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A DUALITY METHOD FOR MICROMAGNETICS*

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Abstract. We present a new method for micromagnetics based on replacing the nonlocal total energy of magnetizations by a new local energy for divergence-free fields and then studying the dual Legendre functional of this new energy restricted on gradient fields. We establish a Fenchel-type duality principle relevant to the minimization for these problems. The dual functional may be written as a convex integral functional of gradients, and its minimization problem will be solved by standard minimization procedures in the calculus of variations. Special emphasis is placed on the analysis of existence/nonexistence, depending on the applied field and the physical domain. In particular, we describe a precise procedure to check the existence of magnetization of minimal energy for ellipsoid domains.

Key words. micromagnetics, energy minimizers, duality method, divergence-free fields

AMS subject classifications. 49J45, 49K20, 49M29, 78M30, 82D40

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1. Introduction.

In the static theory of micromagnetics, observable magnetic properties of a ferromagnetic material are characterized by the equilibrium states of a total micromagnetic energy, including several contributions of different energies; see [6, 19]. Mathematical studies of micromagnetics have been extensively conducted by many authors based on such a theory; see [2, 8, 10, 11, 12, 13, 15, 16, 18, 22, 24, 25, 27].

For large ferromagnetic materials, it has been justified in [10] (see also [16, 28]) that the total micromagnetic energy can be approximated by the following simple form (ignoring the so-called exchange energy):

$$I(m) = \int_\Omega \varphi(m(x))\,dx - \int_\Omega H(x)\cdot m(x)\,dx + \frac{1}{2}\int_{\mathbf{R}^n} |F_m(z)|^2\,dz, \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with piecewise smooth boundary occupied by the ferromagnetic material, m is the magnetization vector satisfying

$$|m(x)| = 1 \text{ a.e. } x\in\Omega, \tag{1.2}$$

and $F_m \in L^2(\mathbf{R}^n;\mathbf{R}^n)$ is the induced magnetic field on the whole $\mathbf{R}^n$ determined by the simplified Maxwell's equations:

$$\operatorname{curl} F_m = 0, \quad \operatorname{div}(-F_m + m\chi_\Omega) = 0 \text{ in } \mathbf{R}^n. \tag{1.3}$$

Here φ is the density of anisotropy energy that is minimized along preferred crystallographic directions, and $H \in L^2(\Omega;\mathbf{R}^n)$ is a given external applied field. The first

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term in the energy $I(m)$ is called the *anisotropy energy*, the second term is called the *external interaction energy*, and the last term is called the *magnetostatic energy*.

Our main concern of this paper is to understand the existence of magnetizations m of minimum energy. Due to the saturation constraint $|m| = 1$ and the anisotropy energy, this existence is not granted because of the nonconvex nature of the problem, so a more careful analysis should be carried out. In this paper, we focus on establishing a necessary and sufficient condition for the existence of minimizers for energy $I(m)$ based on a new variational problem of gradient type. The new energy functional is strictly convex and is obtained by the method used in our previous work [25] (see also [16, section 6]) with a new idea of the Fenchel-type duality principle in convex analysis.

To discuss the main ideas, first note that, by (1.3), the magnetostatic energy can be expressed as a variational problem

$$\frac{1}{2}\int_{\mathbf{R}^n} |F_m|^2 = \min_{\mathrm{div} G=0} \frac{1}{2}\int_{\mathbf{R}^n} |m\chi_\Omega - G|^2,$$

where the minimum is taken over all divergence-free fields G in $L^2(\mathbf{R}^n;\mathbf{R}^n)$. Next introduce an auxiliary functional $\mathcal{A}(m,G)$ for $m \in L^\infty(\Omega;\mathbf{S}^{n-1}), G \in L^2(\mathbf{R}^n;\mathbf{R}^n)$ by

$$\mathcal{A}(m,G) = \int_\Omega \varphi(m) - \int_\Omega H(x)\cdot m + \frac{1}{2}\int_{\mathbf{R}^n} |m\chi_\Omega - G|^2, \tag{1.4}$$

which leads to $I(m) = \min_{\mathrm{div} G=0} \mathcal{A}(m,G)$. Therefore

$$\inf_{|m|=1} I(m) = \inf_{|m|=1}\left[\inf_{\mathrm{div} G=0} \mathcal{A}(m,G)\right] = \inf_{\mathrm{div} G=0}\left[\inf_{|m|=1} \mathcal{A}(m,G)\right].$$

Now, for fixed G in $L^2(\mathbf{R}^n;\mathbf{R}^n)$, define

$$J(G) = \inf_{|m|=1} \mathcal{A}(m,G),$$

where the infimum (in fact a minimum) is taken over all $m \in L^\infty(\Omega;\mathbf{S}^{n-1})$. Then one easily has

$$\inf_{|m|=1} I(m) = \inf_{\mathrm{div} G=0} J(G);$$

furthermore, the minimization problem of $I(m)$ over $m \in L^\infty(\Omega;\mathbf{S}^{n-1})$ is equivalent to the minimization problem of $J(G)$ over all divergence-free fields $G \in L^2(\mathbf{R}^n;\mathbf{R}^n)$ (see Proposition 3.2 below for the precise statement).

It is for the minimization of $J(G)$ on divergence-free fields that the Fenchel-type duality principle in convex analysis [4, 26] comes into play, which involves the dual functional J^* or the Legendre transform of J. By the duality and relaxation principles (Theorems 2.2 and 2.3) proved later, only the restriction of J^* to the gradient fields plays an essential role in the minimization problem for J; this restriction gives the new convex variational problem we seek.

An elementary computation shows that

$$J(G) = \int_\Omega \psi(x, G(x))\,dx + \frac{1}{2}\int_{\Omega^c} |G|^2\,dx, \tag{1.5}$$

where

$$\psi(x,\xi) = \frac{1}{2}(|\xi|^2 + 1) - \Phi(\xi + H(x));$$

here and throughout the paper, Φ denotes the convex function associated with the anisotropy function φ defined by

$$\Phi(\eta) = \max_{h\in \mathbf{S}^{n-1}} [\eta \cdot h - \varphi(h)] \quad (\eta \in \mathbf{R}^n). \tag{1.6}$$

As seen later, the dual functional J^* of J is given in terms of the convex conjugate or the Legendre transform $\psi^*(x,\xi)$ of function $\psi(x,\xi)$ defined above; it turns out that $\psi^*(x,\xi) = \frac{1}{2}|\xi|^2 + \Phi(\xi + H(x))$ (see the computation later). Therefore, the dual functional J^* restricted to the gradient fields can be written as

$$L(u) = \int_\Omega \Phi(\nabla u(x) + H(x))\, dx + \frac{1}{2}\int_{\mathbf{R}^n} |\nabla u(x)|^2\, dx \tag{1.7}$$

$\forall u \in H^1_{loc}(\mathbf{R}^n)$ with $\nabla u \in L^2(\mathbf{R}^n;\mathbf{R}^n)$.

Note that $L(u + c) = L(u)$ for all constants $c \in \mathbf{R}$. To fix the idea, we define the linear space $\mathcal{X}$ by

$$\mathcal{X} = \left\{ u \in H^1_{loc}(\mathbf{R}^n) \;\Big|\; \nabla u \in L^2(\mathbf{R}^n;\mathbf{R}^n),\ \int_{\partial\Omega} \Gamma u\, dS = 0 \right\}, \tag{1.8}$$

where $\Gamma u = u|_{\partial\Omega}$ is the well-defined trace in $H^{1/2}(\partial\Omega)$ (see [1]). It is easily seen that L is strictly convex on $\mathcal{X}$. Hence L has a *unique minimizer* on $\mathcal{X}$; we denote this unique minimizer by $\bar{u} = \bar{v}\chi_\Omega + \bar{w}\chi_{\Omega^c}$. Certainly, this function $\bar{u}$ depends on the domain Ω, the anisotropy function φ (in terms of function Φ), and the applied field $H(x)$. It will be shown later that $\bar{u}$ is uniquely determined by its boundary data $\bar{g} = \bar{u}|_{\partial\Omega}$ and, in particular, that $\bar{w}$ is harmonic on Ω^c.

A dual formulation closely related to the functional $L(u)$ has been used recently in [20] to approach some regularity problems for thin films and has also previously been derived in [16]. In fact, functional $L(-u)$ here agrees with the functional defined by [16, Formula (6.9)]. Note that if one defines

$$-P^* = \inf_{m\in L^\infty(\Omega;\mathbf{S}^{n-1})} I(m); \quad P = \min_{u\in\mathcal{X}} L(u),$$

then it has been established in [16] that $P^* \le P$ and the equality was in doubt there. As a byproduct of the results of this paper, we can actually confirm the equality of this estimate.

THEOREM 1.1. *It follows that*

$$\inf_{m\in L^\infty(\Omega;\mathbf{S}^{n-1})} I(m) = -\min_{u\in\mathcal{X}} L(u). \tag{1.9}$$

This result follows directly from Theorems 2.2 and 2.3, together with Proposition 3.2 below.

The main result of the paper is to establish a necessary and sufficient condition for the existence of minimizers of energy $I(m)$ in terms of the unique minimizer $\bar{u} = \bar{v}\chi_\Omega + \bar{w}\chi_{\Omega^c}$ of functional $L(u)$. To state our main result, we need the following set defined for each $\eta \in \mathbf{R}^n$:

$$\Sigma(\eta) = \{h \in \mathbf{S}^{n-1} \mid \Phi(\eta) = \eta\cdot h - \varphi(h)\}. \tag{1.10}$$

We can now state our main result of the paper; the proof will be given in the following sections.

THEOREM 1.2. *Let $\bar u = \bar v\chi_\Omega + \bar w\chi_{\Omega^c} \in \mathcal{X}$ be the unique minimizer of functional L defined above. Then, the energy $I(m)$ has a minimizer if and only if there exists a function $\tilde G \in L^2(\Omega; \mathbf{R}^n)$ that satisfies*

$$\begin{cases} \operatorname{div}[\tilde G\chi_\Omega + (\nabla \bar w)\chi_{\Omega^c}] = 0 & \text{on } \mathbf{R}^n, \\ \tilde G(x) \in \nabla \bar v(x) + \Sigma(\nabla \bar v(x) + H(x)) & \text{a.e. } x \in \Omega. \end{cases} \tag{1.11}$$

In this case, $\bar m(x) = \tilde G(x) - \nabla \bar v(x)$ is a minimizer of energy I.

Note that since $\bar w$ is harmonic in Ω^c, the first condition in (1.11) is equivalent to $\operatorname{div}\tilde G = 0$ in $H^{-1}(\Omega)$ and $\tilde G \cdot \nu|_{\partial\Omega} = \frac{\partial \bar w}{\partial \nu}$, where ν is the outward unit normal to the boundary $\partial\Omega$ of domain Ω and both $\tilde G \cdot \nu$ and $\frac{\partial \bar w}{\partial \nu}$ are defined as elements in $H^{-1/2}(\partial\Omega)$ (see, e.g., [29, page 9]).

