# ON THE VECTORIAL HAMILTON-JACOBI SYSTEM

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We discuss some recent developments in the study of regularity and stability for a first order Hamilton-Jacobi system: $\nabla u(x) \in K$, where $K$ is a closed set of $n \times m$-matrices and $u$ is a map from a domain $\Omega \subset R^m$ to $R^n$. For regularity of solutions, we obtain a higher integrability from a very weak integral coercivity condition known as the $L^p$-mean coercivity. For the stability, we study $W^{1,p}$-sequences $\{u_j\}$ for which $\{\nabla u_j\}$ converges weakly and approaches the set $K$ in some point-wise sense, and describe a new approach to study the weak limits by the so-called $W^{1,p}$-quasiconvex hull of $K$. Computation of quasiconvex hulls is usually extremely hard, but some important new developments in the nonlinear partial differential equations turn out to be greatly useful for our study.

## 1 Introduction

Many partial differential equations arising from problems in analysis, geometry and mechanics can be written as a first order Hamilton-Jacobi system:

$$\nabla u(x) \in K, \quad a.e.\ x \in \Omega \subseteq R^m, \tag{1}$$

where $u: \Omega \to R^n$ and $K$ is a subset of $M^{n\times m}$, the set of all real $n \times m$ matrices. Here $\nabla u(x)$ denotes the Jacobian matrix or the gradient of map $u$:

$$(\nabla u)_{ij} = \partial u^i / \partial x_j; \quad 1 \le i \le n,\ 1 \le j \le m.$$

A systematic study of Hamilton-Jacobi equations (when $n = 1$) has been largely developed in P. L. Lions [21]; while for the vectorial cases where $m, n \geq 2$, some recent attempt for such a study has been made by Dacorogna and Marcellini [8,9], Müller and Šverák [25], Šverák [29,30], Yan [33,35], and Yan and Zhou [36,37,38].

In this note, I report on some recent developments in the study of regularity and stability concerning the system (1). For regularity, the ultimate goal is to obtain the *right* conditions on $K$ which guarantee that the solutions be smooth, for example, the $C^{1,\alpha}$-regularity for $p$-Laplacians of K. Uhlenbeck [32]; for stability, we are interested in certain convergence behaviors of the sequences that satisfy (1) approximately.

A sequence $\{u_j\}$ is called *an approximating sequence* of (1) if there exists a non-negative continuous function $f$ vanishing exactly on $K$ such that

$$\lim_{j\to\infty}\int_{\Omega'} f(\nabla u_j(x))\,dx = 0 \quad \text{for all } \Omega' \subset\subset \Omega. \tag{2}$$

An important problem is to study weakly convergent approximating sequences of (1) and their weak limits in the Sobolev space $W^{1,p}(\Omega; R^n)$ consisting of all $L^p$-integrable maps with $L^p$-integrable gradients. The notion of strong and weak convergence in $W^{1,p}(\Omega; R^n)$ is defined as usual and denoted by "$\to$" and "$\rightharpoonup$", respectively. In particular, we say $K$ is $W^{1,p}$*-stable* if for any weakly (weakly* if $p = \infty$) convergent approximating sequence in $W^{1,p}(\Omega; R^n)$ the weak limit $u_0$ satisfies (1); we say $K$ is $W^{1,p}$*-compact* if every weakly convergent approximating sequence of (1) converges strongly in $W^{1,1}_{loc}(\Omega; R^n)$. From the definition, it is obvious that every $W^{1,p}$-compact set is $W^{1,p}$-stable.

An algebraic structure pertaining to both regularity and stability for (1) is the so-called *rank-one connections* in $K$, which, by definition, are the closed line segments connecting any two matrices in $K$ that differ by a rank-one matrix [3,31]. A necessary condition for $K$ to be $W^{1,\infty}$-stable is that $K$ contains all its rank-one connections, while a necessary condition for $K$ to be $W^{1,\infty}$-compact is that $K$ contains no rank-one connections. Moreover, if $K$ has a rank-one connection, some solutions to (1) may have discontinuous gradients [25].

