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On two-dimensional ferromagnetism

Pablo Pedregal
ETSI Industriales, Universidad de Castilla-La Mancha,
13071 Ciudad Real, Spain (pablo.pedregal@uclm.es)

Baisheng Yan
Department of Mathematics, Michigan State University,
East Lansing, MI 48824, USA (yan@math.msu.edu)

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We present a new method for solving the minimization problem in ferromagnetism. Our method is based on replacing the non-local non-convex total energy of magnetization by a new local non-convex energy of divergence-free fields. Such a general method works in all dimensions. However, for the two-dimensional case, since the divergence-free fields are equivalent to the rotated gradients, this new energy can be written as an integral functional of gradients and hence the minimization problem can be solved by some recent non-convex minimization procedures in the calculus of variations. We focus on the two-dimensional case in this paper and leave the three-dimensional situation to future work. Special emphasis is placed on the analysis of the existence/non-existence depending on the applied field and the physical domain.

1. Introduction

The model of micromagnetism seeks the magnetization $m : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$ of a body occupying the region Ω by minimizing the energy functional

$$I(m) = \tfrac{1}{2}\alpha \int_\Omega |\nabla m(x)|^2 \,\mathrm{d}x + \int_\Omega \varphi(m(x)) \,\mathrm{d}x - \int_\Omega H \cdot m(x) \,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^N} |F(z)|^2 \,\mathrm{d}z \tag{1.1}$$

among all admissible magnetizations m satisfying

$$m \in L^\infty(\Omega), \quad |m(x)| = 1 \text{ a.e. } x \in \Omega, \tag{1.2}$$

where $F \in L^2(\mathbb{R}^N; \mathbb{R}^N)$ is the unique field determined by the simplified Maxwell equations:

$$\operatorname{curl} F = 0, \qquad \operatorname{div}(-F + m\chi_\Omega) = 0 \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Here $\alpha > 0$ is a material constant, φ is the density of anisotropy that is minimized along preferred crystallographic directions, H is a given external applied field (typically constant) and F is the induced magnetic field on the whole $\mathbb{R}^N$ related to m via Maxwell's equations (1.3). The first term in the energy $I(m)$ is called the *exchange energy*, the second term the *anisotropy energy*, the third term the *external interaction energy* and the last term is a non-local energy and is usually called the *magnetostatic energy*. The non-locality and non-convexity of the total energy not only present a major and challenging mathematical problem but also provide a concrete example for some other physical problems of a similar nature. Therefore, the model has been at the centre of much of current active research (see [2, 6, 7, 9–15, 17–19] for more references and [14] for a recent comprehensive survey on the model and related problems).

Here we are only concerned with existence of minimizers of the energy I and their dependence on the given applied field under various sets of assumptions on exchange and anisotropy energies. We will not study the important issues on possible microstructure when minimizers do not exist or are not unique.

We will restrict our attention to the two-dimensional case, $N = 2$, deferring the three-dimensional case to a future work. We will proceed in three steps of increasing complexity.

(i) The soft case. Here we neglect both the exchange energy and the anisotropy energy, and focus on the problem

$$\text{minimize in } m : I(m) = \frac{1}{2}\int_{\mathbb{R}^2} |F(z)|^2 \,\mathrm{d}z - \int_{\Omega} H \cdot m(x)\,\mathrm{d}x \tag{P}$$

subject to

$$m \in L^{\infty}(\Omega), \quad |m(x)| = 1 \text{ a.e. } x \in \Omega,$$
$$\operatorname{curl} F = 0, \quad \operatorname{div}(-F + m\chi_{\Omega}) = 0 \quad \text{in } \mathbb{R}^2.$$

(ii) The hard case. Here we consider the effect of a hard anisotropy, either uniaxial or cubic, but still neglect the exchange energy. In the uniaxial case, we will postulate anisotropy in the particular form

$$\varphi(m) = \beta(1 - |m \cdot e|), \tag{1.4}$$

where $\beta > 0$ and $e \in S^1$ is the unique preferred direction (easy axis); in the biaxial or cubic case, we assume that

$$\varphi(m) = \min\{\beta_1(1 - |m \cdot e_1|),\ \beta_2(1 - |m \cdot e_2|)\}, \tag{1.5}$$

where $\beta_1, \beta_2 > 0$ and $e_1, e_2 \in S^1$ are two independent preferred directions (easy axes). Here β, β_1 and β_2 represent the strengths of the anisotropies. Indeed, we will treat the particular case where e_1 and e_2 are mutually orthogonal, and the strength of anisotropy is the same in both directions $\beta_1 = \beta_2 = \beta$.

(iii) The full case. Here we assume that $\alpha > 0$ and try to understand the interplay among the different contributions to the energy.

The main qualitative result concerning the existence of minimizers in all these cases can be summarized in the following theorem. More specific, quantitative results are given later.

Theorem 1.1. *According to each case listed above, we have the following results.*

(i) *In the soft case, $I(m)$ always has a minimizer for all domains Ω and all given fields H.*

(ii) *In the hard case, there exists a proper, non-empty set $\mathcal{H} \subsetneq \mathbb{R}^2$ (depending on the domain Ω and the anisotropy density φ) such that $I(m)$ has a minimizer if $H \notin \mathcal{H}$ and does not have any minimizers if $H \in \mathcal{H}$; in the uniaxial case $0 \in \mathcal{H}$ and in the biaxial case $0 \notin \mathcal{H}$.*

(iii) *In the full case, $I(m)$ always has a minimizer.*

Our general procedure for attacking these problems (which is also valid regardless of dimension) is as follows. First we introduce a new field, $G = -F + m\chi_\Omega$ and a new augmented energy

$$\begin{aligned}\mathcal{A}(m, G) &= \tfrac{1}{2}\alpha \int_\Omega |\nabla m|^2 + \int_\Omega \varphi(m) - \int_\Omega H \cdot m + \frac{1}{2}\int_{\mathbb{R}^N} |m\chi_\Omega - G|^2 \\ &= \int_\Omega [\tfrac{1}{2}\alpha|\nabla m|^2 + \varphi(m) - (H + G)\cdot m + \tfrac{1}{2}(|G|^2 + 1)] + \frac{1}{2}\int_{\Omega^c} |G|^2.\end{aligned} \tag{1.6}$$

We then define a new energy for $G \in L^2(\mathbb{R}^N; \mathbb{R}^N)$ by setting

$$J(G) = \inf_{m\in L^\infty(\Omega),\, |m|=1} \mathcal{A}(m, G). \tag{1.7}$$

Here we do not distinguish between the cases $\alpha = 0$ and $\alpha > 0$, but we use the convention that, for the admissible m,

$$\int_\Omega |\nabla m|^2 \,\mathrm{d}x = \infty$$

unless $m \in H^1(\Omega)$. Therefore, if $\alpha > 0$, $J(G) = \inf_{m\in H^1(\Omega),\, |m|=1} \mathcal{A}(m, G)$.

The following easy facts establish our general philosophy that solving the minimization problem for energy I over unit directions in $\Omega \subset \mathbb{R}^N$ is equivalent to solving the minimization problem for the new energy J over divergence-free fields on $\mathbb{R}^N$. We will apply this general philosophy to the two-dimensional case in this paper and to the three-dimensional situation in a future work.

Proposition 1.2.

(i) *For any $m \in L^\infty(\Omega)$ with $|m(x)| = 1$ almost everywhere (a.e.),*

$$I(m) = \min_{G\in L^2(\mathbb{R}^N;\mathbb{R}^N),\, \operatorname{div} G=0} \mathcal{A}(m, G).$$

Moreover, if $I(m) < \infty$, then the minimizer $G = G_m$ is unique and satisfies

$$\operatorname{div} G_m = 0, \qquad \operatorname{curl}(m\chi_\Omega - G_m) = 0; \tag{1.8}$$

hence $F = F_m = m\chi_\Omega - G_m$ is the unique solution to Maxwell's equations (1.3).

(ii) *For any $G \in L^2(\mathbb{R}^N; \mathbb{R}^N)$, $J(G)$ is attained by minimizers; that is,*

$$J(G) = \min_{m\in L^\infty(\Omega),\, |m|=1} \mathcal{A}(m, G).$$

We denote by $\Sigma(G)$ the set of all these minimizers.

(iii) *It follows that*

$$\inf_{m\in L^\infty(\Omega),\, |m|=1} I(m) = \inf_{G\in L^2(\mathbb{R}^N;\mathbb{R}^N),\, \operatorname{div} G=0} J(G). \tag{1.9}$$

Moreover, if $\bar m \in L^\infty(\Omega)$, $|\bar m| = 1$, *is a minimizer of* I, *then the function* $G_{\bar m}$ *determined by the Maxwell equation (1.8) is a minimizer of* J*; if* $\bar G \in L^2(\mathbb{R}^N;\mathbb{R}^N)$, $\operatorname{div} \bar G = 0$, *is a minimizer of* J, *then any function* $\tilde m \in \Sigma(\bar G)$ *is a minimizer of* I.