To put the condition (1.11) in further perspective, let $\tilde w \in H^1(\Omega)$ be the unique solution to the Neumann problem

$$\Delta \tilde w = 0 \text{ in } \Omega, \quad \frac{\partial \tilde w}{\partial \nu} = \frac{\partial \bar w}{\partial \nu} \text{ on } \partial\Omega, \quad \int_{\partial\Omega} \tilde w \, dS = 0 \tag{1.12}$$

(notice the appearance of $\bar w$ in $\tilde w$), and let $\mathbf{S}(x)$ be a set-valued function defined by

$$\mathbf{S}(x) = \nabla \bar v(x) - \nabla \tilde w(x) + \Sigma(\nabla \bar v(x) + H(x)) \quad \text{a.e. } x \in \Omega. \tag{1.13}$$

Define $G(x) = \tilde G(x) - \nabla \tilde w(x)$. Then we see that condition (1.11) is equivalent to the following condition for function $G \in L^2(\Omega; \mathbf{R}^n)$:

$$\begin{cases} \operatorname{div}(G\chi_\Omega) = 0 & \text{on } \mathbf{R}^n, \\ G(x) \in \mathbf{S}(x) & \text{a.e. } x \in \Omega. \end{cases} \tag{1.14}$$

Constrained problems like (1.14) for divergence-free fields with constant set-valued functions $\mathbf{S}(x) = \mathbf{S}$ have been recently studied by many authors; see, e.g., [3, 8]. In such cases, when $n = 3$, it has been shown that the problem (1.14) has a solution if and only if either $0 \in \mathbf{S}$ or else there exists a set $\mathbf{F} \subseteq \mathbf{S}$ with $\dim(\operatorname{span}\mathbf{F}) \geq 2$ such that $0 \in \operatorname{ri}(\operatorname{con}\mathbf{F})$ (the relative interior of the convex hull of $\mathbf{F}$); see also Theorem 3.7 below.

Note that the set $\mathbf{S}(x)$ in our problem (1.14) depends heavily on the anisotropy energy (in terms of Φ), the applied field $H(x)$, and the specimen domain Ω and may not be a constant set. For an applied field $H(x)$ which is simply an L^2-function, we do not expect any better regularity for the minimizer $\bar u$ and thus for the set $\mathbf{S}(x)$ even with special domains. (See more discussions at the end of section 3 following Theorem 3.7.) In section 4, for constants H and ellipsoid domains Ω, we will show that the set $\mathbf{S}(x)$ involved is in fact constant, and therefore we can characterize the precise condition for the existence and nonexistence for minimizing magnetizations of micromagnetic energies with soft, uniaxial and biaxial anisotropy energy.

2. Preliminaries and general results. Let $\mathcal{H}$ be the usual real Hilbert space $L^2(\mathbf{R}^n; \mathbf{R}^n)$ with inner product and norm defined by $G \cdot F = \int_{\mathbf{R}^n} G(x) \cdot F(x)\, dx$ and $\|G\| = (\int_{\mathbf{R}^n} |G(x)|^2\, dx)^{1/2}$, respectively.

2.1. Notation and definitions of convex analysis. Let $p\colon \mathcal{H} \to \mathbf{R}$ be a given functional on $\mathcal{H}$. We review some notation and definitions in convex analysis (see, e.g., [4, 26]).

The *(convex) conjugate* or the *Legendre transform* p^* of p and the *convexification* $p^\#$ of p are, respectively, defined by

$$p^*(G) = \sup_{F\in\mathcal{H}} \{F\cdot G - p(F)\}, \quad p^\#(G) = \sup_{F\in\mathcal{H}} \{F\cdot G - p^*(F)\};$$

that is, $p^\# = (p^*)^*$. Both are convex functionals on $\mathcal{H}$, and it also follows that $p^* = (p^\#)^*$. The *subdifferential* of p at $G \in \mathcal{H}$ is defined to be the set

$$\partial p(G) = \{F \in \mathcal{H} \mid p(A) \geq p(G) + F\cdot(A-G) \ \forall\ A \in \mathcal{H}\}.$$

Clearly $\partial p(G)$ is a convex subset of $\mathcal{H}$, and $0 \in \partial p(G)$ if and only if $p(G)$ is the absolute minimum of p on $\mathcal{H}$. We also have the following property:

$$F \in \partial p(G) \text{ if and only if } p^*(F) = F\cdot G - p(G). \tag{2.1}$$

Moreover, if q is a convex functional on $\mathcal{H}$, then

$$\|F\| \leq \sup_{\|A\|\leq 1} \{q(G+A) - q(G)\} \quad \forall\ F \in \partial q(G). \tag{2.2}$$

2.2. Integral functionals and representations on $\mathcal{H}$. We consider an integral functional on $\mathcal{H}$ of the form

$$p(G) = \int_{\mathbf{R}^n} \Psi(x, G(x))\, dx, \tag{2.3}$$

where $\Psi(x,\xi)$ is a function measurable in $x \in \mathbf{R}^n$ for each $\xi \in \mathbf{R}^n$ and continuous in $\xi \in \mathbf{R}^n$ for almost every $x \in \mathbf{R}^n$ and satisfies the following conditions:

$$c_0|\xi|^2 - c_1(x) \leq \Psi(x,\xi) \leq c_2|\xi|^2 + c_3(x), \tag{2.4}$$

$$|\Psi(x,\xi) - \Psi(x,\eta)| \leq c_4(|\xi| + |\eta| + c_5(x))|\xi - \eta|, \tag{2.5}$$

where c_0, c_2, c_4 are given positive constants and c_1, $c_3 \in L^1(\mathbf{R}^n)$ and $c_5 \in L^2(\mathbf{R}^n)$ are given functions.

Under these conditions, functional p is a local Lipschitz continuous functional on $\mathcal{H}$ and satisfies $p(G) \to \infty$ if $\|G\| \to \infty$. We show that for such a functional p the convex functionals p^* and $p^\#$ can be represented by Ψ. To do so, we introduce the following notation:

$$\Psi^*(x,\lambda) = \sup_{\xi\in\mathbf{R}^n} \{\xi\cdot\lambda - \Psi(x,\xi)\}, \quad \Psi^\#(x,\lambda) = \sup_{\xi\in\mathbf{R}^n} \{\xi\cdot\lambda - \Psi^*(x,\xi)\},$$
$$\partial\Psi(x,\lambda) = \{\beta \in \mathbf{R}^n \mid \Psi(x,\eta) \geq \Psi(x,\lambda) + \beta\cdot(\eta-\lambda)\ \forall\ \eta \in \mathbf{R}^n\}.$$

From the condition (2.4) above, it follows easily that

$$\tilde{c}_0|\xi|^2 - \tilde{c}_1(x) \leq \Psi^*(x,\xi), \quad \Psi^\#(x,\xi) \leq \tilde{c}_2|\xi|^2 + \tilde{c}_3(x), \tag{2.6}$$

where $\tilde{c}_0$, $\tilde{c}_2$ are some positive constants and $\tilde{c}_1$, $\tilde{c}_3 \in L^1(\mathbf{R}^n)$ are some functions. Moreover, by the growth condition (2.4), for each $\xi \in \mathbf{R}^n$ and almost every $x \in \mathbf{R}^n$, it follows that (see, e.g., [23, Lemma 4.2]) there exists a probability measure $\mu_{x,\xi}$ on $\mathbf{R}^n$ such that

$$\xi = \int_{\mathbf{R}^n} \lambda\, d\mu_{x,\xi}(\lambda), \quad \Psi^\#(x,\xi) = \int_{\mathbf{R}^n} \Psi(x,\lambda)\, d\mu_{x,\xi}(\lambda). \tag{2.7}$$

We summarize the representation results in the following lemma.

LEMMA 2.1. *It follows that*

$$p^*(G) = \int_{\mathbf{R}^n} \Psi^*(x, G(x))\, dx, \quad p^{\#}(G) = \int_{\mathbf{R}^n} \Psi^{\#}(x, G(x))\, dx. \tag{2.8}$$

Moreover, given any $F, G \in \mathcal{H}$, the relation $F \in \partial p^{\#}(G)$ holds if and only if $G \in \partial p^*(F)$, which is also equivalent to one of the following two conditions:

$$F(x) \in \partial \Psi^{\#}(x, G(x)), \quad G(x) \in \partial \Psi^*(x, F(x)) \quad \forall \text{ a.e. } x \in \mathbf{R}^n. \tag{2.9}$$

Furthermore, there exists a constant C such that the condition

$$\|F\| \le C\,(\|G\|^2 + 1) \tag{2.10}$$

holds for all $G \in \mathcal{H}$ and all $F \in \partial p^{\#}(G) \cup \partial p^*(G)$.

Proof. Given $G \in \mathcal{H}$, for any $F \in \mathcal{H}$, it follows that

$$F \cdot G - p(F) = \int_{\mathbf{R}^n} [F(x) \cdot G(x) - \Psi(x, F(x))] \le \int_{\mathbf{R}^n} \Psi^*(x, G(x)),$$

and hence $p^*(G) \le \int_{\mathbf{R}^n} \Psi^*(x, G(x))\, dx$. To prove the opposite inequality, note that one can select a measurable function $F\colon \mathbf{R}^n \to \mathbf{R}^n$ such that

$$\Psi^*(x, G(x)) = F(x) \cdot G(x) - \Psi(x, F(x)) \quad \forall \text{ a.e. } x \in \mathbf{R}^n.$$

The growth conditions on Ψ and Ψ^* imply $F \in L^2(\mathbf{R}^n; \mathbf{R}^n)$. Integrate the above identity over $\mathbf{R}^n$; we obtain $\int_{\mathbf{R}^n} \Psi^*(x, G(x))\, dx = F \cdot G - p(F) \le p^*(G)$. This proves $p^*(G) = \int_{\mathbf{R}^n} \Psi^*(x, G(x))\, dx$. The representation for $p^{\#}$ can be proved similarly using the proved representation of p^*. The equivalence of $F \in \partial p^{\#}(G)$ and $G \in \partial p^*(F)$ follows easily by (2.1). Using the representation of p^*, by elementary proofs, one can show that $G \in \partial p^*(F)$ if and only if $G(x) \in \partial \Psi^*(x, F(x))$ for a.e. $x \in \mathbf{R}^n$; this proves the stated equivalence of either pointwise condition of (2.9). Finally, (2.10) follows from (2.2) and (2.6). ∎

2.3. Subspaces of divergence-free and curl-free fields in $\mathcal{H}$. In the following, we denote by $\mathcal{K}$, the closed linear subspace of $\mathcal{H}$, defined by

$$\mathcal{K} = \{G \in \mathcal{H} \mid \operatorname{div} G = 0 \text{ in the sense of distributions}\};$$

that is, $G \in \mathcal{K}$ if and only if $G \in L^2(\mathbf{R}^n; \mathbf{R}^n)$ and

$$\int_{\mathbf{R}^n} G(x) \cdot \nabla \zeta(x)\, dx = 0 \quad \forall\ \zeta \in C_0^\infty(\mathbf{R}^n).$$

It is known that the orthogonal space $\mathcal{K}^\perp$ of $\mathcal{K}$ is exactly given by

$$\mathcal{K}^\perp = \{F \in \mathcal{H} \mid \operatorname{curl} F = 0 \text{ in the sense of distributions}\};$$

that is, $F \in \mathcal{K}^\perp$ if and only if $F \in L^2(\mathbf{R}^n; \mathbf{R}^n)$ and, for all $i, j = 1, 2, \ldots, n$,

$$\int_{\mathbf{R}^n} (F^i \zeta_{x_j} - F^j \zeta_{x_i})\, dx = 0 \quad \forall\ \zeta \in C_0^\infty(\mathbf{R}^n).$$

These spaces can be also easily characterized by the Fourier transforms on $\mathbf{R}^n$; see, e.g., [14]. For any dimension n, it is known [29, page 14] that each $F \in \mathcal{K}^\perp$ can be represented as $F = \nabla u$ for some function $u \in H^1_{loc}(\mathbf{R}^n)$; this function u is also unique in the space $\mathcal{X}$ introduced earlier. Note that if $n = 3$, then $G = \nabla \times M \in \mathcal{K}\ \forall\ M \in H^1(\mathbf{R}^3; \mathbf{R}^3)$.

Moreover, if Ω is a bounded domain with piecewise smooth boundary $\partial\Omega$, then $G \cdot \nu|_{\partial\Omega}$ can be well defined as an element in $H^{-1/2}(\partial\Omega)\ \forall\ G \in \mathcal{K}$; see [29, page 9].

2.4. The duality principle. Let p be the integral functional defined above which satisfies the stated conditions. We prove the following duality principle; this result actually follows from Fenchel's duality theorem [4, Theorem 2.5, Chapter 3, page 206]. However, we present a direct proof based on a more elementary approach.

