{A}'(\vec{u})\rangle = 0$ *for all* $\vec{u} \in \vec{U}_\delta$.

Proof. (i)–(iii) are direct consequences of the definition of $\mathcal{A}$. (iv) follows from differentiating (i) with respect to θ at $\theta = 0$. □

For any given $\vec{u} \in \vec{U}_\delta$, we can solve the differential equation

$$\begin{cases} \dfrac{d\vec{S}(\lambda)}{d\lambda} = -P^{-1}\mathcal{A}'\big(\vec{S}(\lambda)\big), \\ \vec{S}(0) = \vec{u}. \end{cases} \tag{5.11}$$

Since $P^{-1}\mathcal{A}'$ is Lipschitz continuous from $\vec{U}_\delta$ to X, a unique solution $\vec{S}(\lambda) = \vec{S}(\lambda, \vec{u})$ of (5.11) exists in $|\lambda| < \sigma(\vec{u})$ for all $\vec{u} \in \vec{U}_\delta$ and it can be shown that $\sigma(\vec{u}) \geqslant \sigma_0 > 0$ for all $\vec{u} \in \vec{U}_\gamma$ if $0 < \gamma < \delta$.

Proposition 5.7. *There exists a smooth deformation $\vec{S} = (S_1, S_2)\colon (-\sigma_0, \sigma_0) \times \vec{U}_\gamma \to \vec{U}_\delta$ such that*

(i) $\vec{S}(0, \vec{u}) = \vec{u}$,
(ii) $\frac{\partial \vec{S}}{\partial \lambda}(0, \vec{u}) = -P^{-1}\mathcal{A}'(\vec{u})$,
(iii) $\mathcal{Q}(\vec{S}(\lambda, \vec{u})) = \mathcal{Q}(\vec{u})$,
(iv) $K_{\omega_0}(S_1(\lambda(\vec{u}), \vec{u})) = 0$ *for a curve* $\lambda = \lambda(\vec{u})$, $\vec{u} \in \vec{U}_\gamma$.

Proof. That (iii) is true follows from (iv) of Proposition 5.6 by differentiating with respect to λ. We only need to prove (iv). Using (ii) and $-P^{-1}\mathcal{A}'(\vec{\phi}_{\omega_0}) = -\vec{y}_0 = -(y_1, y_2)$, we have

$$\frac{\partial S_1}{\partial \lambda}(0, \vec{\phi}_{\omega_0}) = -y_1.$$

By (iii) of Proposition 5.5 we have

$$\left.\frac{\partial K_{\omega_0}(S_1(\lambda, \vec{u}))}{\partial \lambda}\right|_{\lambda=0, \vec{u}=\vec{\phi}_{\omega_0}} = -\big\langle K'_{\omega_0}(\phi_{\omega_0}), y_1\big\rangle \neq 0.$$

By the implicit function theorem, we know that for $\vec{u}$ near $\vec{\phi}_{\omega_0}$ there is a solution $\lambda = \lambda(\vec{u})$ such that

$$K_{\omega_0}\big(S_1\big(\lambda(\vec{u}), \vec{u}\big)\big) = K_{\omega_0}\big(S_1(0, \vec{\phi}_{\omega_0})\big) = K_{\omega_0}(\phi_{\omega_0}) = 0.$$

This $\lambda(\vec{u})$ can be extended to $\vec{U}_\gamma$ from the invariance of $\vec{U}_\gamma$ under $\{e^{i\theta}\}$. $\square$

We now compute the energy along the deformation flow $\vec{S}(\lambda,\vec{u})$. Let $E(\lambda,\vec{u})=\mathcal{E}(\vec{S}(\lambda,\vec{u}))$. Then by (ii) of Proposition 5.7

$$\frac{\partial E}{\partial\lambda}(0,\vec{u})=\left\langle\mathcal{E}'(\vec{u}),-P^{-1}\mathcal{A}'(\vec{u})\right\rangle=\mathcal{B}(\vec{u}),\quad \vec{u}\in\vec{U}_\gamma. \tag{5.12}$$