In order to obtain the optimal conditions for $W^{1,p}$-stability of the sets, we introduce an important concept of $W^{1,p}$*-quasiconvex hulls* or simply $p$-quasiconvex hulls of the sets using Morrey's notion of *quasiconvex* functions [23]. Our $p$-quasiconvex hulls generalize the usual quasiconvex hulls introduced in the study of microstructures of certain elastic materials using the theory of *Young measures* [3,20,29,31]. As we shall see later, $W^{1,p}$-stability for (1) is essentially determined by the $p$-quasiconvex hull of the set $K$.

In many problems, however, computing the $p$-quasiconvex hulls is an infinite dimensional problem and usually is extremely difficult for a given set. Nevertheless, some new developments in the nonlinear PDEs including methods of compensated compactness [6,10,24,31] of F. Murat and L. Tartar and nonlinear Hodge decompositions of T. Iwaniec [17,19] can be very useful in these studies.

## 2 $W^{1,p}$-Quasiconvex Hulls and $W^{1,p}$-Stability

Given $f : M^{n\times m} \to R^1$, define the *quasiconvexification* $f^{qc}$ of $f$ to be

$$f^{qc}(A) = \inf_{\phi\in C_0^\infty(\Omega,R^n)} \frac{1}{|\Omega|}\int_\Omega f(A+\nabla\phi(x))\,dx, \quad A \in M^{n\times m}. \tag{3}$$

We say $f$ is *quasiconvex* if $f^{qc} \equiv f$. It is well-known that $f^{qc}$ is independent of the domain $\Omega$ and, under some mild conditions, is also quasiconvex [7]. By Jensen's inequality, every convex function is quasiconvex, but V. Šverák [28] has shown that the class of quasiconvex functions is strictly larger if, e.g., $n,\, m \geq 3$.

Let $K$ be a subset of $M^{n\times m}$ and $1 \leq p \leq \infty$. We denote by $Q_p^+(K)$ the set of all quasiconvex continuous functions $f$ on $M^{n\times m}$ which satisfy $f|_K = 0$ and

$$0 \leq f(X) < C_f\,(|X|^p + 1), \qquad X \in M^{n\times m} \tag{4}$$

for a constant $C_f < \infty$. If $p = \infty$, condition (4) simply means $0 \leq f(X) < \infty$. Let $Z(f)$ denote the zero set of $f$. We now define the $p$-quasiconvex hulls.

**Definition 2.1** *The set*

$$Q_p(K) = \cap\{Z(f) \mid f \in Q_p^+(K)\}$$

*is called the* $W^{1,p}$- or $p$-quasiconvex hull *of* $K$. *We say* $K$ *is* $p$-quasiconvex *if* $Q_p(K) = K$.

*Remarks.* 1. It is well-known that [1,4] the functional $I_S(u) = \int_S f(\nabla u)$ is weakly lower semicontinuous on $W^{1,p}(\Omega; R^n)$ for any $f \in Q_p^+(K)$ and measurable set $S \subseteq \Omega$.

2. If $f(X)$ is a continuous and quasiconvex function on $M^{n\times m}$ satisfying $0 \leq f(X) < C\,(|X|^p + 1)$ and $Z(f) \neq \emptyset$, then $K = Z(f)$ is $p$-quasiconvex.

3. It is easily seen that $Q_q(K) \subseteq Q_p(K)$ for all $1 \leq p < q \leq \infty$. Moreover, if $\mathrm{conv}(K)$ is the closed convex hull of $K$, then $Q_1(K) \subseteq \mathrm{conv}(K)$. So, all $Q_p(K)$ are compact sets if $K$ is bounded.

Using the Chacon biting convergence lemma [5], we can prove the following result [35].

**Theorem 2.2** *Suppose* $\{u_j\}$ *is an approximating sequence of* (1) *and* $u_j \rightharpoonup u_0$ *weakly in* $W^{1,p}(\Omega; R^n)$ (weakly * if $p = \infty$). *Then* $\nabla u_0(x) \in Q_p(K)$ *for almost every* $x \in \Omega$. *Therefore,* $K$ *is* $W^{1,p}$*-stable if it is p-quasiconvex.*

For bounded sets, using a Luzin type approximation result [1,20,39], we can prove the following necessary and sufficient condition for $W^{1,p}$-stability of compact sets [35].