THEOREM 2.2. *It follows that*

$$\min_{G\in\mathcal{K}} p^{\#}(G) = -\min_{F\in\mathcal{K}^{\perp}} p^{*}(F). \tag{2.11}$$

Moreover, $\bar{G}\in\mathcal{K}$ *is a minimizer of* $p^{\#}$ *if and only if* $\mathcal{K}^{\perp}\cap\partial p^{\#}(\bar{G})\neq\emptyset$; *in this case, any* $\bar{F}\in\mathcal{K}^{\perp}\cap\partial p^{\#}(\bar{G})$ *is a minimizer of* p^{*}. *Conversely,* $\bar{F}\in\mathcal{K}^{\perp}$ *is a minimizer of* p^{*} *if and only if* $\mathcal{K}\cap\partial p^{*}(\bar{F})\neq\emptyset$; *in this case, any* $\bar{G}\in\mathcal{K}\cap\partial p^{*}(\bar{F})$ *is a minimizer of* $p^{\#}$.

Proof. By the standard direct method of the calculus of variations, both minimization problems in (2.11) have minimizers. Let $\bar{G}\in\mathcal{K}$ be a minimizer of $p^{\#}$ over $\mathcal{K}$. Consider the functional

$$p_\epsilon(G) = p^{\#}(G) + \frac{1}{2}\|G-\bar{G}\|^2 + \frac{1}{2\epsilon}\|\mathcal{P}G\|^2,$$

where $\mathcal{P}\colon\mathcal{H}\to\mathcal{K}^{\perp}$ is the orthogonal projection. Standard methods show there exists a unique $G_\epsilon\in\mathcal{H}$ such that $p_\epsilon(G_\epsilon)\le p_\epsilon(G)\ \forall\ G\in\mathcal{H}$. Let $F_\epsilon = \bar{G}-G_\epsilon-\frac{1}{\epsilon}\mathcal{P}G_\epsilon$. We claim $F_\epsilon\in\partial p^{\#}(G_\epsilon)$. To see this, given any $G\in\mathcal{H}$ and $t\in\mathbf{R}$, let $h(t) = p^{\#}(G_\epsilon+tG)-p^{\#}(G_\epsilon)-F_\epsilon\cdot tG$. We need to show $h(1)\ge 0$. Note that h is convex and $h(0)=0$. Hence $h(t)\le h(1)t\ \forall\ 0\le t\le 1$. Therefore, by the minimality of p_ϵ at G_ϵ, it follows that

$$0\le p_\epsilon(G_\epsilon+tG)-p_\epsilon(G_\epsilon)\le h(1)t+\frac{t^2}{2}\|G\|^2+\frac{t^2}{2\epsilon}\|\mathcal{P}G\|^2 \quad \forall\ t\in(0,1);$$

this implies $h(1)\ge 0$, and hence $F_\epsilon\in\partial p^{\#}(G_\epsilon)$. Note that $\{G_\epsilon\}$ is bounded in $\mathcal{H}$. Assume $G_\epsilon\rightharpoonup\tilde{G}$ as $\epsilon\to 0$ via a subsequence. First, from

$$\frac{1}{2\epsilon}\|\mathcal{P}G_\epsilon\|^2\le p_\epsilon(G_\epsilon)-p^{\#}(G_\epsilon)\le p^{\#}(\bar{G})-p^{\#}(G_\epsilon)\le C<\infty,$$

we have $\mathcal{P}\tilde{G}=0$, and hence $\tilde{G}\in\mathcal{K}$. Hence, we have

$$\begin{aligned} p^{\#}(\bar{G}) &\le p^{\#}(\tilde{G})\le p^{\#}(\tilde{G})+\frac{1}{2}\|\tilde{G}-\bar{G}\|^2\\ &\le \liminf_{\epsilon\to 0}\left(p^{\#}(G_\epsilon)+\frac{1}{2}\|G_\epsilon-\bar{G}\|^2\right)\\ &\le \liminf_{\epsilon\to 0} p_\epsilon(G_\epsilon)\le p^{\#}(\bar{G}). \end{aligned}$$

From this we have $\bar{G}=\tilde{G}$ and $G_\epsilon\to\bar{G}$ strongly in $\mathcal{H}$. By (2.10) in Lemma 2.1 above, the sets $\{\partial p^{\#}(G_\epsilon)\}$ are uniformly bounded, and hence we assume $F_\epsilon\rightharpoonup\tilde{F}$, where F_ϵ is defined above. Note that, for all $G\in\mathcal{K}$, we have

$$\tilde{F}\cdot G=\lim_{\epsilon\to 0}F_\epsilon\cdot G=\lim_{\epsilon\to 0}(G_\epsilon-\bar{G})\cdot G=0,$$

and hence $\tilde{F}\in\mathcal{K}^{\perp}$. Moreover, for all $G\in\mathcal{H}$, one has $p^{\#}(G)\ge p^{\#}(G_\epsilon)+F_\epsilon\cdot(G-G_\epsilon)$. We thus infer that $\tilde{F}\in\partial p^{\#}(\bar{G})$, since $G_\epsilon\to\bar{G}$. This proves $\tilde{F}\in\partial p^{\#}(\bar{G})\cap\mathcal{K}^{\perp}\neq\emptyset$.

On the other hand, let $\bar F$ be any element in $\mathcal{K}^\perp \cap \partial p^\#(\bar G)$. Then, by (2.1), $p^*(\bar F) = (p^\#)^*(\bar F) = \bar G \cdot \bar F - p^\#(\bar G) = -p^\#(\bar G)$. Note that, for all $F \in \mathcal{K}^\perp$, it follows that $p^*(F) \geq \bar G \cdot F - p^\#(\bar G) = -p^\#(\bar G) = p^*(\bar F)$. Hence $\bar F$ is a minimizer of p^* over $\mathcal{K}^\perp$. This also proves the equality (2.11). The converse conclusion follows by the duality: $(p^\#)^* = p^*$, $(\mathcal{K}^\perp)^\perp = \mathcal{K}$. ∎

The dual functional p^* restricted to $\mathcal{K}^\perp$ is equivalent to the following convex functional of gradient type on the space $\mathcal{X}$:

$$l(u) = \int_{\mathbf{R}^n} \Psi^*(x, \nabla u(x))\, dx.$$

Therefore, the minimization of $p^\#$ on $\mathcal{K}$ reduces to minimization of the variational functional l on $\mathcal{X}$.

2.5. The relaxation principle. Let p be the integral functional defined by (2.3) above with Ψ satisfying the growth conditions (2.4) and (2.5). Using some techniques and results in [14, 23], we can establish the relaxation principle for minimization problems of p and $p^\#$ on the subspace $\mathcal{K}$.

THEOREM 2.3. *It follows that*

$$\inf_{G \in \mathcal{K}} p(G) = \min_{G \in \mathcal{K}} p^\#(G).$$

Proof. Let $\tilde G \in \mathcal{K}$ be a minimizer of $p^\#$ on $\mathcal{K}$. Given any $\delta > 0$, let $D = B_R(0)$ be a sufficiently large ball in $\mathbf{R}^n$ such that

$$\int_{D^c} \Psi(x, \tilde G)\, dx \leq \delta, \quad \int_D \Psi^\#(x, \tilde G)\, dx \leq p^\#(\tilde G) + \delta. \tag{2.12}$$

By (2.7), there exists a family of probability measures $\{\nu_x = \mu_{x, \tilde G(x)}\}_{x \in D}$ on $\mathbf{R}^n$ such that

$$\tilde G(x) = \int_{\mathbf{R}^n} \lambda d\nu_x(\lambda), \quad \Psi^\#(x, \tilde G(x)) = \int_{\mathbf{R}^n} \Psi(x, \lambda)\, d\nu_x(\lambda), \quad \text{a.e. } x \in D.$$

Note that $\mathrm{div}\tilde G = 0$ in $H^{-1}(D)$. From assumption (2.4) above, it also follows that

$$\int_D \int_{\mathbf{R}^n} |\lambda|^2\, d\nu_x(\lambda)\, dx < \infty.$$

Therefore, by [14, Example 4.5(b)] (see also [23, Theorem 10.10]), $\{\nu_x\}$ is generated by a 2-equi-integrable sequence of divergence-free fields $\{G_j\}$ in $L^2(D; \mathbf{R}^n)$; hence, $\mathrm{div} G_j = 0$ in $H^{-1}(D)$, $G_j \rightharpoonup \tilde G$ in $L^2(D; \mathbf{R}^n)$, and

$$\lim_{j\to\infty} \int_D \Psi(x, G_j) dx = \int_D \int_{\mathbf{R}^n} \Psi(x, \lambda)\, d\nu_x(\lambda)\, dx = \int_D \Psi^\#(x, \tilde G) dx. \tag{2.13}$$

Let $U_j(x) = \chi_D(x)\,(G_j(x) - \tilde G(x))$. Then $U_j \in \mathcal{H} = L^2(\mathbf{R}^n; \mathbf{R}^n)$. Clearly, $U_j \rightharpoonup 0$ in $L^2(\mathbf{R}^n; \mathbf{R}^n)$, and $\mathrm{div} U_j = 0$ in $H^{-1}(D)$. Moreover, the sequence $\{|U_j|^2\}$ is equi-integrable. Hence, by an argument similar to the proof of [23, Lemma 10.4], it follows that $\mathrm{div} U_j \to 0$ in $H^{-1}(\mathbf{R}^n)$. We now decompose $U_j = W_j + \nabla \zeta_j$ in $\mathcal{H}$, where $W_j \in \mathcal{K}$ and $V_j = \nabla \zeta_j \in \mathcal{K}^\perp$. Note that

$$\|U_j\|^2 = \|W_j\|^2 + \|\nabla \zeta_j\|^2,$$
$$\|\nabla \zeta_j\|^2 = \int_{\mathbf{R}^n} U_j \cdot \nabla \zeta_j\, dx = -\langle \mathrm{div} U_j, \zeta_j \rangle.$$

Hence $\|\nabla\zeta_j\| \le \|\mathrm{div} U_j\|_{H^{-1}(\mathbf{R}^n)} \to 0$ as $j \to \infty$. Note that, by (2.5),

$$|\Psi(x, W_j + \tilde{G}) - \Psi(x, U_j + \tilde{G})| \le c_4(|\tilde{G}| + |U_j| + |W_j| + c_5(x))\,|\nabla\zeta_j|.$$

Therefore $|p(W_j + \tilde{G}) - p(U_j + \tilde{G})| \to 0$ as $j \to \infty$. Since

$$p(U_j + \tilde{G}) = \int_D \Psi(x, G_j)\,dx + \int_{D^c} \Psi(x, \tilde{G})\,dx,$$

by (2.12) and (2.13), it follows that

$$\lim_{j\to\infty} p(U_j + \tilde{G}) = \int_D \Psi^{\#}(x, \tilde{G}) + \int_{D^c} \Psi(x, \tilde{G}) \le \min_{G\in\mathcal{K}} p^{\#}(G) + 2\delta.$$

Consequently,

$$\lim_{j\to\infty} p(W_j + \tilde{G}) \le \min_{G\in\mathcal{K}} p^{\#}(G) + 2\delta.$$

Since $W_j + \tilde{G} \in \mathcal{K}$, it follows finally that

$$\inf_{G\in\mathcal{K}} p(G) \le \min_{G\in\mathcal{K}} p^{\#}(G) + 2\delta$$

$\forall\ \delta > 0$. Hence $\inf_{G\in\mathcal{K}} p(G) = \min_{G\in\mathcal{K}} p^{\#}(G)$. This completes the proof. ∎

2.6. Existence of minimizers of *p*. We now give a necessary and sufficient condition for the existence of minimizers of functional p on $\mathcal{K}$ in terms of minimizers of p^* on $\mathcal{K}^\perp$.

