A Theorem on Improving Regularity of Minimizing Sequences by Reverse Hölder Inequalities

BAISHENG YAN & ZHENGFANG ZHOU

1. Introduction

The use of reverse Hölder inequalities pioneered by Gehring's celebrated lemma [5] in the theory of quasiconformal mappings has been well adapted in the calculus of variations for obtaining regularity of minimizers of integral functionals with certain natural growth conditions [6]. In this paper we elaborate upon some ideas of our recent paper [16] to prove a theorem on improving regularity of minimizing sequences of a family of integral functionals that do not satisfy the usual growth conditions but satisfy instead a uniform integral coercivity condition as given by (1.4) below. As an important application, we also prove a stability result on the strong convergence of the so-called *weakly almost conformal mappings* in $W^{1,p}(\Omega; \mathbf{R}^n)$ for certain p below the dimension n. See also [4; 7; 11; 13; 14; 16].

We begin with some notation. Let $\mathcal{M}^{n\times m}$ be the space of all real $n \times m$-matrices with norm $|X|$ defined by $|X|^2 = \operatorname{tr}(X^T X)$. For $p \geq 1$ and a domain D in $\mathbf{R}^m$, let $W^{1,p}(D; \mathbf{R}^n)$ be the usual Sobolev space of L^p-integrable maps $u: D \to \mathbf{R}^n$ having L^p-integrable gradients $(\nabla u)_{ij} = \partial u^i/\partial x_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

Let $\mathcal{K}$ be a closed subset of $\mathcal{M}^{n\times m}$, and let $d_{\mathcal{K}}(X) = \inf_{A\in\mathcal{K}}|X - A|$ be the distance function to $\mathcal{K}$. In this paper, we shall always assume that $d_{\mathcal{K}}$ satisfies the following condition:

$$d_{\mathcal{K}}(\lambda X) \leq K_0(d_{\mathcal{K}}(X) + 1), \quad X \in \mathcal{M}^{n\times m}, \ 0 \leq \lambda \leq 1. \tag{1.1}$$

Note that condition (1.1) is satisfied if $\mathcal{K}$ is a cone or a bounded set.

We consider the integral functionals $I_p(u; D)$ defined by

$$I_p(u; D) = \int_D d_{\mathcal{K}}^p(\nabla u(x))\,dx. \tag{1.2}$$

The natural admissible space for $I_p(u; D)$ is $W^{1,p}(D; \mathbf{R}^n)$, but we shall often consider $I_p(u; D)$ for all $u \in W^{1,1}_{\mathrm{loc}}(D; \mathbf{R}^n)$.

Throughout this paper, we assume that $1 \leq \alpha \leq \beta < \infty$ are given numbers, that $\Omega \subset\subset D_0$ are bounded smooth domains in $\mathbf{R}^m$, and that u_0 is a given map in $W^{1,\alpha}_{\mathrm{loc}}(D_0; \mathbf{R}^n)$ satisfying

Received October 17, 1996. Revision received March 25, 1997.
Michigan Math. J. 44 (1997).

$$\nabla u_0(x) \in \mathcal{K} \quad \text{a.e. } x \in D_0. \tag{1.3}$$

We shall also assume that $\mathcal{K}$ satisfies the following condition.

UNIFORM INTEGRAL COERCIVITY. There exist constants $\Gamma_0 > 0$ and $\Gamma_1 \geq 0$ depending on n, α, β, and $\mathcal{K}$ such that, for every ball $B \subset \mathbf{R}^m$,

$$\int_B d_{\mathcal{K}}^p(\nabla\phi) \geq \Gamma_0 \int_B (|\nabla\phi|^p - \Gamma_1) \quad \forall p \in [\alpha, \beta], \quad \phi \in C_0^\infty(B; \mathbf{R}^n). \tag{1.4}$$

We remark that condition (1.4) is satisfied for all compact sets $\mathcal{K}$. Note also that from (1.4) one easily sees that, for any bounded domain D and $\phi \in C_0^\infty(D; \mathbf{R}^n)$,

$$\int_D d_{\mathcal{K}}^p(\nabla\phi) \geq \Gamma_0 \int_D |\nabla\phi|^p - (\Gamma_0\Gamma_1 + d_{\mathcal{K}}^p(0))|B| \tag{1.5}$$

for all $p \in [\alpha, \beta]$ and balls B containing D.