**Theorem 2.3** *Let* $K$ *be a compact set. Then, the p-quasiconvex hulls* $Q_p(K)$ *are all the same for* $1 \leq p \leq \infty$ *and will be denoted by* $Q(K)$. *Moreover,* $K$ *is* $W^{1,p}$*-stable if and only if* $K$ *is* quasiconvex, *i.e.,* $Q(K) = K$.

## 3 Integral Growth Conditions and $W^{1,p}$-Compactness

We now study the $W^{1,p}$-compactness based on the integral growth condition similar to the uniformly strict quasiconvexity of Evans and Gariepy [14]. The following results have been proved in Yan [35].

**Theorem 3.1** *Let $1 < p < \infty$ and $f \in Q_p^+(K)$ with $Z(f) = K$. Suppose for each $\epsilon > 0$ there exists $C_\epsilon > 0$ such that*

$$\int_D |\nabla\phi| \le \epsilon \int_D \Big(|\nabla\phi|^p + 1\Big) + C_\epsilon \int_D f(A + \nabla\phi) \tag{5}$$

*holds for all $A \in K$ and $\phi \in W_0^{1,p}(D; R^n)$, where $D$ is a fixed cube in $\Omega$. Then $K$ is $W^{1,q}$-compact for all $q > p$.*

The usefulness of this theorem is that we only need to check the condition (5) for all $A \in K$ and this condition is much weaker than the uniformly strict quasiconvexity [14].

**Theorem 3.2** *Let $K$ be compact, and let $D \subseteq \Omega$ be a cube and $1 < p < \infty$. Then, $K$ is $W^{1,1}$-compact if and only if for each $\epsilon > 0$ there exists a constant $C_\epsilon < \infty$ such that*

$$\int_D \Big(d_K(A) + |\nabla\phi|\Big) \le \epsilon|D| + C_\epsilon \int_D d_K^p(A + \nabla\phi) \tag{6}$$

*holds for all $A \in M^{n\times m}$ and $\phi \in W_0^{1,p}(D; R^n)$.*

## 4 Special Structures in $p$-Quasiconvex Hulls

In order to compute $Q_p(K)$, we need some special structures besides those consisting of rank-one connections mentioned earlier in the introduction.

For any set $K$ and number $p \in [1, \infty]$, let $\beta_p(K)$ be the set of all matrices $A$ for which (1) has a solution $u \in W^{1,p}(\Omega; R^n)$ that satisfies $u(x) = u_A(x) \equiv Ax$ on $\partial\Omega$, and let $\omega_p(K)$ be the set of all matrices $A$ for which there exists an approximating sequence $\{u_j\}$ of (1) such that $u_j \rightharpoonup u_A$ weakly in $W^{1,p}(\Omega; R^n)$ (weakly * if $p = \infty$).

Directly from the definition, we can easily prove the following.

**Theorem 4.1** *(a) $\beta_p(K) \subseteq \omega_p(K) \cap Q_p(K)$ and furthermore, if $K$ is $W^{1,p}$-compact then the only solution to (1) with $u = u_A$ on $\partial\Omega$ is $u \equiv u_A$.*

*(b) $\omega_\infty(K) \subseteq Q_p(K)$ and $\omega_\infty(K)$ contains all rank-one connections in $K$.*

*(c) If $K$ is compact, then $Q(K) = \omega_\infty(K)$.*

*Remark.* There has been some recent study of the set $\beta_p(K)$ defined above by Dacorogna and Marcellini [8,9] using the Baire category theory, and by Müller and Šverák [25] using Gromov's idea of convex integration.

## 5 $L^p$-Mean Coercivity and Higher Regularity

We consider the case when $K$ is a closed cone in $M^{n\times m}$, i.e., $\lambda K \subseteq K$ for all $\lambda \geq 0$. Let $d_K$ be the corresponding distance function and let $B$ be the unit ball in $R^m$. Define

$$\mu(p;K) = \inf \left\{ \int_B d_K^p(\nabla\phi) \,\Big|\, \phi \in C_0^\infty(B;R^n),\ \|\nabla\phi\|_{L^p(B)} = 1 \right\}. \tag{7}$$

We say that $K$ satisfies the $L^p$*-mean coercivity* provided that $\mu(p;K) > 0$.