THEOREM 2.4. *A function $G \in \mathcal{K}$ is a minimizer of p if and only if there exists a minimizer $F \in \mathcal{K}^\perp$ of p^* on $\mathcal{K}^\perp$ such that*

$$\Psi(x, G(x)) = F(x)\cdot G(x) - \Psi^*(x, F(x)) \quad \forall \text{ a.e. } x \in \mathbf{R}^n. \tag{2.14}$$

Proof. Note that $G \in \mathcal{K}$ is a minimizer of p if and only if G is a minimizer of $p^{\#}$ over $\mathcal{K}$ and satisfies $p(G) = p^{\#}(G)$. Note that $p(G) = p^{\#}(G)$ if and only if $\Psi(x, G(x)) = \Psi^{\#}(x, G(x))$ for a.e. $x \in \mathbf{R}^n$. By Theorem 2.2, $G \in \mathcal{K}$ is a minimizer of $p^{\#}$ if and only if there exists a minimizer $F \in \mathcal{K}^\perp$ of p^* belonging to $\partial p^{\#}(G)$, which, by (2.1) and Lemma 2.1, is equivalent to the pointwise condition $\Psi^{\#}(x, G(x)) = F(x)\cdot G(x) - \Psi^*(x, F(x))$ a.e. $x \in \mathbf{R}^n$; together with $\Psi(x, G(x)) = \Psi^{\#}(x, G(x))$, this is equivalent to the pointwise relation (2.14) above. ∎

3. The duality method for micromagnetics. As in the introduction, assume that $\varphi\colon \mathbf{S}^{n-1} \to \mathbf{R}$ is a given function representing the anisotropy energy density, Ω is a given bounded domain with piecewise smooth boundary occupied by the ferromagnetic material, and $H \in L^2(\Omega; \mathbf{R}^n)$ is a given applied magnetic field.

Consider the micromagnetic energy introduced above:

$$I(m) = \int_\Omega [\varphi(m(x)) - H(x)\cdot m(x)]\,dx + \frac{1}{2}\int_{\mathbf{R}^n} |F_m|^2\,dx,$$

where $F_m \in \mathcal{K}^\perp$ is defined by Maxwell's equation (1.3) above.

As before, we consider the important convex function defined by (1.6)

$$\Phi(\eta) = \sup_{h\in\mathbf{S}^{n-1}} (h\cdot\eta - \varphi(h))$$

and the set $\Sigma(\eta) = \{h \in \mathbf{S}^{n-1} \mid \eta \cdot h - \varphi(h) = \Phi(\eta)\}$. Then Φ is Lipschitz continuous and satisfies

$$|\Phi(\eta) - \Phi(\lambda)| \leq |\eta - \lambda|. \tag{3.1}$$

We also have the following useful result.

LEMMA 3.1. *Let $\partial\Phi(\eta)$ denote the subdifferential set of Φ at η. Then $\partial\Phi(\eta) \subseteq \mathbf{B}^{n-1}$, the closed unit ball in $\mathbf{R}^n$, and $\Sigma(\eta) = \mathbf{S}^{n-1} \cap \partial\Phi(\eta)\ \forall\ \eta \in \mathbf{R}^n$. Moreover, $\xi \in \Sigma(\eta)$ if and only if $|\xi| = 1$ and $\Phi(\eta+\xi) = \Phi(\eta) + 1$.*

Proof. By the Lipschitz condition (3.1), it follows easily that $\partial\Phi(\eta) = \{\xi \in \mathbf{R}^n \mid \Phi(\lambda) \geq \Phi(\eta) + \xi \cdot (\lambda - \eta)\ \forall\, \lambda \in \mathbf{R}^n\} \subseteq \mathbf{B}^{n-1}$. Now let $\xi \in \Sigma(\eta)$. Then $\Phi(\eta) = \eta \cdot \xi - \varphi(\xi)$, and, hence, for all $\lambda \in \mathbf{R}^n$, it follows that $\Phi(\lambda) \geq \lambda \cdot \xi - \varphi(\xi) = \Phi(\eta) + \xi \cdot (\lambda - \eta)$, which shows that $\xi \in \partial\Phi(\eta)$, and hence $\Sigma(\eta) \subseteq \mathbf{S}^{n-1} \cap \partial\Phi(\eta)$. To show the opposite inclusion, assume $\xi \in \mathbf{S}^{n-1} \cap \partial\Phi(\eta)$. Then $|\xi| + \Phi(\eta) \geq \Phi(\xi + \eta) \geq \Phi(\eta) + \xi \cdot \xi$. Since $|\xi| = 1$, these inequalities imply $\Phi(\xi + \eta) = \Phi(\eta) + 1$. Let $h \in \Sigma(\xi + \eta)$. Then $\Phi(\xi + \eta) = (\xi + \eta) \cdot h - \varphi(h)$, and $\Phi(\eta) \geq \eta \cdot h - \varphi(h) = \Phi(\xi + \eta) - \xi \cdot h$. Hence $\Phi(\eta) + 1 = \Phi(\xi + \eta) \leq \Phi(\eta) + \xi \cdot h \leq \Phi(\eta) + 1$, and thus equality holds everywhere. This implies $\xi \cdot h = 1$, and hence $\xi = h \in \mathbf{S}^{n-1}$ and $\Phi(\eta) = \Phi(\xi+\eta) - \xi \cdot h = \eta \cdot \xi - \varphi(\xi)$, which shows that $\xi \in \Sigma(\eta)$. Hence $\Sigma(\eta) = \mathbf{S}^{n-1} \cap \partial\Phi(\eta)$. The proof also establishes the last statement of the result. ∎

Remark 3.1. (i) From this result, we see that if Φ is differentiable at a point η_0, then $\Sigma(\eta_0) = \partial\Phi(\eta_0) = \{\Phi'(\eta_0)\}$.

(ii) We also remark that if $\xi \in \Sigma(\eta)$, then $\partial\Phi(\xi + \eta) = \Sigma(\xi + \eta) = \{\xi\}$. To see this, given any $h \in \partial\Phi(\xi + \eta)$, by the monotonicity of set-valued function $\lambda \mapsto \partial\Phi(\lambda)$, we have $(h - \xi) \cdot \xi \geq 0$; hence $h \cdot \xi \geq 1$, which implies $h = \xi$.

Recall that $\psi(x, \xi) = \frac{1}{2}(|\xi|^2 + 1) - \Phi(\xi + H(x))$ and that the energy $J(G)$ defined before can be written as

$$J(G) = \int_{\mathbf{R}^n} \Psi(x, G(x))\, dx, \tag{3.2}$$

where the density function Ψ is given by

$$\Psi(x, \xi) = \chi_\Omega(x)\psi(x, \xi) + \frac{1}{2}\chi_{\Omega^c}(x)|\xi|^2. \tag{3.3}$$

We first have the following elementary but important result; its proof can been found in [25, Proposition 2.1].

PROPOSITION 3.2. *It always follows that*

$$\inf_{m \in L^\infty(\Omega; \mathbf{S}^{n-1})} I(m) = \inf_{G \in \mathcal{K}} J(G). \tag{3.4}$$

Moreover, if $\bar{m} \in L^\infty(\Omega; \mathbf{S}^{n-1})$ is a minimizer of I, then the function $G_{\bar{m}} \in \mathcal{K}$ determined uniquely by $\bar{m}\chi_\Omega - G_{\bar{m}} \in \mathcal{K}^\perp$ is a minimizer of J; conversely, if $\bar{G} \in \mathcal{K}$ is a minimizer of J, then any function $\bar{m} \in L^\infty(\Omega; \mathbf{S}^{n-1})$ satisfying $\bar{m}(x) \in \Sigma(\bar{G}(x) + H(x))$ a.e. $x \in \Omega$ is a minimizer of I.

It is easily seen that $\Psi(x, \xi)$ satisfies the growth conditions (2.4) and (2.5) above and that

$$\Psi^*(x, \lambda) = \chi_\Omega(x)\, \psi^*(x, \lambda) + \frac{1}{2}\chi_{\Omega^c}(x)\, |\lambda|^2, \tag{3.5}$$

$$\Psi^\#(x, \lambda) = \chi_\Omega(x)\, \psi^\#(x, \lambda) + \frac{1}{2}\chi_{\Omega^c}(x)\, |\lambda|^2. \tag{3.6}$$

Therefore,

$$J^{\#}(G) = \int_{\Omega} \psi^{\#}(x, G(x))\, dx + \frac{1}{2} \int_{\Omega^c} |G(x)|^2\, dx, \tag{3.7}$$

$$J^{*}(G) = \int_{\Omega} \psi^{*}(x, G(x))\, dx + \frac{1}{2} \int_{\Omega^c} |G(x)|^2\, dx. \tag{3.8}$$

We now compute the conjugate function $\psi^*(x,\lambda)$ for function $\psi(x,\xi) = \frac{1}{2}(|\xi|^2 + 1) - \Phi(\xi + H(x))$. By the definition of Φ, we compute that

$$\begin{aligned}
\psi^*(x,\lambda) &= \sup_{\xi \in \mathbf{R}^n} \{\lambda \cdot \xi - \psi(x,\xi)\} \\
&= \sup_{\xi \in \mathbf{R}^n} \left\{ \lambda \cdot \xi - \frac{1}{2}(|\xi|^2 + 1) + \Phi(\xi + H(x)) \right\} \\
&= \sup_{\xi \in \mathbf{R}^n} \left\{ \lambda \cdot \xi - \frac{1}{2}|\xi|^2 - \frac{1}{2} + \max_{h \in \mathbf{S}^{n-1}} \{(\xi + H(x)) \cdot h - \varphi(h)\} \right\} \\
&= \sup_{\xi \in \mathbf{R}^n} \left[\max_{h \in \mathbf{S}^{n-1}} \left\{ (\lambda + h) \cdot \xi - \frac{1}{2}|\xi|^2 - \frac{1}{2} + H(x) \cdot h - \varphi(h) \right\} \right] \\
&= \sup_{h \in \mathbf{S}^{n-1}} \left[\sup_{\xi \in \mathbf{R}^n} \left\{ (\lambda + h) \cdot \xi - \frac{1}{2}|\xi|^2 - \frac{1}{2} + H(x) \cdot h - \varphi(h) \right\} \right] \\
&= \sup_{h \in \mathbf{S}^{n-1}} \left[\frac{1}{2}|\lambda + h|^2 - \frac{1}{2} + H(x) \cdot h - \varphi(h) \right] \\
&= \sup_{h \in \mathbf{S}^{n-1}} \left[\frac{1}{2}|\lambda|^2 + (\lambda + H(x)) \cdot h - \varphi(h) \right] \\
&= \frac{1}{2}|\lambda|^2 + \Phi(\lambda + H(x)).
\end{aligned}$$

Therefore the functional J^* restricted to the gradient fields reduces to the following functional:

$$L(u) = \int_{\Omega} \Phi(\nabla u(x) + H(x))\, dx + \frac{1}{2} \int_{\mathbf{R}^n} |\nabla u(x)|^2\, dx, \tag{3.9}$$

which is defined earlier in the introduction.

We now have the following important result on the functional $L(u)$.