Proof. The proof of this proposition is straightforward. It shows that finding optimal magnetic configurations amounts, after all, to minimizing the energy $\mathcal{A}(m,G)$ recursively on m or on G, depending on which is more convenient. The main observation here is that, for given m, the minimization in divergence-free vector fields G takes place for

$$\frac{1}{2}\int_{\mathbb{R}^N} |m\chi_\Omega - G|^2.$$

It is well known that the unique solution is determined by the Helmholtz projection

$$\operatorname{div} G = 0, \qquad \operatorname{curl}(m\chi_\Omega - G) = 0.$$

On the other hand, for G fixed, the minimization on m is a regular variational problem for $\alpha > 0$ (coercive and convex). If $\alpha = 0$, then optimal fields m arise by pointwise minimizing the integrand

$$\min_{|m|=1} (\varphi(m) - (H+G)m).$$

Part (iii) is a direct consequence of parts (i) and (ii). □

The case involving the derivative of field m presents an easier problem regarding the existence of minimizers by virtue of the following result, although the concrete computations of minimizers seem a much harder non-local problem [2, 6].

PROPOSITION 1.3. *Let* $\alpha > 0$. *Then the energy* J *defined above is sequentially weakly lower semicontinuous and coercive on* $L^2(\mathbb{R}^N;\mathbb{R}^N)$, *and hence* J *possesses a minimizer over the divergence-free fields; therefore, the energy* I *has minimizers* $\bar m \in \Sigma(\bar G)$, *where* $\bar G$ *is any minimizer of* J. *Furthermore, any such pair* $(\bar m, \bar G)$ *is also a minimizer for the augmented energy* $\mathcal{A}$ *defined above.*

Proof. Assume that $G_j \rightharpoonup G_0$ weakly in $L^2(\mathbb{R}^N;\mathbb{R}^N)$. By proposition 1.2(ii), there exist minimizers $m_j \in L^\infty(\Omega)$ with $|m_j(x)| = 1$ a.e. such that

$$J(G_j) = \mathcal{A}(m_j, G_j) \leqslant M, \quad j = 0, 1, \dots,$$

where $M < \infty$ is some constant. Since $\alpha > 0$, it follows that $m_j \in H^1(\Omega)$ and $\|\nabla m_j\|_{L^2(\Omega)} \leqslant M'$ for another constant M'. We can assume, without loss of generality, that $m_j \rightharpoonup m^*$ weakly in $H^1(\Omega)$ and $m_j(x) \to m^*(x)$ and hence $|m^*(x)| = 1$ for almost every $x \in \Omega$. Note that

$$\mathcal{A}(m_j, G_j) = \tfrac{1}{2}\alpha \int_\Omega |\nabla m_j|^2 + \int_\Omega \varphi(m_j) - \int_\Omega H\cdot m_j + \frac{1}{2}\int_{\mathbb{R}^N} |m_j\chi_\Omega - G_j|^2.$$

Taking the liminf as $j \to \infty$, we have

$$\begin{aligned}\liminf_{j\to\infty} J(G_j) &= \liminf_{j\to\infty} \mathcal{A}(m_j, G_j) \\ &\geqslant \tfrac{1}{2}\alpha \int_\Omega |\nabla m^*|^2 + \int_\Omega \varphi(m^*) - \int_\Omega H\cdot m^* + \frac{1}{2}\int_{\mathbb{R}^N} |m^*\chi_\Omega - G_0|^2 \\ &= \mathcal{A}(m^*, G_0) \\ &\geqslant J(G_0).\end{aligned}$$

This proves the lower semicontinuity of J on $L^2(\mathbb{R}^N;\mathbb{R}^N)$. Moreover, it follows easily that (even when $\alpha = 0$)

$$J(G) \geqslant \tfrac{1}{4}\|G\|^2_{L^2(\mathbb{R}^N;\mathbb{R}^N)} - C,$$

where C is a constant depending only on φ and H. This proves the coercivity of J. The rest of the proof follows easily. □

REMARK 1.4. If dimension $N = 2$, we can use the angle as an independent variable to represent the direction of magnetization m as

$$m = (\cos\theta, \sin\theta), \quad \theta : \Omega \to \mathbb{R}.$$

Notice that $|\nabla m|^2 = |\nabla\theta|^2$. If we introduce $\boldsymbol{v} = (v_1, v_2) = (\theta, u)$, where $G = \nabla^\perp u$, then the augmented energy $\mathcal{A}(m, G)$ above can be written as a variational integral:

$$\mathcal{A}(m, G) = \mathcal{E}(\boldsymbol{v}) = \int_{\mathbb{R}^2} W(x, \boldsymbol{v}(x), \nabla\boldsymbol{v}(x))\,\mathrm{d}x, \tag{1.10}$$

where $W(x, \boldsymbol{v}, A)$ is a function of $x, \boldsymbol{v} \in \mathbb{R}^2$ and the 2×2 matrix A defined by

$$W(x, \boldsymbol{v}, A) = \tfrac{1}{2}|A_2|^2 + \chi_\Omega(x)[\varphi(\cos v_1, \sin v_1) - (H + A_2^\perp)\cdot(\cos v_1, \sin v_1) + \tfrac{1}{2}\alpha|A_1|^2] \tag{1.11}$$

with A_1 and A_2 respectively denoting the first and second rows of matrix A. In fact, if $\alpha > 0$, this density function $W(x, \boldsymbol{v}, A)$ is convex in A, and hence the minimization problem in this case can be solved easily by the standard direct method of calculus of variations.

The rest of the discussion is devoted to the case when there is no exchange energy. Again our discussion is valid regardless of dimension. Let $\alpha = 0$ and write

$$\mathcal{A}(m, G) = \int_\Omega \rho(m(x), G(x))\,\mathrm{d}x + \frac{1}{2}\int_{\Omega^c} |G(z)|^2\,\mathrm{d}z; \quad \Omega^c = \mathbb{R}^N \setminus \Omega,$$

where

$$\rho(h, \xi) = \varphi(h) - (H + \xi)\cdot h + \tfrac{1}{2}(|\xi|^2 + 1).$$

Define the function

$$\Psi(\xi) = \min_{h\in S^{N-1}} \rho(h, \xi) = \min_{h\in S^{N-1}} [\varphi(h) - (H + \xi)\cdot h + \tfrac{1}{2}(|\xi|^2 + 1)] \tag{1.12}$$

and its minimal set

$$\sigma(\xi) = \{h \in S^{N-1} \mid \rho(h, \xi) = \Psi(\xi)\}. \tag{1.13}$$

The following proposition is a more explicit way of writing the one above for the case when $\alpha = 0$.

Proposition 1.5. *Let $\alpha = 0$. Then the energy $J(G)$ defined above is given by*

$$J(G) = \int_{\Omega} \Psi(G(x))\,\mathrm{d}x + \frac{1}{2}\int_{\Omega^c} |G(z)|^2\,\mathrm{d}z = \mathcal{A}(m_G, G),$$

where $m_G \in L^\infty(\Omega)$ is any function satisfying

$$m_G(x) \in \sigma(G(x)) \quad a.e.\ x \in \Omega; \tag{1.14}$$

all such functions m_G constitute the solution set $\Sigma(G)$ defined in proposition 1.2.

We now discuss the two-dimensional case. Assume dimension $N = 2$. Then every divergence-free field G can be written as a rotated gradient:

$$G(x) = T\nabla u(x) = \nabla^\perp u(x),$$

where $\nabla u(x) = (u_{x_1}, u_{x_2})$ and $T(a,b) = (a,b)^\perp = (-b,a)$. Therefore, if $\alpha = 0$, the energy $J(G)$ can be written as a variational integral

$$J(G) = E(u) = \int_{\mathbb{R}^2} \Phi(x, \nabla u(x))\,\mathrm{d}x, \tag{1.15}$$

where $u : \mathbb{R}^2 \to \mathbb{R}$ is measurable with distributional gradient ∇u belonging to $L^2(\mathbb{R}^2;\mathbb{R}^2)$ and

$$\Phi(x,\lambda) = \chi_\Omega(x)\psi(\lambda) + \chi_{\Omega^c}(x)\tfrac{1}{2}|\lambda|^2, \quad \lambda \in \mathbb{R}^2,$$

where $\psi(\lambda) = \Psi(\lambda^\perp)$.