Lemma 5.8. *For $\gamma>0$ and $\sigma_0>0$ small enough we have*

$$E(\lambda,\vec{u})<\mathcal{E}(\vec{u})+\lambda\mathcal{B}(\vec{u}),\quad 0<|\lambda|<\sigma_0,\ \vec{u}\in\vec{U}_\gamma. \tag{5.13}$$

Proof. A simple calculation of second derivatives $\partial_\lambda^2E(\lambda,\vec{u})$ and $\partial_\lambda^2\mathcal{Q}(\vec{S}(\lambda,\vec{u}))$ shows that at $\lambda=0$ and $\vec{u}=\vec{\phi}_{\omega_0}$

$$\partial_\lambda^2E(0,\vec{\phi}_{\omega_0})=\left\langle\left[\mathcal{E}''(\vec{\phi}_{\omega_0})-\omega_0\mathcal{Q}''(\vec{\phi}_{\omega_0})\right]\vec{y}_0,\vec{y}_0\right\rangle\leqslant d''(\omega_0)<0$$

from part (ii) of Proposition 5.5. Therefore, (5.12) and the Taylor expansion give

$$E(\lambda,\vec{u})<\mathcal{E}(\vec{u})+\lambda\mathcal{B}(\vec{u})$$

for $\vec{u}$ near $\vec{\phi}_{\omega_0}$ and λ near 0. This proves (5.13) again from the invariance under $e^{i\theta}$. $\square$

Proposition 5.9. *Let $\lambda=\lambda(\vec{u})$ be the curve determined in* (iv) *of Proposition* 5.6*. For all $\vec{u}\in\vec{U}_\gamma$ with $\mathcal{E}(\vec{u})<\mathcal{E}(\vec{\phi}_{\omega_0})$ and $\mathcal{Q}(\vec{u})=\mathcal{Q}(\vec{\phi}_{\omega_0})$, we have $\lambda(\vec{u})\neq0$ and*

$$\mathcal{E}(\vec{\phi}_{\omega_0})<\mathcal{E}(\vec{u})+\lambda(\vec{u})\mathcal{B}(\vec{u}). \tag{5.14}$$

Proof. By Lemma 5.1,

$$E(\lambda,\vec{u})\geqslant J_{\omega_0}\big(S_1(\lambda,\vec{u})\big)+\omega_0\mathcal{Q}\big(\vec{S}(\lambda,\vec{u})\big)=J_{\omega_0}\big(S_1(\lambda,\vec{u})\big)+\omega_0\mathcal{Q}(\vec{\phi}_{\omega_0}). \tag{5.15}$$

By (iv) of Proposition 5.7 and definition of $d(\omega)$ we have $J_{\omega_0}(S_1(\lambda,\vec{u}))\geqslant d(\omega_0)$, thus

$$E(\lambda,\vec{u})\geqslant d(\omega_0)+\omega_0\mathcal{Q}(\vec{\phi}_{\omega_0})=\mathcal{E}(\vec{\phi}_{\omega_0}).$$

We claim that $\lambda(\vec{u})\neq0$. In fact, if $\lambda(\vec{u})=0$, we would have

$$E(\lambda,\vec{u})=\mathcal{E}(\vec{u})<\mathcal{E}(\vec{\phi}_{\omega_0}),$$

which is a contradiction to (5.15). The proof is done by combining (5.15) with the previous lemma. $\square$

In what follows, let $e_0=\mathcal{E}(\vec{\phi}_{\omega_0})$, $q_0=\mathcal{Q}(\vec{\phi}_{\omega_0})$ and $\vec{G}_{\omega_0}=S_{\omega_0}(\phi_{\omega_0})$. Define

$$\mathcal{S}^\pm=\left\{\vec{u}\in\vec{U}_\gamma\setminus\vec{G}_{\omega_0}\,\middle|\,\mathcal{E}(\vec{u})<e_0,\ \mathcal{Q}(\vec{u})=q_0,\ \pm\mathcal{B}(\vec{u})>0\right\}.$$

Lemma 5.10. *The sets $\mathcal{S}^\pm$ are invariant under the flow* (5.4)*. In particular, if $\vec{u}_0\in\mathcal{S}^\pm$, then $\pm\mathcal{B}(\vec{u}(t))>0$ for all $t>0$ such that $\vec{u}(s)\in\vec{U}_\gamma\setminus\vec{G}_{\omega_0}$ for $0\leqslant s\leqslant t$.*