Theorem 3.3. *There exist unique functions $\bar{u} \in \mathcal{X}$ and $\bar{F} = \nabla \bar{u} \in \mathcal{K}^{\perp}$ such that*

$$L(\bar{u}) = \min_{u \in \mathcal{X}} L(u) = \min_{F \in \mathcal{K}^{\perp}} J^*(F) = J^*(\bar{F}).$$

Moreover, there exists unique boundary data $\bar{g} \in H^{1/2}(\partial\Omega)$ with $\int_{\partial\Omega} \bar{g}\, dS = 0$ such that $\bar{u}$ is given by $\bar{u} = \chi_{\Omega}\bar{v} + \chi_{\Omega^c}\bar{w}$, where $\bar{w} = \omega(\bar{g})$ is the unique solution to the Dirichlet problem

$$\Delta \bar{w} = 0 \quad \text{in } \Omega^c, \quad \bar{w}|_{\partial\Omega} = \bar{g}, \tag{3.10}$$

and $\bar{v}$ is the unique minimizer of the variational problem

$$\int_{\Omega} \psi^*(x, \nabla \bar{v})\, dx = \min_{v \in H^1(\Omega),\, v|_{\partial\Omega} = \bar{g}} \int_{\Omega} \psi^*(x, \nabla v)\, dx. \tag{3.11}$$

Proof. The uniqueness of $\bar{u}$ follows from the fact that L is *strictly* convex on $\mathcal{X}$. The existence and characterization of minimizers follow from the general results proved in [25]. For example, the existence of $\bar{u}$ can be proved by the standard direct method of calculus of variations as in the proof of [25, Proposition 2.1], the existence of unique $\bar{g}$ follows from [25, Proposition 2.2], and the characterization of $\bar{u}$ in terms of $\bar{g}$ is obtained similarly as in [25, Theorem 2.5]. ∎

Note that the convex functional $J^\#$ may not have a unique minimizer on $\mathcal{K}$. However, in terms of the unique boundary data $\bar{g}$ and function $\bar{u} = \chi_\Omega \bar{v} + \chi_{\Omega^c}\bar{w} \in \mathcal{X}$ determined in Theorem 3.3, we have the following characterization of minimizers of $J^\#$.

THEOREM 3.4. *A function $\bar{G} \in \mathcal{K}$ is a minimizer of $J^\#$ on $\mathcal{K}$ if and only if $\bar{G} = \chi_\Omega(x)\tilde{G}(x) + \chi_{\Omega^c}(x)\nabla\bar{w}$, where $\tilde{G} \in L^2(\Omega;\mathbf{R}^n)$ is any function satisfying*

$$\begin{cases} \operatorname{div}\tilde{G} = 0 & \text{in } H^{-1}(\Omega), \\ \tilde{G}\cdot\nu = \frac{\partial\bar{w}}{\partial\nu} & \text{on } \partial\Omega, \\ \tilde{G}(x) \in \partial\psi^*(x,\nabla\bar{v}(x)) & \text{a.e. } x\in\Omega. \end{cases} \tag{3.12}$$

Proof. Since J^* has the unique minimizer $\bar{F} = \nabla\bar{u}$ on $\mathcal{K}^\perp$, by Theorem 2.2, $\bar{G} \in \mathcal{K}$ is a minimizer of $J^\#$ on $\mathcal{K}$ if and only if $\bar{G} \in \partial J^*(\bar{F})$. By Lemma 2.1, this last condition is equivalent to $\bar{G}(x) \in \partial\Psi^*(x,\bar{F}(x))$ for a.e. $x \in \mathbf{R}^n$, which gives $\bar{G}(x) = \bar{F}(x) = \nabla\bar{w}(x)$ for a.e. $x \in \Omega^c$ and $\bar{G}(x) \in \partial\psi^*(x,\nabla\bar{v}(x))$ for a.e. $x \in \Omega$. The first two conditions in (3.12) are equivalent to $G = \chi_\Omega(x)\tilde{G}(x) + \chi_{\Omega^c}(x)\nabla\bar{w} \in \mathcal{K}$. This proves the theorem. ∎

From Theorems 2.4 and 3.4, we see that a function $\bar{G} \in \mathcal{K}$ is a minimizer of J on $\mathcal{K}$ if and only if $\bar{G} = \chi_\Omega(x)\tilde{G}(x) + \chi_{\Omega^c}(x)\nabla\bar{w}$, where $\tilde{G} \in L^2(\Omega;\mathbf{R}^n)$ is any function satisfying

$$\begin{cases} \operatorname{div}\tilde{G} = 0 & \text{in } H^{-1}(\Omega), \\ \tilde{G}\cdot\nu = \frac{\partial\bar{w}}{\partial\nu} & \text{on } \partial\Omega, \\ \tilde{G}(x) \in \partial\psi^*(x,\nabla\bar{v}(x)) & \text{a.e. } x\in\Omega, \\ \psi(x,\tilde{G}(x)) = \psi^\#(x,\tilde{G}(x)) & \text{a.e. } x\in\Omega. \end{cases} \tag{3.13}$$

To study the last two pointwise conditions in (3.13), given $x \in \Omega$, $\xi \in \mathbf{R}^n$, let $A(x,\xi) = \{\eta \in \mathbf{R}^n \mid \eta \in \partial\psi^*(x,\xi),\ \psi(x,\eta) = \psi^\#(x,\eta)\}$.

LEMMA 3.5. *$A(x,\xi) = \xi + \Sigma(\xi + H(x))$.*

Proof. Since $\eta \in \partial\psi^*(x,\xi)$ if and only if $\psi^\#(x,\eta) = \eta\cdot\xi - \psi^*(x,\xi)$, we easily see that $\eta \in A(x,\xi)$ if and only if $\psi(x,\eta) = \eta\cdot\xi - \psi^*(x,\xi)$. Since $\psi(x,\eta) = \frac{1}{2}(|\eta|^2 + 1) - \Phi(\eta + H(x))$ and $\psi^*(x,\xi) = \frac{1}{2}|\xi|^2 + \Phi(\xi + H(x))$, it follows that $\eta \in A(x,\xi)$ if and only if $\Phi(\eta + H(x)) - \Phi(\xi + H(x)) = \frac{1}{2}(|\eta - \xi|^2 + 1)$. However, by (3.1), $|\Phi(\eta + H(x)) - \Phi(\xi + H(x))| \le |\eta - \xi|\ \forall\ \eta, \xi \in \mathbf{R}^n$; therefore $\eta \in A(x,\xi)$ if and only if $\Phi(\eta + H(x)) - \Phi(\xi + H(x)) = |\eta - \xi| = 1$, which, by Lemma 3.1, is equivalent to $\eta - \xi \in \Sigma(\xi + H(x))$; that is, $\eta \in \xi + \Sigma(\xi + H(x))$. The proof is complete. ∎

Finally, we obtain the main theorem, Theorem 1.2, stated in the introduction.

THEOREM 3.6. *Let $\bar{u} = \chi_\Omega\bar{v} + \chi_{\Omega^c}\bar{w} \in \mathcal{X}$ be the minimizer of functional L defined above by* (3.9). *Then J has a minimizer on $\mathcal{K}$ if and only if there exists a function $\tilde{G} \in L^2(\Omega;\mathbf{R}^n)$ which satisfies*

$$\begin{cases} \operatorname{div}\tilde{G} = 0 & \text{in } H^{-1}(\Omega), \\ \tilde{G}\cdot\nu = \frac{\partial\bar{w}}{\partial\nu} & \text{on } \partial\Omega, \\ \tilde{G}(x) \in \nabla\bar{v}(x) + \Sigma(\nabla\bar{v}(x) + H(x)) & \text{a.e. } x\in\Omega. \end{cases} \tag{3.14}$$

In this case, $\bar{G} = \chi_\Omega(x)\tilde{G} + \chi_{\Omega^c}(x)\nabla\bar{w}$ *is a minimizer of* J *on* $\mathcal{K}$*; moreover,* $\bar{m}(x) = \tilde{G}(x) - \nabla\bar{v}(x)$ *a.e. on* Ω *is a minimizer of energy* I.

Proof. The condition (3.14) follows from condition (3.13) and Lemma 3.5. By Remark 3.1(ii), if $\tilde{G}(x) \in \nabla\bar{v}(x) + \Sigma(\nabla\bar{v}(x) + H(x))$, then $\Sigma(\tilde{G}(x) + H(x)) = \{\tilde{G}(x) - \nabla\bar{v}(x)\}$. Finally, by Proposition 3.2, $\bar{m}(x) = \tilde{G}(x) - \nabla\bar{v}(x)$ is a minimizer of I. ∎

As in the introduction, define $G(x) = \tilde{G}(x) - \nabla\tilde{w}(x)$, where $\tilde{w} \in H^1(\Omega)$ is the unique solution to the Neumann problem

$$\Delta\tilde{w} = 0 \text{ in } \Omega, \quad \frac{\partial\tilde{w}}{\partial\nu} = \frac{\partial\bar{w}}{\partial\nu} \text{ on } \partial\Omega, \quad \int_{\partial\Omega} \tilde{w}\, dS = 0. \tag{3.15}$$

Then define the set-valued function $\mathbf{S}(x) = \nabla\bar{v}(x) - \nabla\tilde{w}(x) + \Sigma(\nabla\bar{v}(x) + H(x))$ as in (1.13) above. Therefore, condition (3.14) is equivalent to the following condition for function $G \in L^2(\Omega; \mathbf{R}^n)$:

$$\begin{cases} \operatorname{div}(G\chi_\Omega) = 0 & \text{on } \mathbf{R}^n; \\ G(x) \in \mathbf{S}(x) & \text{a.e. } x \in \Omega. \end{cases} \tag{3.16}$$

The constrained problem (3.16) for divergence-free fields with constant set $\mathbf{S}(x) = \mathbf{S}$ has been recently studied by many authors; see, e.g., [3, 7, 8, 9, 16]. For example, the following result has been proved in [3, 8].

THEOREM 3.7. (cf. [3, Theorem 4.15]; [8, Theorem 6.2]) *Let* $n = 3$*, let* Ω *be any bounded open set in* $\mathbf{R}^3$*, and assume* $\mathbf{S}(x) = \mathbf{S}$ *is any constant bounded set in* $\mathbf{R}^3$*. Then problem* (3.16) *has a solution if and only if either* $0 \in \mathbf{S}$ *or there exists a subset* $\mathbf{F} \subseteq \mathbf{S}$ *such that* $\dim(\operatorname{span}\mathbf{F}) \geq 2$ *and* $0 \in \operatorname{ri}(\operatorname{con}\mathbf{F})$*. Moreover, in this case, a solution* G *can be obtained by* $G = \nabla \times \omega$ *with some* $\omega \in W_0^{1,\infty}(\Omega; \mathbf{R}^3)$.

Remark 3.2. For certain *regular* (e.g., piecewise constant or smooth) nonconstant sets $\mathbf{S}(x)$, we expect a similar condition as above on the set $\mathbf{S}(x)$ for each fixed $x \in \Omega$ to be sufficient for the solvability of (3.16). However, without certain regularity assumption on sets $\mathbf{S}(x)$, such a pointwise condition is not sufficient for the solvability of (3.16). The following counterexample uses an idea in [21].

Example 3.1. Let G be an open dense set of $[0,1]$, with $0 < |G| < 1$. Let $\Omega = (0,1)^3 \subset \mathbf{R}^3$ and $\Omega_0 = G \times G \times (0,1)$. Then Ω_0 is open and dense in Ω, and $|\Omega_0| = |G|^2 \in (0,1)$. Let $\alpha_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$, $\alpha_2 = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$, $\alpha_3 = (0, -1, 0)$ be three unit vectors in $\mathbf{R}^3$. Define the set-valued function $\mathbf{S}(x)$ as follows:

$$\mathbf{S}(x) = \{\alpha_1, \alpha_2, \alpha_3\},\ x \in \Omega_0; \quad \mathbf{S}(x) = \{3\alpha_1, 3\alpha_2, 3\alpha_3\},\ x \in \Omega \setminus \Omega_0.$$

Then, for each fixed $x \in \Omega$, we have that $\dim(\operatorname{span}(\mathbf{S}(x)) = 2$ and $0 \in \operatorname{ri}(\operatorname{con}\mathbf{S}(x))$. But we claim that the following problem

$$\operatorname{div}(G\chi_\Omega) = 0 \quad \text{on } \mathbf{R}^3, \quad G(x) \in \mathbf{S}(x) \quad \text{a.e. } x \in \Omega$$

does not have any solution. To see this, note that any solution G must be given by $G = (G_1, G_2, 0)$ and for a.e. $x_3 \in (0,1)$ the function $G'(x') = (G_1(x', x_3), G_2(x', x_3))$ with $x' \in \Omega' = (0,1) \times (0,1)$ solves

$$\operatorname{div}(G'\chi_{\Omega'}) = 0 \quad \text{on } \mathbf{R}^2, \quad G'(x') \in \mathbf{S}(x') \quad \text{a.e. } x' \in \Omega',$$

where $\mathbf{S}(x')$ is the projection of $\mathbf{S}(x)$ to $\mathbf{R}^2$. Note also that any solution G' must be given by $G' = (\nabla u)^\perp$ for some function $u \in W^{1,\infty}(\Omega')$. This function u will satisfy the Eikonal equation

$$|\nabla u(x')| = f(x'),$$

where $f(x') = 1$ on $\Omega'_0 = G \times G$ and $f(x') = 3$ on $\Omega' \setminus \Omega'_0$. However, it is shown in Müller–Sychev's paper (see [21, page 462]) that such a Lipschitz solution u in Ω' does not exist because u would have to satisfy $|\nabla u(x')| \le 2$ for a.e. $x' \in \Omega'$.