Proposition 1.6. *Let $N = 2$ and $\alpha = 0$. Then any minimizer $\bar{u}$ of E gives a minimizer $\bar{G}$ of J in terms of $\bar{G} = \nabla^\perp u$, and hence gives a minimizer $\bar{m}$ of I in terms of $\bar{m}(x) \in \sigma(\nabla^\perp u(x))$ for almost every $x \in \Omega$.*

It is interesting to note that, in some cases, one can compute the density $\psi(\lambda) = \Psi(\lambda^\perp)$ explicitly. In fact, in the soft case $\varphi \equiv 0$, one can find easily that

$$\psi(\lambda) = \tfrac{1}{2}(|\lambda - H^\perp| - 1)^2 - \tfrac{1}{2}|H|^2 + \lambda \cdot H^\perp.$$

In the uniaxial case with e pointing in the easy direction, one can find

$$\psi(\lambda) = \tfrac{1}{2}(|\lambda|^2 + 1) + \beta - \sqrt{|\lambda - H^\perp|^2 + 2\beta|(\lambda - H^\perp)\cdot e^\perp| + \beta^2},$$

where $\beta > 0$ is the parameter associated with the strength of anisotropy. A similar expression can be given in the biaxial case (see § 4.2).

The analysis of the corresponding variational problems leads to a scalar, non-convex problem for certain integral functionals of gradients on the whole space. In § 2, we study a general variational principle which works explicitly for such functionals on the whole space. In this setting, some existence results (see, for example, [3–5, 7, 8, 20]) under lack of convexity can nevertheless be applied. Then, in §§ 3 and 4, we apply this general principle to the soft and hard cases of two-dimensional ferromagnetism. In the soft case, we have the existence of minimizers for all domains and all applied fields (§ 3), and sometimes we even have explicit solutions. In the hard case, we have results on the non-existence of minimizers in the uniaxial case for some applied fields H (these are already very well known [7, 9, 13]) and the existence of minimizers for other H, and similar results in the cubic (biaxial) case (§ 4). In the case of a unit disc, we have a rather explicit characterization of the set $\mathcal{H}$. We emphasize that the main purpose of this (and the forthcoming) paper is to provides a new method for the micromagnetism problem parallel to the existing methods in [7, 9, 13, 17].

2. A general variational principle

Though the minimization problem in (1.15) is equivalent to the micromagnetics functional only for dimension $N = 2$, it is worth analysing such a functional regardless of dimension. In this section we therefore focus on the minimization problem for variational functionals of the type defined by (1.15); that is,

$$E(u) = \int_{\mathbb{R}^N} \Phi(x, \nabla u(x))\, \mathrm{d}x = \int_{\Omega} \psi(\nabla u(x))\, \mathrm{d}x + \frac{1}{2}\int_{\Omega^{\mathrm{c}}} |\nabla u(x)|^2\, \mathrm{d}x,$$

where N is any dimension, $\Omega \subset \mathbb{R}^N$ is a bounded domain with piecewise smooth boundary and $\psi : \mathbb{R}^N \to \mathbb{R}$ is a locally Lipschitz function satisfying, for some constants $c_2 \geqslant c_0 > 0$, $c_1, c_3 \in \mathbb{R}$,

$$c_0|\lambda|^2 - c_1 \leqslant \psi(\lambda) \leqslant c_2|\lambda|^2 + c_3, \quad \lambda \in \mathbb{R}^N.$$

The natural admissible class for $E(u)$ is the space of functions $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ whose gradient ∇u belongs to $L^2(\mathbb{R}^N; \mathbb{R}^N)$; in this case, the functional E is finite valued. Also, it is easy to see that $E(u) = E(u + c)$ for all constants $c \in \mathbb{R}$. Note that for all $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ the trace $\Gamma u = u|_{\partial\Omega}$ is well defined and, by the trace theorem [1], the trace operator $\Gamma : H^1_{\mathrm{loc}}(\mathbb{R}^N) \to H^{1/2}(\partial\Omega)$ is onto; moreover, Γ is also compact from $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ into $L^2(\partial\Omega)$.

We introduce the following function spaces:

$$X = \left\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \;\middle|\; \nabla u \in L^2(\mathbb{R}^N; \mathbb{R}^N),\ \int_{\partial\Omega} \Gamma u(x)\, \mathrm{d}S = 0 \right\}, \tag{2.1}$$

$$Y = \Gamma(X) = \left\{ g \in H^{1/2}(\partial\Omega) \;\middle|\; \int_{\partial\Omega} g\, \mathrm{d}S = 0 \right\}. \tag{2.2}$$

Define spaces X_1, X_2 to be the restriction of X on Ω and Ω^{c}, respectively. Note that if $u_i \in X_i$ $(i = 1, 2)$ satisfy $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$, then the function $u = \chi_\Omega u_1 + \chi_{\Omega^{\mathrm{c}}} u_2 \in X$.

We study the minimization problem for functional E on the space X. To this end, we define the relaxation functional of E by

$$E^{\#}(u) = \int_{\Omega} \psi^{\#}(\nabla u(x))\, \mathrm{d}x + \frac{1}{2}\int_{\Omega^{\mathrm{c}}} |\nabla u(x)|^2\, \mathrm{d}x, \tag{2.3}$$

where $\psi^{\#}$ is the convexification of ψ.

The following result establishes that the standard relaxation theorem still holds in this case where $E(u)$ is not a standard variational integral on bounded domains.

PROPOSITION 2.1. *It follows that*

$$\inf_{u\in X} E(u) = \min_{u\in X} E^{\#}(u).$$

Proof. Since E is not a standard variational integral defined on bounded domains, we include the proof for completeness. We show first that $\inf_X E = \inf_X E^{\#}$ and then that the infimum of $E^{\#}$ is attained. Assume that $\inf_X E^{\#} < \infty$. Given any $\varepsilon > 0$, let $u_\varepsilon \in X$ be such that $E^{\#}(u_\varepsilon) \leqslant \inf_X E^{\#} + \varepsilon$. Let $g_\varepsilon = \Gamma u_\varepsilon = u_\varepsilon|_{\partial\Omega} \in Y$. By the standard relaxation theorem [16],

$$\inf_{v\in H^1(\Omega),\, v|_{\partial\Omega}=g_\varepsilon} \int_\Omega \psi(\nabla v(x))\,\mathrm{d}x = \inf_{v\in H^1(\Omega),\, v|_{\partial\Omega}=g_\varepsilon} \int_\Omega \psi^{\#}(\nabla v(x))\,\mathrm{d}x.$$

Hence, there exists $v_\varepsilon \in H^1(\Omega)$ with $v_\varepsilon|_{\partial\Omega} = g_\varepsilon$ such that

$$\int_\Omega \psi(\nabla v_\varepsilon)\,\mathrm{d}x \leqslant \inf_{v\in H^1(\Omega),\ v|_{\partial\Omega}=g_\varepsilon} \int_\Omega \psi^{\#}(\nabla v)\,\mathrm{d}x + \varepsilon \leqslant \int_\Omega \psi^{\#}(\nabla u_\varepsilon)\,\mathrm{d}x + \varepsilon.$$

Let $\tilde{u} = \chi_\Omega v_\varepsilon + \chi_{\Omega^c} u_\varepsilon$. Then $\tilde{u} \in X$ and

$$\begin{aligned} E(\tilde{u}) &= \int_\Omega \psi(\nabla v_\varepsilon)\,\mathrm{d}x + \frac{1}{2}\int_{\Omega^c} |\nabla u_\varepsilon|^2\,\mathrm{d}x \\ &\leqslant \int_\Omega \psi^{\#}(\nabla u_\varepsilon)\,\mathrm{d}x + \varepsilon + \frac{1}{2}\int_{\Omega^c} |\nabla u_\varepsilon|^2\,\mathrm{d}x \\ &= E^{\#}(u_\varepsilon) + \varepsilon \\ &\leqslant \inf_X E^{\#} + 2\varepsilon. \end{aligned}$$