Proof. For strong solutions, $\mathcal{E}$ and $\mathcal{Q}$ are conserved, thus if $\vec{u}(s)\in\vec{U}_\gamma\setminus\vec{G}_{\omega_0}$ for all $0\leqslant s\leqslant t$ we have

$$0<e_0-\mathcal{E}\big(\vec{u}(t)\big)<\lambda\big(\vec{u}(t)\big)\mathcal{B}\big(\vec{u}(t)\big),$$

hence $\mathcal{B}(\vec{u}(t))\neq0$. By continuity of the solution, curve $\mathcal{B}(\vec{u}(t))$ has one sign and the same as that of $\mathcal{B}(\vec{u}_0)$. Thus $\mathcal{S}^\pm$ each is invariant. $\square$

Lemma 5.11. *Let $\vec{u}_0\in\mathcal{S}^+$ and let*

$$T_0=\sup\left\{t\,\middle|\,\vec{u}(s)\in\vec{U}_\gamma\setminus\vec{G}_{\omega_0},\ 0\leqslant s<t\right\}\leqslant\infty$$

be the exit time. Then there is $\epsilon_0>0$ such that $\mathcal{B}(\vec{u}(t))\geqslant\epsilon_0$ for all $t<T_0$.

Proof. We have

$$\epsilon_0\equiv e_0-\mathcal{E}(\vec{u}_0)\leqslant e_0-\mathcal{E}\big(\vec{u}(t)\big)\leqslant\lambda\big(\vec{u}(t)\big)\mathcal{B}\big(\vec{u}(t)\big)\quad\forall t\in[0,T_0).$$

Thus $\lambda(\vec{u}(t))\mathcal{B}(\vec{u}(t))\geqslant\epsilon_0>0$. Since we can choose $\lambda(\vec{u})$ so that $|\lambda(\vec{u})|<\sigma_0\leqslant\frac{1}{2}$, thus it follows that $\mathcal{B}(\vec{u}(t))\geqslant2\epsilon_0$ for all $t<T_0$. $\square$

Proposition 5.12. *If $\vec{u}_0 \in \mathcal{S}^+$, the solution to* (5.4) *with initial condition $\vec{u}(0) = \vec{u}_0$ exits $\vec{U}_\gamma \setminus \vec{G}_{\omega_0}$ in finite time $T_0 < \infty$.*

Proof. Apply Eq. (5.4) to $-P^{-1}\mathcal{A}'(\vec{u}(t)) \in X$ and we obtain

$$\frac{d}{dt}\mathcal{A}\big(\vec{u}(t)\big) = \left\langle P\frac{d\vec{u}(t)}{dt}, -P^{-1}\mathcal{A}'\big(\vec{u}(t)\big)\right\rangle = \big\langle \mathcal{E}'\big(\vec{u}(t)\big), -P^{-1}\mathcal{A}'\big(\vec{u}(t)\big)\big\rangle = \mathcal{B}\big(\vec{u}(t)\big).$$

By Lemma 5.11 above, $\mathcal{B}(\vec{u}(t)) \geqslant \epsilon_0$ as long as $\vec{u}(t) \in \vec{U}_\gamma \setminus \vec{G}_{\omega_0}$. So

$$\mathcal{A}\big(\vec{u}(t)\big) - \mathcal{A}(\vec{u}_0) \geqslant \epsilon_0 t.$$

Since $\vec{U}_\gamma$ is bounded and $\mathcal{A}$ is bounded on $\vec{U}_\gamma$, the solution must exit from $\vec{U}_\gamma \setminus \vec{G}_{\omega_0}$ in a finite time. □

To complete the proof of the instability theorem, Theorem 5.3, by Proposition 5.12, we have to show that $\mathcal{S}^+$ is nonempty and contain points arbitrarily close to the orbit $\vec{G}_{\omega_0}$. This will follow from the proof of the only remaining result: Proposition 5.5.