**Theorem 5.1** *If $p > 1$ and $\mu(p;K) > 0$, then there exists $\epsilon \in (0,\, p-1)$ such that $\mu(r;K) > 0$ for all $r \in [p-\epsilon,\, p+\epsilon]$.*

The proof of this theorem relies on the following stability result on the *nonlinear Hodge decompositions* due to Iwaniec [17]. We refer to Iwaniec and Sbordone [19] for a proof of this result using the $L^p$-estimates of nonlinear para-commutators.

**Lemma 5.2** *Let $r > 1$ and $B$ be the open unit ball in $R^m$. Then, for any $u \in W_0^{1,r}(B;R^n)$ and $\epsilon \in (-1,\, r-1)$, the matrix $|\nabla u|^\epsilon \nabla u \in L^{\frac{r}{1+\epsilon}}(B;M^{n\times m})$ can be decomposed as $|\nabla u(x)|^\epsilon \nabla u(x) = \nabla\psi(x) + h(x)$ for a.e. $x \in B$, where $\psi \in W_0^{1,\frac{r}{1+\epsilon}}(B;R^n)$, and $h \in L^{\frac{r}{1+\epsilon}}(R^m;M^{n\times m})$ is a divergence free matrix field satisfying*

$$\|h\|_{L^{\frac{r}{1+\epsilon}}(R^m)} \leq C(m,n,r,\epsilon)\,|\epsilon|\,\|\nabla u\|_{L^r(B)}^{1+\epsilon}. \tag{8}$$

*Moreover, for any constants $1 < r_1 < r_2 < \infty$,*

$$\sup_{|\epsilon| \leq \frac{r_1-1}{r_1+1},\ r_1 \leq r \leq r_2} C(m,n,r,\epsilon) \equiv \alpha(r_1,r_2) < \infty. \tag{9}$$

We now sketch the proof of Theorem 5.1 given in Yan and Zhou [38]. Let

$$r_1 = \frac{\sqrt{8p+1}-1}{2}, \quad \epsilon_0 = \frac{r_1-1}{r_1+1}, \quad r_2 = (1+\epsilon_0)p. \tag{10}$$

Let $|\epsilon| \leq \epsilon_0$ and $r = (1+\epsilon)\,p$; then $r_1 \leq r \leq r_2$. Let $\phi \in C_0^\infty(B;R^n)$. Using Lemma 5.2, we decompose $|\nabla\phi|^\epsilon \nabla\phi \in L^p(B;M^{n\times m})$ as follows:

$$|\nabla\phi(x)|^\epsilon \nabla\phi(x) = \nabla\psi(x) + h(x) \quad a.e.\ x \in B, \tag{11}$$

where $\psi \in W_0^{1,p}(B;R^n)$, $h \in L^p(R^m;M^{n\times m})$ and

$$\|h\|_{L^p(R^m)} \leq \alpha_p\,|\epsilon|\,\|\nabla\phi\|_{L^r(B)}^{1+\epsilon}, \tag{12}$$

where $\alpha_p$ depends only on $p$. Hence,

$$\|\nabla\psi\|_{L^p(B)} \geq (1-\alpha_p|\epsilon|)\,\|\nabla\phi\|^{1+\epsilon}_{L^r(B)}. \tag{13}$$

From (11), we have $\nabla\psi(x) = |\nabla\phi(x)|^{\epsilon}\nabla\phi(x) - h(x)$ and hence

$$d_K(\nabla\psi(x)) \leq |\nabla\phi(x)|^{\epsilon} d_K(\nabla\phi(x)) + |h(x)| \quad \forall\, x\in B.$$