From this it follows that $\inf_X E \leqslant \inf_X E^{\#}$. Hence, $\inf_X E = \inf_X E^{\#}$. To show that $E^{\#}$ has a minimizer on X, we use the standard direct method. Let $\{u_j\}$ be a minimizing sequence of $E^{\#}$. Then $\{\nabla u_j\}$ is a bounded sequence in $L^2(\mathbb{R}^N;\mathbb{R}^N)$. Hence, assume (via a subsequence) $\nabla u_j \rightharpoonup U$ weakly in $L^2(\mathbb{R}^N;\mathbb{R}^N)$ for some function $U \in L^2(\mathbb{R}^N;\mathbb{R}^N)$. The gradient structure of ∇u_j and this weak convergence imply that

$$\int_{\mathbb{R}^N} U(x)\cdot\zeta(x)\,\mathrm{d}x = 0$$

for all test functions $\zeta \in C_0^\infty(\mathbb{R}^N;\mathbb{R}^N)$ with $\operatorname{div}\zeta = 0$. Therefore, U is a gradient; that is, $U = \nabla\bar{u}$ for some function $\bar{u} \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ (see, for example, [21, pp. 13–16]). On the other hand, since

$$\int_{\partial\Omega} \Gamma u_j\,\mathrm{d}S = 0,$$

it follows that $\{u_j\}$ is a bounded sequence in $H^1(\Omega)$. Hence, assume (also via a subsequence) $u_j \rightharpoonup \bar{v}$ weakly in $H^1(\Omega)$. The compactness of the trace operator also implies that

$$\int_{\partial\Omega} \Gamma\bar{v}\,\mathrm{d}S = 0.$$

Since $\nabla\bar{u} = \nabla\bar{v}$ in Ω, it follows that $\bar{u}(x) = \bar{v}(x) + c$ for almost every $x \in \Omega$ and a constant c. Therefore, $\tilde{u} = \bar{u} - c \in X$ and from the lower semicontinuities of both parts of $E^{\#}(u)$ it follows easily that $E^{\#}(\tilde{u}) \leqslant \liminf E^{\#}(u_j) = \inf_X E^{\#}$. Hence, $\tilde{u} \in X$ is a minimizer of $E^{\#}$ on X. $\square$

To characterize the minimizers of $E^{\#}$, we first have the following exterior uniqueness result.

PROPOSITION 2.2. *Let u_1, u_2 be any two minimizers of $E^{\#}$ on X. Then $u_1 = u_2$ on Ω^{c}. Therefore, there exists a unique $\bar{g} \in Y$ such that all minimizers $\bar{u}$ of $E^{\#}$ satisfy $\Gamma \bar{u} = \bar{u}|_{\partial\Omega} = \bar{g}$.*

Proof. We write $E^{\#}(u) = P_1(u) + P_2(u)$, where

$$P_1(u) = \int_{\Omega} \psi^{\#}(\nabla u(x))\,\mathrm{d}x, \qquad P_2(u) = \frac{1}{2}\int_{\Omega^{\mathrm{c}}} |\nabla u(x)|^2\,\mathrm{d}x.$$

Let $f(t) = E^{\#}(tu_1 + (1-t)u_2)$ and $p_i(t) = P_i(tu_1 + (1-t)u_2)$, $i = 1, 2$, for $0 \leqslant t \leqslant 1$. Then all these functions are convex on $[0, 1]$ and $f(t) = p_1(t) + p_2(t)$. Moreover, $f(0) = f(1) = \min_{[0,1]} f$. From this it follows easily that $f(t) = f(0)$ for all $t \in [0, 1]$. Since $p_i(t) \leqslant tp_i(1) + (1-t)p_i(0)$, $i = 1, 2$, the equality $f(t) = f(0) = f(1)$ implies that $p_i(t) = tp_i(1) + (1-t)p_i(0)$ for $i = 1, 2$. In particular, $p_2(\frac{1}{2}) = \frac{1}{2}(p_2(0) + p_2(1))$. This equality implies $\nabla u_1 = \nabla u_2$ on Ω^{c}. Hence, $u_1 = u_2$ on Ω^{c} since

$$\int_{\partial\Omega} \Gamma u_i \,\mathrm{d}S = 0 \quad \text{for } i = 1, 2.$$

In particular, $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$. This proves the proposition. $\square$

If $\psi^{\#}$ is C^1, the minimizers of $E^{\#}$ can be exactly solved as the weak solutions of the Euler–Lagrange equation for $E^{\#}$. From this, we have the following result characterizing the minimizers of $E^{\#}$ and hence the boundary data $\bar{g}$.

THEOREM 2.3. *Let $\psi^{\#}$ be C^1. Then, $\bar{u} = \chi_{\Omega}\bar{v} + \chi_{\Omega^{\mathrm{c}}}\bar{w}$ is a minimizer of $E^{\#}$ if and only if $\bar{v}$, $\bar{w}$ satisfy the following conditions:*

$$\left.\begin{aligned} \bar{w}|_{\partial\Omega} &= \bar{v}|_{\partial\Omega}, \\ \Delta\bar{w} &= 0 && \text{in } \Omega^{\mathrm{c}}, \\ \operatorname{div}[(\psi^{\#})'(\nabla\bar{v})] &= 0 && \text{in } \Omega, \\ \frac{\partial\bar{w}}{\partial n} &= (\psi^{\#})'(\nabla\bar{v})\cdot n && \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.4}$$

where $n(x)$ is the unit outward normal to the boundary $\partial\Omega$ of the interior domain Ω.

Proof. Since $E^{\#}(u + c) = E^{\#}(u)$ for all $c \in \mathbb{R}$, it follows that a function $\bar{u} = \chi_{\Omega}\bar{v} + \chi_{\Omega^{\mathrm{c}}}\bar{w}$ is a minimizer of $E^{\#}$ on X if and only if it is a minimizer of $E^{\#}$ on the linear space $X_0 = \{u \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \mid \nabla u \in L^2(\mathbb{R}^N; \mathbb{R}^N)\}$. Since $E^{\#}$ is convex on X_0, the minimizers of $E^{\#}$ on X_0 are exactly the weak solutions of the Euler–Lagrange equation. Therefore, $\bar{u} = \chi_{\Omega}\bar{v} + \chi_{\Omega^{\mathrm{c}}}\bar{w}$ is a minimizer of $E^{\#}$ if and only if

$$\int_{\Omega} (\psi^{\#})'(\nabla\bar{v})\cdot\nabla\zeta\,\mathrm{d}x + \int_{\Omega^{\mathrm{c}}} \nabla\bar{w}\cdot\nabla\zeta\,\mathrm{d}x = 0 \quad \text{for all } \zeta \in X_0. \tag{2.5}$$

Clearly, this condition is equivalent to the last three conditions in (2.4). $\square$

REMARK 2.4. Certainly, (2.4) is a difficult condition involving the mixed Dirichlet–Neumann boundary conditions and depending heavily on the structures of function $\psi^{\#}$ and domain Ω; nevertheless, it always has solutions. Also, from proposition 2.2, although the solution $\bar{v}$ to (2.4) may not be unique, the solution $\bar{w}$ and hence the boundary trace $\bar{g} = \bar{w}|_{\partial\Omega} = \bar{v}|_{\partial\Omega}$ must be unique.

Using the unique boundary data $\bar{g}$, we can derive a necessary and sufficient condition for minimizers of $E^{\#}$.

THEOREM 2.5. *Let $\bar{g} \in Y$ be the unique boundary data determined in the previous proposition. Then $\bar{u} \in X$ is a minimizer of $E^{\#}$ on X if and only if $\bar{u} = \chi_{\Omega}\bar{v} + \chi_{\Omega^{\mathrm{c}}}\bar{w}$, where $\bar{w} = \omega(\bar{g})$ is the unique solution to the Dirichlet problem:*

$$\Delta\bar{w} = 0 \quad \text{in } \Omega^{\mathrm{c}}, \qquad \bar{w}|_{\partial\Omega} = \bar{g}, \tag{2.6}$$

and $\bar{v}$ is any minimizer of the following problem:

$$\int_{\Omega} \psi^{\#}(\nabla\bar{v})\,\mathrm{d}x = \min_{v\in H^1(\Omega),\, v|_{\partial\Omega}=\bar{g}} \int_{\Omega} \psi^{\#}(\nabla v)\,\mathrm{d}x. \tag{2.7}$$

Proof. First, assume that $\bar{u} \in X$ is a minimizer of $E^{\#}$. Then, by proposition 2.2, $\bar{u}|_{\partial\Omega} = \bar{g}$. Let $E^{\#}(u) = P_1(u) + P_2(u)$ with P_i introduced in the previous proof. Since $E^{\#}(\bar{u}) \leqslant E^{\#}(\bar{u} + \zeta)$ for all $\zeta \in X$, it follows by choosing $\zeta = 0$ on Ω^{c} or on Ω that $\bar{u}$ minimizes each P_i with the given boundary data $\bar{g}$. The minimizer of P_2 must be harmonic and given by the unique solution $\bar{w} = \omega(\bar{g})$ of the Dirichlet problem (2.6) above. Therefore, $\bar{u} = \chi_{\Omega}\bar{v} + \chi_{\Omega^{\mathrm{c}}}\omega(\bar{g})$, where $\bar{v} \in X_1$ is a minimizer of problem (2.7). Now, assume that $\bar{u} = \chi_{\Omega}\bar{v} + \chi_{\Omega^{\mathrm{c}}}\bar{w}$, where $\bar{v} \in X_1$ is a minimizer of problem (2.7) and $\bar{w} = \omega(\bar{g})$ is the solution of the Dirichlet problem (2.6). We want to show $E^{\#}(\bar{u}) = \min_X E^{\#}$. Let $\tilde{u} \in X$ be a minimizer of $E^{\#}$ on X; that is,

$$\min_X E^{\#} = E^{\#}(\tilde{u}) = \int_{\Omega} \psi^{\#}(\nabla\tilde{u})\,\mathrm{d}x + \frac{1}{2}\int_{\Omega^{\mathrm{c}}} |\nabla\tilde{u}|^2\,\mathrm{d}x.$$