Proof of Proposition 5.5. To construct $\vec{y}_0$, we let

$$\vec{\psi}(\omega) = \big(\psi_1(\omega), \psi_2(\omega)\big) = a(\omega)\vec{\phi}_\omega,$$

where $a(\omega) > 0$ is chosen so that $\mathcal{Q}(\vec{\psi}(\omega)) = \mathcal{Q}(\vec{\phi}_{\omega_0}) = q_0$, i.e.,

$$\omega a^2(\omega)\int |\phi_\omega|^2 = \omega_0 \int |\phi_{\omega_0}|^2, \quad \text{i.e.,} \quad a^2(\omega)d'(\omega) = d'(\omega_0). \tag{5.16}$$

From this we easily have

$$2a'(\omega)d'(\omega) = -a(\omega)d''(\omega). \tag{5.17}$$

With $\vec{\psi}(\omega)$ so defined, we set

$$\vec{y}_0 = (y_1, y_2) = \frac{d}{d\omega}\vec{\psi}(\omega)\Big|_{\omega=\omega_0}. \tag{5.18}$$

Since $\mathcal{E}(\vec{\psi}(\omega)) = J_\omega(a(\omega)\phi_\omega) - \omega d'(\omega_0)$, we consider the function

$$g(\omega) = J_\omega\big(a(\omega)\phi_\omega\big) - d(\omega) = \mathcal{E}\big(\vec{\psi}(\omega)\big) + \omega d'(\omega_0) - d(\omega). \tag{5.19}$$

Then $g(\omega_0) = 0$, $g'(\omega_0) = 0$, and simple but long calculations by expanding the term $J_\omega(a(\omega)\phi_\omega)$ in $g(\omega)$ show that

$$g''(\omega_0) = \big(a'(\omega_0)\big)^2 \int \phi_{\omega_0}\big(\phi_{\omega_0} f'_u(x, \phi_{\omega_0}) - f(x, \phi_{\omega_0})\big).$$

Since by assumption (f_3)

$$s\big(sf'_s(x,s) - f(x,s)\big) \leqslant (\theta - 1)sf(x,s) < 0 \quad \forall s > 0.$$

Therefore, by (5.17), $g''(\omega_0) < 0$. Hence by (5.19), $\frac{d}{d\omega}\mathcal{E}(\vec{\psi}(\omega))|_{\omega=\omega_0} = 0$ and

$$\big\langle\big[\mathcal{E}''(\vec{\phi}_{\omega_0}) - \omega_0\mathcal{Q}''(\vec{\phi}_{\omega_0})\big]\vec{y}_0, \vec{y}_0\big\rangle = \frac{d^2}{d\omega^2}\mathcal{E}\big(\vec{\psi}(\omega)\big)\Big|_{\omega=\omega_0} < d''(\omega_0) < 0. \tag{5.20}$$

Thus, for ω close but not equal to ω_0,

$$\mathcal{E}\big(\vec{\psi}(\omega)\big) < \mathcal{E}(\vec{\phi}_{\omega_0}) = e_0. \tag{5.21}$$

We now consider $K_{\omega_0}(\psi_1(\omega)) = K_{\omega_0}(a(\omega)\phi_\omega)$. Further calculations show that

$$\frac{d}{d\omega}K_{\omega_0}\big(\psi_1(\omega)\big)\Big|_{\omega=\omega_0} = -2d'(\omega_0) + a'(\omega_0)\int \phi_{\omega_0}\big(\phi_{\omega_0} f'_u(x, \phi_{\omega_0}) - f(x, \phi_{\omega_0})\big).$$

The term on the right-hand side is not zero since by (5.17) $a'(\omega_0)$ and $d'(\omega_0)$ have the same nonzero sign. This implies that

$$\big\langle K'_{\omega_0}(\phi_{\omega_0}), y_1\big\rangle = \frac{d}{d\omega}K_{\omega_0}\big(\psi_1(\omega)\big)\Big|_{\omega=\omega_0} \neq 0. \tag{5.22}$$