However, there are some cases in micromagnetics where the set $\mathbf{S}(x)$ involved is constant. In the next section, we will see that for ellipsoid domains Ω and constant applied fields H this is always the case. The special case considered in [8, Theorem 6.2] concerns arbitrary domains but constant applied fields. In that case, it is assumed that 0 is in the convex hull of set $Z = \Sigma(H)$, and thus $0 \in \partial\Phi(H)$, which implies that Φ has the absolute minimum at H. Therefore, by the definition of energy $L(u)$, $\bar{u} \equiv 0$ is the unique minimizer of L on $\mathcal{X}$; hence, $\mathbf{S}(x)$ in condition (3.16) becomes constant $\mathbf{S}(x) = Z = \Sigma(H)$. However, this special case with the assumption that $0 \in \mathrm{con}\Sigma(H)$ seems to only work for the case $H = (h^1, h^2, h^3)$ with $h^i = 0$ for at least one $i \in \{1, 2, 3\}$ in most of the micromagnetics examples, as has been already pointed out in [8, Proposition 6.6 and Remark 6.7].

4. Ferromagnetism of ellipsoid domains. To illustrate the above results in some special cases, we assume the domain Ω is an ellipsoid and assume the applied field H is constant.

Let $\Omega = \{x \in \mathbf{R}^n \mid \sum_{i=1}^n x_i^2/a_i < 1\}$, where $a_i > 0$ are some constants. Define positive numbers

$$b_i = \frac{1}{2}\int_0^\infty \frac{\sqrt{a_1 \cdots a_n}\, dt}{(a_i + t)\sqrt{(a_1 + t)\cdots(a_n + t)}}. \tag{4.1}$$

Note that when Ω is the unit ball in $\mathbf{R}^n$, all b_i's are equal to $\frac{1}{n}$.

The following result is well known in the potential theory (see, e.g., [17, pages 192–194]); a proof is given in the appendix below for the convenience of the readers.

THEOREM 4.1. *For each $k = 1, 2, \ldots, n$, the Dirichlet–Neumann problem*

$$\Delta w = 0 \quad in\ \Omega^c, \quad w|_{\partial\Omega} = x_k, \quad \frac{\partial w}{\partial \nu}\Big|_{\partial\Omega} = \left(1 - \frac{1}{b_k}\right)\nu_k, \tag{4.2}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the unit normal on $\partial\Omega$ pointing outward of Ω, has a unique solution $w = w_k$ that satisfies $w \in H^1_{loc}(\Omega^c)$ and $|\nabla w| \in L^2(\Omega^c)$.

This theorem has the following important application closely related to the micromagnetics problem.

LEMMA 4.2. *Let $w_1, \ldots, w_n$ be the functions determined in the previous theorem. For each $k = 1, 2, \ldots, n$, let*

$$u_k(x) = \begin{cases} b_k x_k & x \in \Omega, \\ b_k w_k(x) & x \in \Omega^c. \end{cases}$$

Then $u_k \in H^1_{loc}(\mathbf{R}^n)$, $F^k = \nabla u_k \in L^2(\mathbf{R}^n; \mathbf{R}^n)$, and F^k solves the simplified Maxwell equation

$$\mathrm{curl} F^k = 0, \quad \mathrm{div}(-F^k + e^k \chi_\Omega) = 0 \quad in\ \mathbf{R}^n, \quad k = 1, 2, \ldots, n, \tag{4.3}$$

where $e^k = (0, \ldots, 1, \ldots, 0)$ (only the kth place is 1) is the standard basis vector.

Proof. That $u_k \in H^1_{loc}(\mathbf{R}^n)$ (in fact $u_k \in \mathcal{X}$, the space defined above) follows from the continuity condition $w_k|_{\partial\Omega} = x_k$. Hence $F^k = \nabla u_k = b_k e^k \chi_\Omega + b_k \nabla w_k \chi_{\Omega^c} \in L^2(\mathbf{R}^n; \mathbf{R}^n)$. To prove (4.3), it suffices to verify the divergence equation; that is,

$$\int_{\mathbf{R}^n} F^k \cdot \nabla\phi(x)\, dx = \int_\Omega \phi_{x_k}(x)\, dx \quad \forall\, \phi \in C_0^\infty(\mathbf{R}^n).$$

Note that $F^k = b_k e^k$ on Ω, w_k is harmonic on Ω^c and $\frac{\partial w_k}{\partial \nu}|_{\partial\Omega} = (1 - \frac{1}{b_k})\nu_k$; hence it follows by the divergence theorem that

$$\begin{aligned}\int_{\mathbf{R}^n} F^k \cdot \nabla\phi(x)\, dx &= \int_\Omega b_k \phi_{x_k}(x)\, dx + \int_{\Omega^c} b_k \nabla w_k \cdot \nabla\phi(x)\, dx \\ &= \int_{\partial\Omega} b_k \phi \nu_k\, dS - \int_{\partial\Omega} b_k \phi \frac{\partial w_k}{\partial \nu}\, dS \\ &= \int_{\partial\Omega} b_k \phi \nu_k\, dS - \int_{\partial\Omega} b_k \phi \left(1 - \frac{1}{b_k}\right) \nu_k\, dS \\ &= \int_{\partial\Omega} \phi \nu_k\, dS = \int_\Omega \phi_{x_k}(x)\, dx.\end{aligned}$$

This completes the proof. ∎

From this result, it follows that for each *constant* magnetization $m = (m_1, \ldots, m_n)$ the induced magnetic field F_m determined by the simplified Maxwell equation

$$\text{curl} F_m = 0, \quad \text{div}(-F_m + m\chi_\Omega) = 0 \text{ in } \mathbf{R}^n \tag{4.4}$$

is given by $F_m = \sum_{k=1}^n m_k F^k$ (F^k defined above) and thus remains also a constant $F_m = (b_1 m_1, b_2 m_2, \ldots, b_n m_n)$ on Ω. Therefore the matrix $D = \text{diag}(b_1, \ldots, b_n)$ is usually called the *demagnetizing matrix* for ellipsoid Ω in the literature [5, 6].

As above, we consider the functional

$$L(u) = \int_\Omega \Phi(\nabla u(x) + H)\, dx + \frac{1}{2} \int_{\mathbf{R}^n} |\nabla u(x)|^2\, dx,$$

where Φ is the convex function defined above and H is a constant. We expect the unique minimizer $\bar{u}$ of L on the space $\mathcal{X}$ defined before to have a constant gradient $\nabla \bar{u} = \bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n)$ on Ω; then this $\bar{u}$ must be of the form

$$\bar{u}(x) = (\bar{\lambda} \cdot x)\chi_\Omega(x) + \left(\sum_{k=1}^n \bar{\lambda}_k w_k(x)\right) \chi_{\Omega^c}(x), \tag{4.5}$$

where functions w_k ($k = 1, \ldots, n$) are defined by (4.2). This is indeed the case; we have the following result.

LEMMA 4.3. *Let $\bar{\lambda} \in \mathbf{R}^n$ be the unique minimum point of the strictly convex function $f(\lambda) = \Phi(\lambda + H) + \frac{1}{2}\sum_{k=1}^n \frac{\lambda_k^2}{b_k}$. Then $\bar{u}$ defined by* (4.5) *is the unique minimizer of L on $\mathcal{X}$.*

Proof. Again let $\psi^*(\lambda) = \Phi(\lambda + H) + \frac{1}{2}|\lambda|^2$. Then $\bar{\lambda}$ is the unique minimum point of $\psi^*(\lambda) - \frac{1}{2}\sum_{k=1}^n (1 - \frac{1}{b_k})\lambda_k^2$. Hence (see, e.g., [25, Lemma 2.12])

$$\psi^*(\bar{\lambda} + \eta) \geq \psi^*(\bar{\lambda}) + \bar{q} \cdot \eta \quad \forall\, \eta \in \mathbf{R}^n, \tag{4.6}$$

where $\bar{q} = (q_1, \ldots, q_n)$ with $q_k = (1 - \frac{1}{b_k})\bar{\lambda}_k$. Let $\bar{u}$ be defined by (4.5) using this $\bar{\lambda}$. Note that, by (4.2), one has $\frac{\partial \bar{u}}{\partial \nu}|_{\partial(\Omega^c)} = \bar{q} \cdot \nu$. Therefore, for all $\zeta \in \mathcal{X}$, by (4.6),

$$\begin{aligned} L(\bar{u} + \zeta) &= \int_\Omega \psi^*(\nabla \bar{u} + \nabla \zeta) + \frac{1}{2} \int_{\Omega^c} |\nabla \bar{u} + \nabla \zeta|^2 \\ &= \int_\Omega \psi^*(\nabla \bar{u} + \nabla \zeta) + \frac{1}{2} \int_{\Omega^c} \left(|\nabla \bar{u}|^2 + |\nabla \zeta|^2\right) + \int_{\Omega^c} \nabla \bar{u} \cdot \nabla \zeta \\ &\geq \int_\Omega [\psi^*(\nabla \bar{u}) + \bar{q} \cdot \nabla \zeta] + \frac{1}{2} \int_{\Omega^c} \left(|\nabla \bar{u}|^2 + |\nabla \zeta|^2\right) - \int_{\partial(\Omega^c)} \left(\frac{\partial \bar{u}}{\partial \nu}\right) \zeta \\ &= L(\bar{u}) + \frac{1}{2} \int_{\Omega^c} |\nabla \zeta|^2 + \int_{\partial \Omega} (\bar{q} \cdot \nu)\, \zeta - \int_{\partial(\Omega^c)} (\bar{q} \cdot \nu)\, \zeta \\ &= L(\bar{u}) + \frac{1}{2} \int_{\Omega^c} |\nabla \zeta|^2 \geq L(\bar{u}), \end{aligned}$$

which completes the proof. ∎

Note that the Neumann problem (3.15) in this case becomes

$$\Delta \tilde{w} = 0 \text{ in } \Omega, \quad \frac{\partial \tilde{w}}{\partial \nu} = \bar{q} \cdot \nu \text{ on } \partial\Omega, \quad \int_{\partial\Omega} \tilde{w}\, dS = 0, \tag{4.7}$$

where constant $\bar{q} = \bar{\lambda} - D^{-1}\bar{\lambda}$ is defined in the previous proof and $D = \operatorname{diag}(b_1, \ldots, b_n)$ is the demagnetizing matrix defined above; this problem has unique solution $\tilde{w}(x) = \bar{q} \cdot x$ in $H^1(\Omega)$. Therefore, the important set-valued function $\mathbf{S}(x) = \nabla \bar{v}(x) - \nabla \tilde{w}(x) + \Sigma(\nabla \bar{v}(x) + H)$ defined above becomes a constant set:

$$\mathbf{S}(x) = \mathbf{S}_H = \bar{\lambda} - \bar{q} + \Sigma(\bar{\lambda} + H) = D^{-1}\bar{\lambda} + \Sigma(\bar{\lambda} + H). \tag{4.8}$$

It is easily seen that the unique $\bar{\lambda}$ is also uniquely determined by the condition

$$-D^{-1}\bar{\lambda} \in \partial\Phi(\bar{\lambda} + H). \tag{4.9}$$

Therefore, by Theorem 3.6, existence problems for micromagnetics on the ellipsoid Ω are then equivalent to the problem of finding functions $G \in L^2(\Omega; \mathbf{R}^n)$ that solve the following relation:

$$\begin{cases} \operatorname{div}(G\chi_\Omega) = 0 & \text{on } \mathbf{R}^n, \\ G(x) \in \mathbf{S}_H & \text{a.e. } x \in \Omega \end{cases} \tag{4.10}$$

for $\mathbf{S}_H = D^{-1}\bar{\lambda} + \Sigma(\bar{\lambda} + H)$, where $\bar{\lambda} = \lambda(H)$ is determined by (4.9).