Since $\tilde{u}|_{\partial\Omega} = \bar{g}$, by the assumptions of $\bar{v}$ and $\bar{w}$, the first term on the right-hand side of this equation is no less than

$$\int_{\Omega} \psi^{\#}(\nabla\bar{v})\,\mathrm{d}x$$

and the second term is no less than

$$\frac{1}{2}\int_{\Omega^{\mathrm{c}}} |\nabla\bar{w}|^2\,\mathrm{d}x.$$

Hence, $E^{\#}(\tilde{u}) \geqslant E^{\#}(\bar{u})$. This shows that $E^{\#}(\bar{u}) = E^{\#}(\tilde{u}) = \min_X E^{\#}$. □

We now give the necessary and sufficient condition for the existence of minimizers of the original functional $E(u)$.

THEOREM 2.6. *Let $\bar{g} \in Y$ be the unique boundary data determined in proposition 2.2. Then, the functional E has a minimizer over X if and only if the minimization problem*

$$\inf_{v\in H^1(\Omega),\, v|_{\partial\Omega}=\bar{g}} \int_{\Omega} \psi(\nabla v(x))\,\mathrm{d}x \tag{2.8}$$

has a minimizer. Moreover, a function $\bar u \in X$ is a minimizer of E if and only if

$$\bar u = \chi_\Omega \tilde v + \chi_{\Omega^c}\omega(\bar g), \tag{2.9}$$

where $\tilde v$ is any minimizer of problem (2.8).

Corollary 2.7. *$\bar u \in X$ is a minimizer of E if and only if $\bar u$ is a minimizer of $E^\#$ and*

$$\psi(\nabla \bar u(x)) = \psi^\#(\nabla \bar u(x)) \quad \textit{for all } x \in \Omega. \tag{2.10}$$

Furthermore, if $\psi^\#$ is C^1 and $\bar u = \chi_\Omega \bar v + \chi_{\Omega^c}\bar w$, then $\bar u$ is a minimizer of E if and only if $\bar v$, $\bar w$ satisfy the following conditions:

$$\left.\begin{aligned} \bar w|_{\partial\Omega} &= \bar v|_{\partial\Omega}; && \\ \Delta \bar w &= 0 && \textit{in } \Omega^c; \\ \operatorname{div}[(\psi^\#)'(\nabla \bar v)] &= 0 && \textit{in } \Omega; \\ \frac{\partial \bar w}{\partial n} &= (\psi^\#)'(\nabla \bar v)\cdot n && \textit{on } \partial\Omega; \\ \psi^\#(\nabla \bar v) &= \psi(\nabla \bar v) && \textit{in } \Omega. \end{aligned}\right\} \tag{2.11}$$

Remark 2.8. If both ψ and $\psi^\#$ are C^1, then any function $\bar u$ satisfying condition (2.11) must be a weak solution of the Euler–Lagrange equation for the functional $E(u)$. To see this, we simply note that in this case $(\psi^\#)'(\lambda_0) = \psi'(\lambda_0)$ whenever $\psi^\#(\lambda_0) = \psi(\lambda_0)$. However, as in most non-convex problems, the Euler–Lagrange equation for $E(u)$ alone only provides a necessary condition for possible minimizers of $E(u)$, but it is far from being sufficient for the existence of minimizers. When the condition $\psi(\nabla \bar u) = \psi^\#(\nabla \bar u)$ a.e. in Ω is not satisfied by any minimizers $\bar u$ of $E^\#$ the (non-convex) functional E does not have any minimizers.

The following result provides a sufficient condition for existence of minimizers of E.

Corollary 2.9. *Assume that $\psi^\#$ is affine on each component of the detachment set $\mathcal{D} = \{\lambda \mid \psi^\#(\lambda) < \psi(\lambda)\}$. Then the energy E has minimizers.*

Proof. Under the given condition, we can easily see that the conditions of [3, theorem 2.1] are all satisfied. This implies that the functional

$$F(v) = \int_\Omega \psi(\nabla v(x))\,\mathrm{d}x$$

itself has minimizers for all boundary data g. Let $\tilde v \in X_1$ be any minimizer of F for the unique boundary data $\bar g$ determined above. Then, by theorem 2.6, the function

$$\tilde u = \chi_\Omega \tilde v + \chi_{\Omega^c}\omega(\bar g)$$

is a minimizer of E. □

Finally, we have an explicit result in the case when domain Ω is the unit ball.

THEOREM 2.10. *Let $\Omega = B$ be the unit ball in $\mathbb{R}^N$. Then*

$$\inf_X E = \min_X E^{\#} = |B| \min_{\lambda \in \mathbb{R}^N} \left(\psi^{\#}(\lambda) + \frac{N-1}{2}|\lambda|^2 \right) = |B| \left(\psi^{\#}(\bar{\lambda}) + \frac{N-1}{2}|\bar{\lambda}|^2 \right),$$

where $\bar{\lambda} \in \mathbb{R}^N$ is uniquely determined. In this case, the minimizers of $E^{\#}$ are given by

$$\bar{u} = \chi_B(x)\bar{v}(x) + \chi_{B^c}(x)\frac{\bar{\lambda}\cdot x}{|x|^N},$$

where $\bar{v}$ is any function satisfying

$$\bar{v}|_{\partial B} = \bar{\lambda}\cdot x, \qquad \int_B \psi^{\#}(\nabla \bar{v}(x))\,\mathrm{d}x = |B|\psi^{\#}(\bar{\lambda}).$$

Furthermore, E has a minimizer if and only if either $\psi(\bar{\lambda}) = \psi^{\#}(\bar{\lambda})$ or

$$\bar{\lambda} \in \operatorname{intconv}\{\lambda_1, \lambda_2, \ldots, \lambda_k\} \quad (\textit{the interior of the convex hull of } \lambda_1, \ldots, \lambda_k)$$

for some $\lambda_j \in \mathbb{R}^N$, $j = 1, 2, \ldots, k$, satisfying $\bigcap_{j=1}^k \partial\psi(\lambda_j) \neq \emptyset$.

REMARK 2.11.

(i) If $\psi^{\#}$ is C^1, then $\bar{\lambda}$ is uniquely determined by the algebraic equation

$$(\psi^{\#})'(\bar{\lambda}) + (N-1)\bar{\lambda} = 0. \tag{2.12}$$

(ii) The last condition $\bigcap_{j=1}^k \partial\psi(\lambda_j) \neq \emptyset$ is equivalent to $\psi^{\#}$ being affine on the convex hull $\operatorname{conv}\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$.

We need the following elementary result for proving this theorem.

LEMMA 2.12. *Let f and g be two functions on $\mathbb{R}^N$ and let f be convex. Assume that $f+g$ has a local minimum at λ_0 and g is differentiable at λ_0. Then it follows that*

$$f(\lambda_0 + \eta) \geqslant f(\lambda_0) - g'(\lambda_0)\cdot\eta \quad \textit{for all } \eta \in \mathbb{R}^N.$$

Proof. Suppose the inequality does not hold. Then there exist $\eta_0 \in \mathbb{R}^N$ and $\varepsilon_0 > 0$ such that

$$f(\lambda_0 + \eta_0) < f(\lambda_0) - g'(\lambda_0)\cdot\eta_0 - \varepsilon_0.$$

Let $h(t) = f(\lambda_0 + t\eta_0) - f(\lambda_0) + g'(\lambda_0)\cdot t\eta_0$. Then h is convex in $t \in \mathbb{R}$ and $h(0) = 0, h(1) < -\varepsilon_0$. By convexity, for all $0 < t < 1$,

$$\frac{h(t) - h(0)}{t - 0} \leqslant \frac{h(1) - h(0)}{1 - 0}$$

and hence $h(t) \leqslant -\varepsilon_0 t$ for all $t \in (0,1)$. This implies that

$$f(\lambda_0 + t\eta_0) \leqslant f(\lambda_0) - g'(\lambda_0)\cdot t\eta_0 - \varepsilon_0 t \quad \text{for all } t \in (0,1).$$