Lastly, we need prove that the set $\mathcal{S}^+$ is nonempty and contain points arbitrarily close to $\vec{G}_{\omega_0}$. By (5.16) and (5.21) we have only to prove $\mathcal{B}(\vec{\psi}(\omega))$ changes sign as ω passes ω_0. Since by Proposition 5.9 and (5.16), (5.21),

$$\lambda\big(\vec{\psi}(\omega)\big)\mathcal{B}\big(\vec{\psi}(\omega)\big) > 0$$

for all $\omega \neq \omega_0$ but near ω_0. Differentiating

$$K_{\omega_0}\big(S_1\big(\lambda\big(\vec{\psi}(\omega)\big), \vec{\psi}(\omega)\big)\big) = 0$$

with respect to ω yields that

$$\frac{d}{d\omega}\lambda\big(\vec{\psi}(\omega)\big)\Big|_{\omega=\omega_0} = 1 \neq 0.$$

So $\lambda(\vec{\psi}(\omega))$ changes sign when ω passes ω_0 since $\lambda(\vec{\psi}(\omega_0)) = 0$. This implies that $\mathcal{B}(\vec{\psi}(\omega))$ changes sign as ω passes ω_0. $\square$

6. Applications

In this section, we consider several cases of nonlinearity f or domain Ω where we have orbitally stable standing waves.

Theorem 6.1. *Suppose the lower bound λ_1 of spectrum of operator $-\Delta + m$ is a positive eigenvalue.* (*This is certainly true if the underlying domain Ω is a bounded domain in $\mathbf{R}^n$ or a compact manifold. It is also true if $m(x)$ is a potential, and the operator $-\Delta + m$ has discrete spectrum to the left of a continuous spectrum.*) *Then the* (*NLKG*) *equation* (1.1) *has orbitally stable standing waves for some frequency ω with $\omega^2 \in (0, \lambda_1)$.*

Proof. By Theorem 1.1, it suffices to show that there exists a ω_0 such that $d''(\omega_0) > 0$. Suppose not, then $d''(\omega) \leqslant 0$ for all $\omega \in (0, \sqrt{\lambda_1})$, which implies that $d'(\omega) = -\omega \int |\phi_\omega|^2$ is decreasing for $\omega \in (0, \sqrt{\lambda_1})$. Hence for any $0 < \epsilon < \sqrt{\lambda_1}$ there exists a positive constant $C = C(\epsilon)$ such that $\int |\phi_\omega|^2 \geqslant C$ for $\omega \in (\epsilon, \sqrt{\lambda_1})$. By the definition of $d(\omega)$, we have

$$d(\omega) = I_\omega(\phi_\omega) \geqslant \frac{\theta - 1}{2(\theta + 1)} \int \big(|\nabla\phi_\omega|^2 + \big(m - \omega^2\big)|\phi_\omega|^2\big) \geqslant A_1(\theta, C)\big(\lambda_1 - \omega^2\big), \tag{6.1}$$

where $A_1 = A_1(\theta, C)$ is a positive constant independent of $\omega \in (\epsilon, \sqrt{\lambda_1})$. Next we estimate an upper bound for $d(\omega)$. Let $v(x)$ be the first eigenfunction of $-\Delta + m$, and $v_\delta(x) = \delta v(x)$. We define $\delta = \delta(\omega)$ so that

$$K_\omega(v_\delta) = \delta^2 \int \big(|\nabla v|^2 + \big(m - \omega^2\big)|v|^2\big) + \delta \int v f(x, \delta v) = 0,$$

or

$$\big(\lambda_1 - \omega^2\big) \int v^2 = -\frac{1}{\delta} \int v f(x, \delta v).$$

Then from Lemma 2.2 we have

$$\delta = \delta(\omega) \to 0 \quad \text{as } \omega^2 \to \lambda_1. \tag{6.2}$$

Using (f_3) and the alternative expression for $d(\omega)$ we get

$$d(\omega) \leqslant J_\omega(\delta v) \leqslant \frac{\delta^2}{2} \int \big(|\nabla v|^2 + \big(m - \omega^2\big)|v|^2\big) = \frac{\lambda_1 - \omega^2}{2}\delta^2 \int |v|^2 = A_2(\theta, v)\delta^2\big(\lambda_1 - \omega^2\big),$$

where $A_2(\theta, v)$ is a positive constant independent of ω. Combining (6.1) and (6.3) gives

$$0 < \frac{A_1}{A_2} \leqslant \delta^2,$$

a contradiction to (6.2), and the theorem is proved. $\square$

The second application we consider is for the case $\Omega = \mathbf{R}^n$ with $n \geqslant 2$. We investigate the stability of standing waves for the special (NLKG)

$$u_{tt} - \Delta u + u - |u|^{p-1}u = 0 \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{6.3}$$

which corresponds to $m(x) \equiv 1$ and $f(x, u) = -|u|^{p-1}u$.