From Theorem 3.7, we obtain the following result on the existence of minimizers of micromagnetic energy for ellipsoids.

THEOREM 4.4. *Let $n = 3$ and $\Omega \subset \mathbf{R}^3$ be the ellipsoid with demagnetizing matrix D defined above. For a given constant H, let $\bar{\lambda}$ be determined by* (4.9). *Then problem* (4.10) *has solutions if and only if either $-D^{-1}\bar{\lambda} \in \Sigma(\bar{\lambda} + H)$ or there exists a subset $\mathbf{F} \subseteq \Sigma(\bar{\lambda} + H)$ such that* $\dim(\operatorname{span}\mathbf{F}) \geq 2$ *and* $-D^{-1}\bar{\lambda} \in \operatorname{ri}(\operatorname{con}\mathbf{F})$; *in the case where $-D^{-1}\bar{\lambda} \in \Sigma(\bar{\lambda} + H)$, it has only the trivial solution $G \equiv 0$, and in all other cases it has infinitely many solutions. Moreover, in all these cases, the micromagnetic energy has all minimizers m given by $m(x) = G(x) - D^{-1}\bar{\lambda}$ on Ω with G a solution of* (4.10).

Proof. Existence follows from Theorem 3.7. We show that the solution G to (4.10) is trivial if $-D^{-1}\bar{\lambda} \in \Sigma(\bar{\lambda} + H)$. In this case, since $|G(x) - D^{-1}\bar{\lambda}|^2 = 1$ a.e. on

Ω, we have $|G(x)|^2 = 2G(x) \cdot D^{-1}\bar{\lambda}$, and hence the result follows from the equality $\int_\Omega G(x)\,dx = 0$ that results from the divergence-free condition. Also, a solution in all other existence cases cannot be a constant. Finally, using a Vitali covering argument, it is easily seen that if (4.10) has a nonconstant solution G, then it has infinitely many solutions. For example, in any dimension n, let G be a nonconstant solution to (4.10) and, for any $\epsilon > 0$, by Vitali's covering lemma (e.g., [23, Theorem 7.8]), we can write $\Omega = \cup_{i=1}^\infty (a_i + \epsilon_i \bar{\Omega}) \cup N$, where $0 < \epsilon_i < \epsilon$, $\{a_i + \epsilon_i \bar{\Omega}\}_{i=1}^\infty$ are disjoint and $|N| = 0$. Define

$$G_\epsilon(y) = \begin{cases} G(\frac{y-a_i}{\epsilon_i}) & \text{if } y \in a_i + \epsilon_i \Omega \text{ for some } i = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then $G_\epsilon(y) \in \mathbf{S}_H$ for a.e. $y \in \Omega$ and, for all $\phi \in C_0^\infty(\mathbf{R}^n)$,

$$\begin{aligned} \int_\Omega G_\epsilon(y) \cdot \nabla\phi(y)\,dy &= \sum_{i=1}^\infty \int_{a_i+\epsilon_i\Omega} G\left(\frac{y-a_i}{\epsilon_i}\right) \cdot \nabla\phi(y)\,dy \\ &= \sum_{i=1}^\infty \int_\Omega G(x) \cdot \nabla\phi(a_i + \epsilon_i x)\epsilon_i^n\,dx \\ &= \sum_{i=1}^\infty \int_\Omega G(x) \cdot \nabla\phi_i(x)\,dx, \end{aligned}$$

where $\phi_i(x) \equiv \epsilon_i^{n-1}\phi(a_i + \epsilon_i x) \in C_0^\infty(\mathbf{R}^n)$. Since $\operatorname{div}(G\chi_\Omega) = 0$ on $\mathbf{R}^n$, it follows that $\int_\Omega G \cdot \nabla\psi = 0\ \forall\, \psi \in C_0^\infty(\mathbf{R}^n)$. Hence $\int_\Omega G_\epsilon \cdot \nabla\phi = 0\ \forall\, \phi \in C_0^\infty(\mathbf{R}^n)$. This proves that $\operatorname{div}(G_\epsilon\chi_\Omega) = 0$ on $\mathbf{R}^n$, and hence every G_ϵ is also a solution of (4.10); this completes the proof. ∎

We apply this theorem to study some special cases of the micromagnetic energy for three-dimensional ellipsoids.

4.1. The soft case. In this case, we assume the anisotropy function $\varphi \equiv 0$. The function Φ defined above is then simply $\Phi(\eta) = |\eta|$, and the set $\Sigma(\xi)$ is thus given by

$$\Sigma(0) = \mathbf{S}^2, \quad \Sigma(\xi) = \partial\Phi(\xi) = \{\xi/|\xi|\} \quad (\xi \neq 0).$$

The functional L is given by

$$L(u) = \int_\Omega |\nabla u(x) + H|\,dx + \frac{1}{2}\int_{\mathbf{R}^n} |\nabla u(x)|^2\,dx.$$

Let $\bar{\lambda}$ be uniquely determined by (4.9) above. We also define the following (dual) ellipsoid:

$$\mathbf{E} = \left\{ \xi \in \mathbf{R}^3 \;\middle|\; |D^{-1}\xi|^2 = \sum_{k=1}^3 \xi_k^2/b_k^2 < 1 \right\}.$$

PROPOSITION 4.5. *The soft micromagnetic energy on ellipsoid Ω always has energy minimizers. More specifically, if $H \in \mathbf{R}^3 \setminus \mathbf{E}$, then the minimizer is unique and equals the constant $m = -D^{-1}\bar{\lambda}$; if $H \in \mathbf{E}$, then there are infinitely many minimizers.*

Proof. By (4.9), it follows that $\bar{\lambda} = -H$ if and only if $D^{-1}H \in \partial\Phi(0)$; that is, $H \in \bar{\mathbf{E}}$ in this case. If $H \in \mathbf{R}^3 \setminus \bar{\mathbf{E}}$, then $\bar{\lambda} + H \neq 0$, and thus $-D^{-1}\bar{\lambda} \in \partial\Phi(\bar{\lambda} + H) = \Sigma(\bar{\lambda} + H)$; if $H \in \partial\mathbf{E}$, one still has $-D^{-1}\bar{\lambda} = D^{-1}H \in \mathbf{S}^2 = \Sigma(\bar{\lambda} + H)$. Therefore, if $H \in \mathbf{R}^3 \setminus \mathbf{E}$, then $-D^{-1}\bar{\lambda}$ is always in $\Sigma(\bar{\lambda}+H)$, and, hence, by Theorem 4.4, the minimizing magnetization exists and is the unique constant $m \in \Sigma(-D^{-1}\bar{\lambda}) = \{-D^{-1}\bar{\lambda}\}$. Now let $H \in \mathbf{E}$. Then $-D^{-1}\bar{\lambda} = D^{-1}H$ is in the interior of the convex hull of $\Sigma(\bar{\lambda}+H) = \mathbf{S}^2$. Hence, by Theorem 4.4, problem (4.10) has infinitely many solutions G, and any solution G will give a minimizing magnetization $m(x) = G(x) + D^{-1}H$ a.e. on Ω. This completes the proof. □

Example 4.1. In the case of $H \in \mathbf{E}$, we can in fact construct some solutions. We consider as an example the special case of the unit ball $\Omega = B$ in $\mathbf{R}^n$ but for all dimensions $n \geq 3$.

Since the demagnetizing matrix for the unit ball in $\mathbf{R}^n$ is $D = \frac{I_n}{n}$ (I_n is the $n \times n$ identity matrix), the dual ellipsoid $\mathbf{E}$ above is the ball $\mathbf{E} = \{\xi \in \mathbf{R}^n \,|\, |\xi| < \frac{1}{n}\}$; hence we assume $|H| < \frac{1}{n}$ in the example. In this case, $\bar{\lambda} = -H$, $\mathbf{S}_H = -nH + \mathbf{S}^2$, and hence (4.10) becomes

$$(4.11) \qquad |G(x) + nH| = 1, \quad \operatorname{div} G = 0 \quad \text{in } B, \quad G \cdot x = 0 \quad \text{on } \partial B.$$

Any solution G of this problem will give a minimizing magnetization $m(x) = G(x) + nH$.

We may solve problem (4.11) by reducing it to a two-dimensional (2-D) Eikonal equation problem if $n \geq 3$. For example, assume $H = he^n$ is given in the e^n-direction; the general case can be reduced to this by rotation. We look for solutions in the form of

$$G(x) = u_{x_2}(x)\, e^1 - u_{x_1}(x)\, e^2, \quad x \in B,$$

where $u = u(x) = u(x_1, x_2, x')$, where $x' \in \mathbf{R}^{n-2}$, is to be determined later. Clearly $\operatorname{div} G = 0$ is satisfied. For each $x' \in \mathbf{R}^{n-2}$ with $|x'| < 1$, we solve the Eikonal equation in the disc $D = D(x') = \{(x_1, x_2) \,|\, x_1^2 + x_2^2 < 1 - |x'|^2\}$:

$$(4.12) \qquad |\nabla U(x_1, x_2)| = \sqrt{1 - n^2h^2}, \quad (x_1, x_2) \in D, \quad U|_{\partial D} = 0.$$

This 2-D problem has infinitely many solutions; for instance, the function

$$U(x_1, x_2) = \sqrt{1 - n^2h^2}\left(\sqrt{1 - |x'|^2} - \sqrt{x_1^2 + x_2^2}\right), \quad (x_1, x_2) \in D(x')$$

is a (unique viscosity) solution. We consider this U as a function defined for $x \in \bar{B}$ and denote it by $u(x)$, and let $G(x) = u_{x_2}\, e^1 - u_{x_1}\, e^2$. Then G solves (4.11). Using this special solution, we obtain a minimizer $\bar{m}$ of I in the case $H = he_n$ as

$$\bar{m}(x) = \sqrt{1 - n^2h^2}\left(-\frac{x_2}{\sqrt{x_1^2 + x_2^2}}e^1 + \frac{x_1}{\sqrt{x_1^2 + x_2^2}}e^2\right) + nhe^n.$$

4.2. The uniaxial case. We now assume the anisotropy energy density φ is given by

$$(4.13) \qquad \varphi(h) = \beta(1 - |h \cdot e|),$$

where $\beta > 0$ and $e \in \mathbf{S}^2$ are given constants. Hence $\varphi(h) \geq 0$ and equals 0 if and only if $h \in \{e, -e\}$; these are the so-called *easy axes*.

We choose to use this form of the uniaxial energy function (4.13) rather than the usually used smooth form $\varphi(h) = \beta(1 - |h \cdot e|^2)$ because it renders the easier computations and also captures the main physical features.