On the other hand, since g is differentiable at λ_0, it follows that

$$g(\lambda_0 + t\eta_0) = g(\lambda_0) + g'(\lambda_0)\cdot t\eta_0 + o(t), \quad t \to 0^+.$$

Therefore, we would have $f(\lambda_0 + t\eta_0) + g(\lambda_0 + t\eta_0) < f(\lambda_0) + g(\lambda_0)$ for all sufficiently small $t > 0$. This contradicts the local minimality of $f + g$ at λ_0. □

Proof of theorem 2.10. Let $\bar{u} = \chi_B \bar{v} + \chi_{B^c} \bar{w}$, where $\bar{v}(x) = \bar{\lambda} \cdot x$ and $\bar{w}(x) = (\bar{\lambda} \cdot x)/|x|^N$. Then $\bar{u} \in X$ and we will show that $\bar{u}$ is a minimizer of $E^{\#}$. It is easy to show by elementary calculations that

$$\Delta \bar{w} = 0, \quad \frac{\partial \bar{w}}{\partial n} = (1-N)\bar{\lambda} \cdot x, \quad \int_{B^c} |\nabla \bar{w}|^2 = |B|(N-1)|\bar{\lambda}|^2.$$

Using lemma 2.12 for $f = \psi^{\#}$ and $g = \frac{1}{2}(N-1)|\lambda|^2$, we have

$$\psi^{\#}(\bar{\lambda} + \eta) \geqslant \psi^{\#}(\bar{\lambda}) - (N-1)\bar{\lambda} \cdot \eta \quad \text{for all } \eta \in \mathbb{R}^N. \tag{2.13}$$

Then, for all $\zeta \in X$,

$$\begin{aligned} E^{\#}(\bar{u} + \zeta) &= \int_B \psi^{\#}(\bar{\lambda} + \nabla\zeta)\,\mathrm{d}x + \frac{1}{2}\int_{B^c} |\nabla \bar{w} + \nabla\zeta|^2\,\mathrm{d}x \\ &= \int_B \psi^{\#}(\bar{\lambda} + \nabla\zeta)\,\mathrm{d}x + \frac{1}{2}\int_{B^c} |\nabla \bar{w}|^2\,\mathrm{d}x \\ &\qquad + \int_{B^c} \nabla\bar{w} \cdot \nabla\zeta\,\mathrm{d}x + \frac{1}{2}\int_{B^c} |\nabla\zeta|^2\,\mathrm{d}x \\ &\geqslant \int_B [\psi^{\#}(\bar{\lambda}) - (N-1)\bar{\lambda} \cdot \nabla\zeta]\,\mathrm{d}x + \frac{N-1}{2}|\bar{\lambda}|^2|B| \\ &\qquad - \int_{\partial B} \frac{\partial \bar{w}}{\partial n}\zeta\,\mathrm{d}S + \frac{1}{2}\int_{B^c} |\nabla\zeta|^2\,\mathrm{d}x \\ &= |B|\bigg[\psi^{\#}(\bar{\lambda}) + \frac{N-1}{2}|\bar{\lambda}|^2\bigg] + \frac{1}{2}\int_{B^c} |\nabla\zeta|^2\,\mathrm{d}x \\ &= E^{\#}(\bar{u}) + \frac{1}{2}\int_{B^c} |\nabla\zeta|^2\,\mathrm{d}x. \end{aligned}$$

This proves that

$$E^{\#}(\bar{u}) = |B|\bigg(\psi^{\#}(\bar{\lambda}) + \frac{N-1}{2}|\bar{\lambda}|^2\bigg)$$

is the minimum of $E^{\#}$. The remaining part of the necessary and sufficient condition for the existence of minimizers of E follows from theorem 2.6 and the results of [4,5] (see also [20, theorem 1.1]). $\square$

3. The two-dimensional soft ferromagnetism

We will first apply our general principle to the simple case of two-dimensional soft ferromagnetism. Therefore, assume that $N = 2$ and $\varphi \equiv 0$. In this case, after some elementary calculus computations, the function $\Psi(\xi)$ defined by (1.12) can be written as

$$\Psi(\xi) = \min_{h \in S^1}[-(H + \xi) \cdot h + \tfrac{1}{2}(|\xi|^2 + 1)] = \tfrac{1}{2}(|\xi + H| - 1)^2 - \tfrac{1}{2}|H|^2 - \xi \cdot H,$$

and the set $\sigma(\xi)$ is given by

$$\sigma(-H) = S^1; \qquad \sigma(\xi) = \left\{ \frac{H + \xi}{|H + \xi|} \right\} \quad \text{if } \xi \neq -H.$$

The corresponding functional $E(u)$ defined by (1.15) above is thus

$$E(u)=\int_{\mathbb{R}^2}\Phi(x,\nabla u(x))\,\mathrm{d}x=\int_{\Omega}\psi(\nabla u(x))\,\mathrm{d}x+\frac{1}{2}\int_{\Omega^{\mathrm{c}}}|\nabla u(x)|^2\,\mathrm{d}x,$$

where

$$\psi(\lambda)=\Psi(\lambda^{\perp})=\tfrac{1}{2}(|\lambda-H^{\perp}|-1)^2+\lambda\cdot H^{\perp}-\tfrac{1}{2}|H|^2.$$

The convexification of ψ can be easily computed and expressed in the form

$$\psi^{\#}(\lambda)=\begin{cases}\tfrac{1}{2}(|\lambda-H^{\perp}|-1)^2+\lambda\cdot H^{\perp}-\tfrac{1}{2}|H|^2 & \text{if } |\lambda-H^{\perp}|\geqslant 1,\\ \lambda\cdot H^{\perp}-\tfrac{1}{2}|H|^2 & \text{if } |\lambda-H^{\perp}|\leqslant 1.\end{cases}$$

Note that $\psi^{\#}$ is in fact C^1 and

$$(\psi^{\#})'(\lambda)=\begin{cases}\lambda-\dfrac{\lambda-H^{\perp}}{|\lambda-H^{\perp}|} & \text{if } |\lambda-H^{\perp}|\geqslant 1,\\ H^{\perp} & \text{if } |\lambda-H^{\perp}|\leqslant 1.\end{cases}$$

The detachment set for this function ψ is $\mathcal{D}=\{|\lambda-H^{\perp}|<1\}$, on which the convexification $\psi^{\#}$ is affine. By corollary 2.9, we easily have the following existence result.

Theorem 3.1. *In the two-dimensional soft case, the energy E and hence the energy I always have minimizers. Moreover, a minimizer $\bar{m}$ of I is given in the form of*

$$\bar{m}(x)=\frac{H+\nabla^{\perp}\bar{u}(x)}{|H+\nabla^{\perp}\bar{u}(x)|}\quad \textit{for a.e. } x\in\Omega,$$

where $\bar{u}$ is a minimizer of E.

However, even in this concrete case, finding a minimizer of E may be a very difficult problem, as the necessary and sufficient condition for minimizers illustrated by condition (2.11) is difficult to study. By $(2.11)_3$, only functions $u\in X$ satisfying $|\nabla u(x)-H^{\perp}|\geqslant 1$ for almost every $x\in\Omega$ can be minimizers of energy E.

We will examine two special cases.

3.1. The case of small external applied fields

First, we focus on functions $u\in X$ that satisfy

$$|\nabla u(x)-H^{\perp}|=1\quad \text{a.e. } x\in\Omega. \tag{3.1}$$

For all such functions, the third condition in (2.4) always holds, and the second and fourth conditions in (2.4) reduce to the exterior Neumann boundary problem:

$$\Delta u=0\quad\text{in } \Omega^{\mathrm{c}},\qquad \frac{\partial u}{\partial n}=H^{\perp}\cdot n\quad\text{on } \partial\Omega. \tag{3.2}$$

This problem has a unique solution $u=w_H\in X_2$, which also depends linearly on H. In fact, let $w_i\in X_2$, $i=1,2$, be the unique solution of

$$\Delta w_i=0\quad\text{in } \Omega^{\mathrm{c}},\qquad \frac{\partial w_i}{\partial n}=n_i\quad\text{on } \partial\Omega,$$

where $n = (n_1, n_2)$, as always, is the outward unit normal to the interior domain Ω. Then the solution $u = w_H$ of the Neumann problem (3.2) above can be written as

$$w_H = -h_2 w_1 + h_1 w_2 = H^{\perp} \cdot (w_1, w_2) \quad \text{in } \Omega^{\mathrm{c}}.$$

Let $u_i \in X$ be an extension of w_i to whole $\mathbb{R}^2$, $i = 1, 2$, and define

$$u_H = -h_2 u_1 + h_1 u_2 = H^{\perp} \cdot (u_1, u_2) \quad \text{in } \mathbb{R}^2.$$

Define $\tilde{g} = w_H|_{\partial\Omega} \in Y$. Under condition (3.1), this $\tilde{g}$ will be the unique boundary data $\bar{g}$ as determined in the general setting above. However, equation (3.1) must be combined with the boundary condition

$$u|_{\partial\Omega} = \tilde{g} = w_H|_{\partial\Omega}. \tag{3.3}$$

We assume that Ω is sufficiently smooth so that $u_i \in X \cap C^1(\mathbb{R}^2)$. It is well known [8] that problem (3.1), (3.3) admits (infinitely many) solutions if

$$|\nabla u_H - H^{\perp}| \leqslant 1 \quad \text{in } \Omega. \tag{3.4}$$

We have thus proved the following.