Let $\sigma_0 = \mu(p;K)^{1/p}$. We have by the $L^p$-mean coercivity

$$\begin{aligned}\sigma_0\|\nabla\psi\|_{L^p(B)} &\leq \|d_K(\nabla\psi)\|_{L^p(B)} \\ &\leq \||\nabla\phi|^{\epsilon} d_K(\nabla\phi)\|_{L^p(B)} + \|h\|_{L^p(B)}. \end{aligned} \tag{14}$$

Combining (12)–(14), we have

$$\Big(\sigma_0 - \alpha_p(1+\sigma_0)\,|\epsilon|\Big)\,\|\nabla\phi\|^{1+\epsilon}_{L^r(B)} \leq \||\nabla\phi|^{\epsilon} d_K(\nabla\phi)\|_{L^p(B)}. \tag{15}$$

From this, it follows that if $|\epsilon|\leq\epsilon_0$ is further chosen sufficiently small then

$$\int_B d_K^r(\nabla\phi) \geq C_p\,\|\nabla\phi\|^r_{L^r(B)}, \quad C_p>0, \quad r=(1+\epsilon)\,p, \tag{16}$$

proving the theorem. Indeed, if $\epsilon<0$, using $|\nabla\phi|^{\epsilon} \leq [d_K(\nabla\phi)]^{\epsilon}$, we have

$$\||\nabla\phi|^{\epsilon} d_K(\nabla\phi)\|_{L^p(B)} \leq \|d_K^{1+\epsilon}(\nabla\phi)\|_{L^p(B)} = \|d_K(\nabla\phi)\|^{1+\epsilon}_{L^r(B)},$$

hence (16) follows by using (15) if $|\epsilon|$ is sufficiently small. Let $\epsilon>0$; then, by Hölder's inequality, we have

$$\||\nabla\phi|^{\epsilon} d_K(\nabla\phi)\|_{L^p(B)} \leq \|d_K(\nabla\phi)\|_{L^r(B)}\|\nabla\phi\|^{\epsilon}_{L^r(B)}.$$

So, we still obtain (16) using (15) for all sufficiently small $|\epsilon|$.

The $L^p$-mean coercivity and the Ekeland variational principle [12] enable us to adapt the standard Caccioppoli-type estimates [16] to obtain some higher regularity results using the technique of reverse Hölder inequalities. This technique was first introduced by Gehring [15] in the study of higher integrability of quasiconformal mappings and used later successfully for studying the nonlinear elliptic systems [16,22].

The following results concerning the regularity and stability for system (1) have been proved in Yan and Zhou [37,38].

**Theorem 5.3** *Let $K$ be a closed cone and $S(K) = \{p>1 \mid \mu(p;K)>0\} \neq \emptyset$. Then $Q_p(K)$ is constant for all $p$ belonging to each of the connected components of $S(K)$.*

**Theorem 5.4** *Let $K$ be a closed cone and $[\alpha,\, \beta] \subset S(K)$. Then any solution $u \in W^{1,\alpha}_{loc}(\Omega; R^n)$ to system (1) must belong to $W^{1,\beta}_{loc}(\Omega; R^n)$.*

*Remarks.* 1. An application of these results will be discussed in the next section concerning the regularity and stability of the *weakly quasiregular mappings.*

2. It remains extremely difficult and quite challenging to describe the set $S(K)$ in terms of any intrinsic properties of the cone $K$.

3. Another problem is to find a necessary and sufficient condition such that $K$ is $p$-quasiconvex, which would have a profound impact on the study of Hamilton-Jacobi system (1).

## 6 Weakly Quasiregular Mappings

As an example of our study, we consider the case $m = n$ and the class of closed cones $K_l$ defined by

$$K_l = \{A \in M^{n\times n} \,|\, |A|^n \le l n^{n/2} \det A\}$$

for all $l \ge 1$; the set $K_1$ is called the *conformal set* and denoted by $C_n$. A map $u \in W^{1,p}_{loc}(\Omega; R^n)$ is called (weakly if $p < n$) $l$-quasiregular [17,18,27] if it satisfies $\nabla u(x) \in K_l$ *a.e.* $x \in \Omega$.