In this case the function Φ defined above can be easily found as follows:

$$\begin{aligned}\Phi(\eta) &= \max_{|h|=1} (\eta \cdot h + |h \cdot \beta e| - \beta) \\ &= \max_{|h|=1} \max_{t=\pm 1} \{\eta \cdot h + t\beta e \cdot h\} - \beta \\ &= \max_{t=\pm 1} \max_{|h|=1} (\eta + t\beta e) \cdot h - \beta \\ &= \max_{t=\pm 1} |\eta + t\beta e| - \beta \\ &= (|\eta|^2 + 2\beta|\eta \cdot e| + \beta^2)^{1/2} - \beta.\end{aligned}$$

Hence, Φ is C^∞ if $\eta \cdot e \neq 0$, and hence $\Sigma(\eta) = \partial\Phi(\eta) = \{\Phi'(\eta)\}$ if $\eta \cdot e \neq 0$, where

$$\Phi'(\eta) = \frac{\eta + \beta \mathrm{sgn}(\eta \cdot e) e}{|\eta + \beta \mathrm{sgn}(\eta \cdot e) e|} \quad (\eta \cdot e \neq 0).$$

It can be also easily computed that if $\eta \cdot e = 0$, then

$$\Sigma(\eta) = \left\{ \frac{\eta \pm \beta e}{(|\eta|^2 + \beta^2)^{1/2}} \right\}, \quad \partial\Phi(\eta) = \left\{ \frac{\eta + t\beta e}{(|\eta|^2 + \beta^2)^{1/2}} \;\Big|\; -1 \leq t \leq 1 \right\}.$$

Since $\dim(\mathrm{span}\Sigma(\eta)) \leq 1 \;\forall\; \eta \in \mathbf{R}^3$, by Theorem 4.4, minimizers for the uniaxial micromagnetic energy exist only when $-D^{-1}\bar{\lambda} \in \Sigma(\bar{\lambda}+H)$; this condition is equivalent to $|D^{-1}\bar{\lambda}| = 1$. It seems impossible to write explicitly the set of all H satisfying the condition $|D^{-1}\bar{\lambda}| = 1$, but since $\bar{\lambda} = \lambda(H)$ is a uniquely determined function by (4.9), this set is well defined by

$$\mathbf{U} = \{H \in \mathbf{R}^3 \mid |D^{-1}\bar{\lambda}| = |D^{-1}(\lambda(H))| = 1\}.$$

We have thus the following result.

PROPOSITION 4.6. *The uniaxial micromagnetic energy has a unique constant minimizer $m = -D^{-1}\bar{\lambda}$ if $H \in \mathbf{U}$ and has no minimizers if $H \notin \mathbf{U}$.*

Remark 4.1. By (4.9), it always follows that $\bar{\lambda} = -H$ if $D^{-1}H \in \partial\Phi(0)$. Since $\partial\Phi(0) = \{te \mid -1 \leq t \leq 1\}$ for this energy, one has that the set $\{tDe \mid -1 < t < 1\} \subset \mathbf{R}^3 \setminus \mathbf{U}$ is a nonexistence set and the set $\{\pm De\} \subset \mathbf{U}$ is an existence set.

4.3. The biaxial case. In this case we assume the anisotropy energy function is given by

$$\varphi(h) = \min\{\varphi_1(h), \varphi_2(h)\},$$

where $\varphi_k(h) = \beta_k(1 - |h \cdot e_k|)$ with constants $\beta_k > 0$ and $e_k \in \mathbf{S}^2$ for $k = 1, 2$ and $e_1 \neq \pm e_2$. Therefore the easy axes are $\{\pm e_1, \pm e_2\}$. For simplicity, we also assume $\beta_1 = \beta_2 = \beta$ and $e_1 \cdot e_2 = 0$.

Let $\Phi_k(\eta) = (|\eta|^2 + 2\beta|\eta \cdot e_k| + \beta^2)^{1/2} - \beta$ for $k = 1, 2$. Then it is easily seen that

$$\Phi(\eta) = \max_{|h|=1} (\eta \cdot h - \varphi(h)) = \max\{\Phi_1(\eta), \Phi_2(\eta)\}.$$

LEMMA 4.7. *Φ is C^∞ on the set $\{\eta \in \mathbf{R}^3 \mid \Phi_1(\eta) \neq \Phi_2(\eta)\} = \{\eta \in \mathbf{R}^3 \mid |\eta \cdot e_1| \neq |\eta \cdot e_2|\}$. Moreover,*

$$\Sigma(\eta) = \left\{ \frac{\eta + \beta \mathrm{sgn}(\eta \cdot e_1) e_1}{|\eta + \beta \mathrm{sgn}(\eta \cdot e_1) e_1|}, \; \frac{\eta + \beta \mathrm{sgn}(\eta \cdot e_2) e_2}{|\eta + \beta \mathrm{sgn}(\eta \cdot e_2) e_2|} \right\}$$

if $|\eta \cdot e_1| = |\eta \cdot e_2| \neq 0$*, and*

$$\Sigma(\eta) = \left\{ \frac{\eta \pm \beta e_1}{(|\eta|^2 + \beta^2)^{1/2}},\ \frac{\eta \pm \beta e_2}{(|\eta|^2 + \beta^2)^{1/2}} \right\}$$

if $|\eta \cdot e_1| = |\eta \cdot e_2| = 0$; *that is,* $\eta \in \{e_1, e_2\}^\perp$.

Proof. Note that if $\Phi_1(\eta) > \Phi_2(\eta)$, that is, $|\eta \cdot e_1| > |\eta \cdot e_2|$, then $\eta \cdot e_1 \neq 0$, and hence $\Phi(\eta) = \Phi_1(\eta)$ is C^∞ at this η. Now assume $\Phi_1(\eta) = \Phi_2(\eta)$. Then the formulas for $\Sigma(\eta)$ follow from an easily-checked identity $\Sigma(\eta) = \Sigma_1(\eta) \cup \Sigma_2(\eta)$ under this condition, where $\Sigma_k(\eta)$ is the set for function Φ_k $(k = 1, 2)$. This completes the proof. ∎

Note that the set $\Sigma(\eta)$ contains more than two points only when $\eta \in \{e_1, e_2\}^\perp$, and in this case note that $\dim(\mathrm{span}\Sigma(\eta)) = 2$ and

$$\mathrm{ri}(\mathrm{con}\Sigma(\eta)) = \left\{ \frac{\eta + te_1 + se_2}{(|\eta|^2 + \beta^2)^{1/2}} \;\middle|\; t,\, s \in \mathbf{R},\ |t| + |s| < \beta \right\}.$$

As above, let $\bar\lambda = \lambda(H)$ be the function defined by (4.9), and define

$$\mathbf{U}_1 = \{H \in \mathbf{R}^3 \,|\, |D^{-1}\bar\lambda| = |D^{-1}(\lambda(H))| = 1\}$$

and

$$\mathbf{U}_2 = \left\{H \in \mathbf{R}^3 \;\middle|\; \bar\lambda + H \in \{e_1, e_2\}^\perp;\ -D^{-1}\bar\lambda \in \mathrm{ri}(\mathrm{con}\Sigma(\bar\lambda + H))\right\}.$$

Again it is impossible to write explicit formulas for the sets $\mathbf{U}_1$ and $\mathbf{U}_2$; however, using them, we can characterize the precise condition for existence and nonexistence of minimizing magnetizations.

PROPOSITION 4.8. *The biaxial micromagnetic energy has a unique constant minimizer* $m = -D^{-1}\bar\lambda$ *if* $H \in \mathbf{U}_1$ *and has infinitely many minimizers if* $H \in \mathbf{U}_2$*, and it has no minimizers if* $H \notin (\mathbf{U}_1 \cup \mathbf{U}_2)$.

Remark 4.2. Similar to the previous remark, by (4.9), it always follows that $\bar\lambda = -H$ if $D^{-1}H \in \partial\Phi(0)$. Since in this case $\partial\Phi(0) = \mathrm{con}(\Sigma(0)) = \{te_1 + se_2 \,|\, |t| + |s| \leq 1\}$, one has that the set $\{tDe_1 + sDe_2 \,|\, |t| + |s| < 1\} \subset \mathbf{U}_2$ is an existence set, but the set $\{tDe_1 + sDe_2 \,|\, |t| + |s| = 1,\ ts \neq 0\} \subset \mathbf{R}^3 \setminus (\mathbf{U}_1 \cup \mathbf{U}_2)$ is a nonexistence set.

5. Appendix. We give a direct proof of Theorem 4.1 by showing that the Dirichlet–Neumann problem (4.2) has a desired solution, which can be constructed explicitly (see (5.8) below). To this end, introduce the function

$$F(x, r) = \sum_{i=1}^n \frac{x_i^2}{a_i + r} - 1. \tag{5.1}$$

Given $x \in \Omega^c$, there exists a unique $r = r(x) \geq 0$ such that

$$F(x, r(x)) = \sum_{i=1}^n \frac{x_i^2}{a_i + r} - 1 = 0. \tag{5.2}$$

This function $r(x)$ has been known as one of the *ellipsoidal coordinates* of x and is smooth on Ω^c and vanishes exactly on $\partial\Omega$, and hence $\nu(x) = \nabla r(x)/|\nabla r(x)|$ for $x \in \partial\Omega$. Differentiating $F(x, r(x)) = 0$ with respect to x_j yields

$$\frac{2x_j}{a_j + r(x)} = \left(\sum_{i=1}^n \frac{x_i^2}{(a_i + r(x))^2} \right) r_{x_j} := M r_{x_j}, \quad |\nabla r|^2 = \frac{4}{M}. \tag{5.3}$$

Differentiating again yields

$$\frac{2}{a_j + r(x)} - \frac{2x_j}{(a_j + r(x))^2} r_{x_j} = M r_{x_j x_j} + M_{x_j} r_{x_j},$$

from which and the definition of M in (5.3) it follows that

$$\Delta r = \frac{1}{M} \left(\sum_{i=1}^{n} \frac{2}{a_i + r(x)} \right). \tag{5.4}$$

We now look for solution w to (4.2) in the form of $w(x) = P(r(x))x_k$ on $x \in \Omega^c$, where $P(r) = P_k(r)$ is a function on $r \geq 0$ with $P(0) = 1$ and $P(\infty) = 0$, to be determined. It is easy to compute that, using (5.3), (5.4),

$$w_{x_i} = P'(r) x_k r_{x_i} + P(r)\delta_{ik}, \tag{5.5}$$

$$\Delta w = P''(r)|\nabla r|^2 x_k + P'(r)(\Delta r) x_k + 2P'(r) r_{x_k}, \tag{5.6}$$

$$= \frac{2x_k}{M} \left[2P''(r) + P'(r) \left(\frac{2}{a_k + r} + \sum_{i=1}^{n} \frac{1}{a_i + r} \right) \right]. \tag{5.7}$$

Solving the ordinary differential equation

$$2P''(r) + P'(r) \left(\frac{2}{a_k + r} + \sum_{i=1}^{n} \frac{1}{a_i + r} \right) = 0,$$

with $P(0) = 1$ and $P(\infty) = 0$, will yield a solution

$$P(r) = \frac{1}{2b_k} \int_r^\infty \frac{\sqrt{a_1 a_2 \cdots a_n}\, dt}{(a_k + t)\sqrt{(a_1 + t) \cdots (a_n + t)}},$$

where b_k is defined by (4.1) above. Hence $P(0) = 1$ and $P'(0) = -\frac{1}{2a_k b_k}$. For this function P it follows that the function

$$w(x) = \frac{x_k}{2b_k} \int_{r(x)}^\infty \frac{\sqrt{a_1 a_2 \cdots a_n}\, dt}{(a_k + t)\sqrt{(a_1 + t) \cdots (a_n + t)}} \quad (x \in \Omega^c) \tag{5.8}$$

satisfies $\Delta w = 0$ in Ω^c, $w|_{\partial\Omega} = x_k$, and, by (5.5),

$$\frac{\partial w}{\partial \nu}\Big|_{\partial\Omega} = (2a_k P'(0) + P(0))\nu_k = \left(1 - \frac{1}{b_k} \right) \nu_k.$$

Moreover, it can be shown that $\nabla w \in L^2(\Omega^c)$. This proves Theorem 4.1.

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