Proposition 3.2. *If condition (3.4) holds, then the minimizers of E are given by*

$$u = \chi_{\Omega} v + \chi_{\Omega^{\mathrm{c}}} w_H,$$

where $w_H \in X_2$ is the unique solution of problem (3.2) and v is any function satisfying

$$|\nabla v(x) - H^{\perp}| = 1 \quad \text{in } \Omega, \qquad v|_{\partial\Omega} = w_H|_{\partial\Omega}.$$

In particular, when there is no applied field: $H = 0$, then the distance function $v(x) = \operatorname{dist}(x, \partial\Omega)$ provides a minimizer $u = \chi_{\Omega}(x) \operatorname{dist}(x, \partial\Omega)$.

Note that

$$|\nabla u_H(x) - H^{\perp}| = |h_1||\nabla u_2(x) - (0,1)| + |h_2||\nabla u_1(x) - (1,0)| \leqslant C|H|, \quad x \in \Omega,$$

where C is a constant depending only on the domain Ω. Therefore, condition (3.4) is always satisfied if $|H|$ is sufficiently small.

3.2. The case of the unit disc

Let $\Omega = B$, the unit disc in $\mathbb{R}^2$. By theorem 2.10 and remark 2.11, the unique boundary data $\bar{g} = \bar{\lambda} \cdot x$ and the corresponding exterior harmonic function $\omega(\bar{g})(x) = (\bar{\lambda} \cdot x)/|x|^2$, where $\bar{\lambda} \in \mathbb{R}^2$ is the unique solution of

$$(\psi^{\#})'(\bar{\lambda}) + \bar{\lambda} = 0.$$

Solving this equation yields $\bar{\lambda} = -H^{\perp}$ if $|H| \leqslant \frac{1}{2}$ and $\bar{\lambda} = -H^{\perp}/2|H|$ if $|H| \geqslant \frac{1}{2}$. Therefore, by theorem 2.6, we have the following.

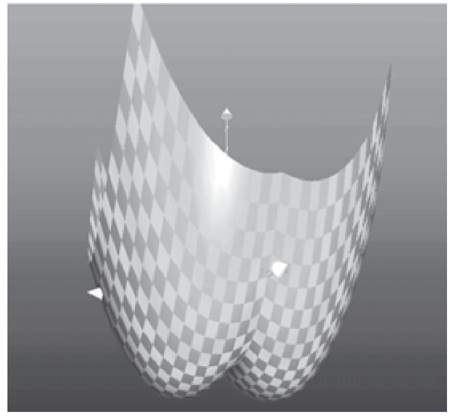

Figure 1. The graph of $h(t,s)$.

Proposition 3.3.

(i) *If* $|H|<\frac{1}{2}$, *then the minimizers of* E *are given non-uniquely by*

$$u=\chi_B(x)v(x)-\chi_{B^c}(x)\frac{H^\perp\cdot x}{|x|^2},$$

where v *is any function satisfying*

$$|\nabla v(x)-H^\perp|=1\quad in\ B,\qquad v|_{\partial B}=-H^\perp\cdot x. \tag{3.5}$$

(ii) *If* $|H|\geqslant\frac{1}{2}$, *then the (unique) minimizer of* E *is given by*

$$\bar{u}=-\chi_B(x)\frac{H^\perp\cdot x}{2|H|}-\chi_{B^c}(x)\frac{H^\perp\cdot x}{2|H||x|^2}.$$

4. The two-dimensional hard ferromagnetism

4.1. The uniaxial case

As mentioned in § 1, we assume that the anisotropy density φ is given by

$$\varphi(m)=\beta(1-|m\cdot e|),$$

where $\beta>0$ and $e\in S^1$ are given. In this case, we can easily compute the function Ψ defined above as follows:

$$\begin{aligned}\Psi(\xi)&=\min_{h\in S^1}[\beta-|\beta e\cdot h|-(H+\xi)\cdot h+\tfrac{1}{2}(|\xi|^2+1)]\\&=\tfrac{1}{2}(|\xi|^2+1)+\beta-\max\{|\xi+H+\beta e|,\ |\xi+H-\beta e|\}\\&=\tfrac{1}{2}(|\xi|^2+1)+\beta-\max_{\pm}\{|\xi+H\pm\beta e|\}.\end{aligned}$$

In this case, the density function ψ is given by

$$\psi(\lambda)=\Psi(\lambda^\perp)=\tfrac{1}{2}(|\lambda|^2+1)+\beta-\max_{\pm}\{|\lambda-(H\pm\beta e)^\perp|\}.$$

Let

$$E(u) = \int_{\Omega} \psi(\nabla u(x))\,\mathrm{d}x + \frac{1}{2}\int_{\Omega^{c}} |\nabla u(x)|^2\,\mathrm{d}x.$$

In order to study the minimization problem for $E(u)$, we need to find the convexification $\psi^{\#}$. Writing $\lambda = te^{\perp} + se + H^{\perp}$ with $(t,s) \in \mathbb{R}^2$, we can express $\psi(\lambda)$ in terms of (t,s) as follows:

$$\psi(\lambda) = h(t,s) + (e^{\perp}\cdot H^{\perp})t + (e\cdot H^{\perp})s + \beta + \tfrac{1}{2}(1+|H|^2),$$

where the nonlinear part $h(t,s)$ is given by

$$h(t,s) = \tfrac{1}{2}(t^2+s^2) - \sqrt{(|t|+\beta)^2+s^2}.$$

(See figure 1 for a graph of h.)

It suffices to compute the convexification $h^{\#}$ of h on $\mathbb{R}^2$. Note that h satisfies

$$h(-t,-s) = h(t,-s) = h(-t,s) = h(t,s), \quad (t,s)\in\mathbb{R}^2.$$

We can restrict the computation to the first quadrant $Q = \{t \geqslant 0, s \geqslant 0\}$. Note that on Q

$$h_t(t,s) = t - \frac{t+\beta}{\sqrt{(t+\beta)^2+s^2}}, \qquad h_s(t,s) = s - \frac{s}{\sqrt{(t+\beta)^2+s^2}}.$$

We want to know where $h(t,s)$ is increasing regarding t and s individually. So we consider the domain $\Gamma \subset Q$ defined by $h_t(t,s) \geqslant 0$ and $h_s(t,s) \geqslant 0$: Γ is the non-shaded area of Q in Figure 3(a). Then $\Gamma = \{(t,s)\in Q \mid t\sqrt{(t+\beta)^2+s^2} \geqslant t+\beta\}$. Write $\Gamma = \{(t,s)\in Q \mid s\geqslant 0,\ \gamma(s)\leqslant t<\infty\}$, where $t=\gamma(s)$, $s>0$, is the inverse function of function

$$s = \sigma(t) = \frac{t+\beta}{t}\sqrt{1-t^2}, \quad 0 < t \leqslant 1.$$

Easy calculations show that $h(t,s)$ is convex on Γ and increasing on t and on s. Let $\delta(s) = h(\gamma(s),s)$ for $s\geqslant 0$. Then $\delta(s)$ is C^{∞} on $s\geqslant 0$ and $\delta'(s) = h_s(\gamma(s),s) > 0$ for $s>0$, and hence δ is increasing on $s\geqslant 0$. We now define a function $\tilde{h}$ on Q by

$$\tilde{h}(t,s) = \begin{cases} h(\gamma(s),s), & s\geqslant 0,\ 0\leqslant t\leqslant\gamma(s),\\ h(t,s), & s\geqslant 0,\ \gamma(s)\leqslant t<\infty.\end{cases}$$

Then it follows easily that $\tilde{h}$ is convex on Q and non-decreasing in each of $t\geqslant 0$ and $s\geqslant 0$. We extend $\tilde{h}(t,s)$ onto $(t,s)\in\mathbb{R}^2$ according to the property

$$\tilde{h}(-t,-s) = \tilde{h}(t,-s) = \tilde{h}(-t,s) = \tilde{h}(t,s), \quad (t,s)\in\mathbb{R}^2.$$

Then this new function $\tilde{h}$ is convex on $\mathbb{R}^2$ and we have the following.