In order to study the regularity and stability of (weakly) quasiregular mappings, we shall try to compute the set $S(K_l)$. Consider the function

$$F_l(X) = \max\{0,\, |X|^n - l n^{n/2} \det X\}.$$

It is easily seen that $F_l \ge 0$ is $n$-homogeneous, quasiconvex and $Z(F_l) = K_l$, and we also have that $\int_B F_l(\nabla\phi) \ge \int_B |\nabla\phi|^n$ for all $\phi \in C^\infty_0(B; R^n)$. From this, using the homogeneity, we obtain that there is a $\Gamma > 0$ such that

$$\int_B d^n_{K_l}(\nabla\phi) \ge \Gamma \int_B |\nabla\phi|^n, \quad \forall\, \phi \in C^\infty_0(B; R^n).$$

Hence, by definition and Theorems 5.3 and 5.4, we have the following regularity and stability result.

**Theorem 6.1** *For all $l \ge 1$, it follows that $n \in S(K_l)$ and $Q_p(K_l) = K_l$ for $p \in [n-\epsilon,\, n+\epsilon] \subset S(K_l)$ for some $\epsilon > 0$. Therefore, we have both the stability and higher regularity for weakly quasiregular mappings in $W^{1,p}_{loc}$ with $p$ slightly below the dimension $n$.*

*Remarks.* 1. Observe that the nonlinear homothety $u_l(x) = x|x|^{(1-l)/l}$ satisfies $\nabla u_l(x) \in K_l$ for $x \ne 0$, and $u_l \in W^{1,n}(B; R^n)$, but $u_l \notin W^{1,\frac{nl}{l-1}}(B; R^n)$. By Theorem 5.4, there must be a $q \in [n,\, nl/(l-1)]$ such that $q \notin S(K_l)$.

2. On a basis of Theorem 5.4 and a conjecture made by Iwaniec [17] concerning regularity of weakly quasiregular mappings, we conjecture that $S(K_l) = \left(\frac{nl}{l+1}, \frac{nl}{l-1}\right)$ for all $n$ and $l \geq 1$. For even dimensions $n$, it is indeed true that $S(C_n) = (n/2, \infty)$, but question concerning all other cases remains open [18,26,34].

3. This conjecture is closely related to an open problem on the $p$-Laplacian equation regarding whether the $C^{1,\alpha}$-regularity result of Uhlenbeck [32] holds for weak solutions in $W^{1,q}$ with $q > p - 1$.

The following theorem gives the compactness results concerning the weakly conformal mappings proved in Müller, Šverák and Yan [26] and in Yan and Zhou [36].

**Theorem 6.2** *There exists a* $p_n \in [n/2, n)$, $p_n = n/2$ *if* $n$ *is even, such that if* $p \geq p_n$ *and if* $u_j \rightharpoonup u_0$ *in* $W^{1,p}(\Omega; R^n)$ *and* $\int_\Omega d^p_{C_n}(\nabla u_j) \to 0$ *then* $u_j \to u_0$ *strongly in* $W^{1,1}_{loc}(\Omega; R^n)$.

*Remarks.* 1. If $p \geq n$, the result follows from a strong convergence result of Evans and Gariepy [14] by a theory of *strictly uniformly polyconvex* functions. However, as proved in Yan [34], such functions do not exist if $p < n$.

2. If $p_n \leq p < n$, Theorem 6.2 generalizes the classical stability result of quasiregular mappings [27]: If $\{u_j\}$ is a sequence of $l_j$-quasiregular mappings bounded in $W^{1,n}$ and $l_j \to 1$, then $\{u_j\}$ converges strongly to a Möbius map in $W^{1,1}_{loc}(\Omega; R^n)$. In this case, it is easily seen that the conformal energy $I_p(u_j) = \int_\Omega d^p_{C_n}(\nabla u_j)$ approaches 0.

3. The result in the even dimension case is sharp. A key to the proof is the fact that for $n = 2l$ the conformality of a matrix can be (almost) characterized by a *linear* condition that involves $l \times l$-minors [11,18]. This characterization gives a nonlinear version of Cauchy-Riemann equations and hence enables us to use the elliptic estimates and compensated compactness [6,26,31].

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