Theorem 6.2. *Let $1 < p < \frac{n+2}{n-2}$. Then* (*NLKG*) *equation* (6.3) *always has orbitally unstable ground state standing waves, and it has orbitally stable ground state standing waves if $1 < p < 1 + 4/n$.*

Proof. Since $1 < p < \frac{n+2}{n-2}$, the semilinear elliptic equation

$$-\Delta v + v - v^p = 0 \tag{6.4}$$

has a unique positive radial symmetric solution $v = \phi_0 \in H$ on the whole space $\mathbf{R}^n$ [10]. For any real ω with $0 < |\omega| < 1$, let

$$\phi_\omega(x) = \left(1-\omega^2\right)^{\frac{1}{p-1}} \phi_0\left(\sqrt{1-\omega^2}x\right), \quad x \in \mathbf{R}^n.$$

Then ϕ_ω is the unique positive radial symmetric solution of equation

$$-\Delta\phi_\omega + \left(1-\omega^2\right)\phi_\omega - \phi_\omega^p = 0$$

on $\mathbf{R}^n$. Thus we can use it to calculate the minimal energy $d(\omega)$ defined by (1.5) as follows

$$\begin{aligned} d(\omega) = J_\omega(\phi_\omega) &= \frac{1}{2}\int \left(|\nabla\phi_\omega|^2 + \phi_\omega^2\right) - \frac{1}{p+1}\int \phi_\omega^{p+1} \\ &= \frac{(1-\omega^2)^{\frac{p+1}{p-1}}}{2} \int \left(\left|\nabla\phi_0\left(\sqrt{1-\omega^2}x\right)\right|^2 + \phi_0^2\left(\sqrt{1-\omega^2}x\right)\right) dx - \frac{(1-\omega^2)^{\frac{p+1}{p-1}}}{p+1} \int \phi_0^{p+1}\left(\sqrt{1-\omega^2}x\right) dx \\ &= \left(1-\omega^2\right)^{\frac{p+1}{p-1}-\frac{n}{2}} \int \left(\frac{|\nabla\phi_0(x)|^2 + \phi_0^2(x)}{2} - \frac{1}{1+p}\left|\phi_0(x)\right|^{p+1}\right) dx \\ &= \left(1-\omega^2\right)^{\frac{p+1}{p-1}-\frac{n}{2}} d(0) = \left(1-\omega^2\right)^\alpha d(0), \end{aligned}$$

where $\alpha = \frac{p+1}{p-1} - \frac{n}{2}$. Taking the second derivative, we find

$$d''(\omega) = 2\alpha\left[-1 + (2\alpha-1)\omega^2\right]\left(1-\omega^2\right)^{\alpha-2} d(0).$$

Note that $d(0) = (\frac{1}{2} - \frac{1}{p+1})\|\phi_0\|_H^2 > 0$, and $\alpha > 1$ since $1 < p < \frac{n+2}{n-2}$. Therefore $d''(\omega) < 0$ if $|\omega|$ is small enough, which shows that the orbit $S_\omega(\phi_\omega)$ is not orbitally stable. Now, if $\omega^2 < 1$, then

$$\left\{\omega \mid d''(\omega) > 0\right\} = \left\{\omega \;\middle|\; 0 < \frac{1}{2\alpha-1} < \omega^2 < 1\right\}.$$

If $1 < p < 1 + 4/n$, the set on right-hand side is nonempty, and hence we have orbitally stable ground state standing waves. □

Remark 6.1. The same stability result for standing waves of (NLKG) equation (6.3) was obtained by Shatah in [14], but the approach in [14] cannot handle the case $n = 2$ due to the usage of a different functional K_ω which is not well defined when $n = 2$.

Acknowledgments

The authors are grateful to Professor Walter Strauss for his interest and many suggestions and encouragements during the preparation of the manuscript of this work. We would also like to thank an anonymous referee for several helpful comments.

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