PROPOSITION 4.1. *The convexification $h^{\#}$ of h is equal to the extended function $\tilde{h}$. Therefore, the convexification $\psi^{\#}$ is given by $\psi^{\#}(\lambda) = \tilde{h}(t,s) + (\lambda - H^{\perp})\cdot H^{\perp} + \beta + \frac{1}{2}(1+|H|^2)$ and satisfies*

$$(\psi^{\#})'(\lambda) = \tilde{h}_t(t,s)e^{\perp} + \tilde{h}_s(t,s)e + H^{\perp}, \tag{4.1}$$

where $\lambda = te^{\perp}+se+H^{\perp}$.

REMARK 4.2. Note that $\psi^{\#}$ is not affine on the detachment set $\mathcal{D} = \{\lambda \mid \psi^{\#}(\lambda) < \psi(\lambda)\}$ and hence we cannot apply corollary 2.9. In fact, in this case, the minimizers may not exist. For example, assume that $H = 0$. Then $(\psi^{\#})'(0) = 0$; hence, by (2.4), the unique boundary data $\bar{g} = 0$. Moreover, any minimizer $\bar{u}$ of E must satisfy

$$\nabla \bar{u}(x) \in \{e^{\perp}, -e^{\perp}\} \quad \text{a.e. } x \in \Omega; \quad \bar{u}|_{\partial\Omega} = 0,$$

which is certainly impossible. Therefore, $E(u)$ does not have any minimizers for any domain Ω if $H = 0$. This was initially shown in [13], and further pursued and explored in many other works (see, in particular, [7]).

However, for some other H, the minimizers may exist. We study the special case for the unit disc. In this case, to determine the boundary data $\bar{g}$, we need to solve the equation $(\psi^{\#})'(\lambda) + \lambda = 0$. Writing $H^{\perp} = ae^{\perp} + be$ and using (4.1), this equation is equivalent to

$$\tilde{h}_t(t,s) + t + 2a = 0, \qquad \tilde{h}_s(t,s) + s + 2b = 0.$$

The solution $(\bar{t}, \bar{s})$ is uniquely determined by (a, b) and hence by H. Define the set

$$\mathcal{H} = \{H \in \mathbb{R}^2 \mid h(\bar{t}, \bar{s}) > \tilde{h}(\bar{t}, \bar{s})\}.$$

Then, it is easy to see that the line $\{se \mid s \in \mathbb{R}\} \subset \mathcal{H}$ and the half-rays $\{te^{\perp} \mid |t| \geqslant \frac{1}{2}\} \subset \mathbb{R}^2 \setminus \mathcal{H}$. Hence, $\mathcal{H} \subsetneq \mathbb{R}^2$.

PROPOSITION 4.3. *Let $\Omega = B$ be the unit disc. Then the uniaxial ferromagnetic energy $I(m)$ has (unique) minimizer if $H \notin \mathcal{H}$ and has no minimizers if $H \in \mathcal{H}$.*

Proof. Since $\tilde{h}$ and hence $\psi^{\#}$ are not affine on any open sets, by theorem 2.10, E and hence I have a minimizer if and only if $h(\bar{t}, \bar{s}) = \tilde{h}(\bar{t}, \bar{s})$. This proves the result. □

REMARK 4.4. This result says that if the applied field is in the direction of the easy axis, then there are no global minimizers, but the preferred magnetizations can easily form a microstructure to minimize the total energy. If the applied field is large and orthogonal to the easy direction, then forming the microstructure will cost more energy and the global energy minimizer exists.

4.2. The biaxial case

For the cubic (biaxial) situation we will set

$$\varphi(m) = \beta \min\{1 - |m \cdot e|, 1 - |m \cdot e^{\perp}|\},$$

where e is a unit vector pointing in the direction of one of the easy axes. Pursuing the computations in the uniaxial case with the same notation, one finds the expression

$$\psi(\lambda) = h(t,s) + (e^{\perp} \cdot H^{\perp})t + (e \cdot H^{\perp})s + \beta + \tfrac{1}{2}(1 + |H|^2),$$

where the nonlinear part $h(t, s)$ is given by

$$h(t,s) = \tfrac{1}{2}(t^2 + s^2) - \max\{\sqrt{(|t| + \beta)^2 + s^2},\ \sqrt{(|s| + \beta)^2 + t^2}\}.$$

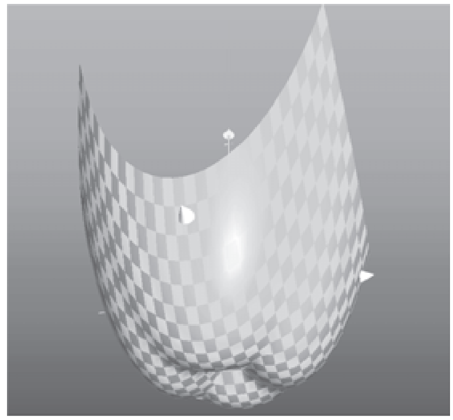

Figure 2. The graph of $h(t,s)$.

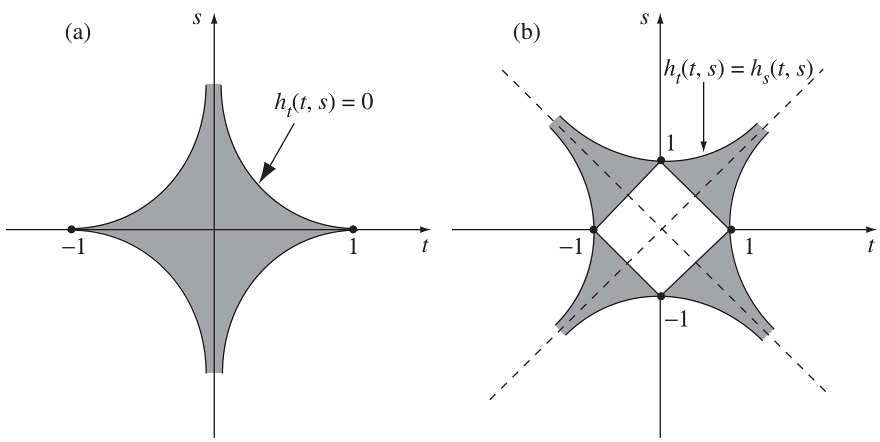

Figure 3. Detachment sets. (a) The uniaxial case (see figure 1). (b) The biaxial case (see figure 2). Shaded areas represent the non-existence of minimizers.

In addition to the symmetries that we had in the uniaxial case, we also have $h(t,s) = h(s,t)$. We can calculate that the detachment set is the region bounded by curves $h_t(t,s) = h_s(t,s)$ and also that the convexification $\tilde{h}$ of h is constant ($\tilde{h} = \min h$) on the square $Q = \{|t| + |s| \leqslant 1\}$ and is affine only on each line segment of $|t| + |s| = c$, $c \geqslant 1$, between the curves mentioned (see figure 3(b)).

Again, we consider the case when $\Omega = B$ is the unit disc. Let $\bar{\lambda} = \bar{t}e^{\perp} + \bar{s}e + H^{\perp}$ be the solution of $(\psi^{\#})'(\bar{\lambda}) + \bar{\lambda} = 0$ with $(\bar{t}, \bar{s})$ uniquely determined by H, and define the following sets:

$$\mathcal{H}_1 = \{H \in \mathbb{R}^2 \mid h(\bar{t}, \bar{s}) = \tilde{h}(\bar{t}, \bar{s})\}, \qquad \mathcal{H}_2 = \{H \in \mathbb{R}^2 \mid (\bar{t}, \bar{s}) \in Q\}.$$

Then, we have the following existence and non-existence results.

Proposition 4.5. *Let $\Omega = B$ be the unit disc. Then the biaxial ferromagnetic energy $I(m)$ has a unique minimizer if $H \in \mathcal{H}_1$, infinitely many minimizers if $H \in \mathcal{H}_2$ and no minimizers if $H \notin \mathcal{H}_1 \cup \mathcal{H}_2$.*

Proof. Since the open set where $\tilde{h}$ is affine is only the square $\{(t,s) \in \mathbb{R}^2 \mid |t| + |s| < 1\}$, by theorem 2.10, E and hence I have a minimizer if and only if either $h(\bar{t},\bar{s}) = \tilde{h}(\bar{t},\bar{s})$ or $|\bar{t}| + |\bar{s}| < 1$. In the first case we have the unique minimizer and in the second case we have infinitely many minimizers. This proves the result. □

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