The main result of this paper is the following theorem.

THEOREM 1.1. *There exists a constant* $\bar{\varepsilon} > 0$ *depending only on* n, α, β, *and* $\mathcal{K}$ *such that, for any sequence* $\{u_j\}$ *in* $W^{1,\alpha}(\Omega; \mathbf{R}^n)$ *that converges weakly to* u_0 *and satisfies* $\lim_{j\to\infty} I_\alpha(u_j; \Omega) = 0$, *there exist a sequence* $\{v_k\}$ *in* $W_{\mathrm{loc}}^{1,\beta+\bar{\varepsilon}}(D_0; \mathbf{R}^n)$ *and a subsequence* $\{u_{j_k}\}$ *such that* $v_k = u_0$ *in* $D_0 \setminus \Omega$, $v_k \rightharpoonup u_0$ *in* $W_{\mathrm{loc}}^{1,\beta+\bar{\varepsilon}}(D_0; \mathbf{R}^n)$, *and*

$$\lim_{k\to\infty} \left(I_{\beta+\bar{\varepsilon}}(v_k; D_0) + \int_\Omega |\nabla v_k - \nabla u_{j_k}| \right) = 0. \tag{1.6}$$

REMARKS. (1) The new sequence $\{v_k\}$ is not only a minimizing sequence of the functional $I_\alpha(u; \Omega)$ but also a minimizing sequence for all functionals $I_p(u; \Omega)$ with $p \in [\alpha, \beta + \bar{\varepsilon}]$; moreover, it has a higher integrability than $\{u_j\}$.

(2) From this theorem we obtain that the *uniform* L^p*-coercivity* (1.4) implies a *higher regularity* for solutions of first-order system (1.3) in $W_{\mathrm{loc}}^{1,\alpha}(D_0; \mathbf{R}^n)$, that is, $u_0 \in W_{\mathrm{loc}}^{1,\beta+\bar{\varepsilon}}(D_0; \mathbf{R}^n)$. Also, by the Sobolev embedding, if $\beta \geq n$ then the sequence $\{v_k\}$ can be chosen in the Hölder space $C_{\mathrm{loc}}^{0,\mu}(D_0; \mathbf{R}^n)$ for some $\mu \in (0, 1)$.

(3) As an important application, we shall prove a new strong convergence theorem (Theorem 5.1) for the weakly almost conformal mappings in $W^{1,p}(\Omega; \mathbf{R}^n)$ with certain $p < n$. See also [11].

The paper is organized as follows. In Section 2 we prove a preliminary lemma that will be used in later sections. In Section 3, we use a version of Ekeland's variational principle and Caccioppoli-type estimates to obtain the reverse Hölder inequality with increasing supports; then a higher regularity follows from the well-known Gehring's lemma. We then prove the main result, Theorem 1.1, in Section 4. Finally, we give an application in Section 5 by proving a strong convergence for the weakly almost conformal mappings.

ACKNOWLEDGMENTS. The authors would like to acknowledge stimulating discussions with Michael Frazier regarding the reverse Hölder inequality. We also thank the referee for many useful suggestions.

2. A Useful Lemma

Before we proceed with the proof of Theorem 1.1, we prove the following useful lemma.

LEMMA 2.1. *Let $p \geq 1$, and let $\{u_j\}$ be a sequence in $W^{1,p}(\Omega; \mathbf{R}^n)$ that converges weakly to u and satisfies $\lim_{j\to\infty} I_p(u_j; \Omega) = 0$. Suppose the weak limit u extends to a map in $W^{1,p}(D; \mathbf{R}^n)$ for some D with $\Omega \subset\subset D$ that satisfies $\nabla u(x) \in \mathcal{K}$ for a.e. $x \in D$. Then, there exist a sequence $\{w_k\}$ in $W^{1,p}(D; \mathbf{R}^n)$ and a subsequence $\{u_{j_k}\}$ such that $w_k = u$ in $D \setminus \Omega$, $w_k \rightharpoonup u$ weakly in $W^{1,p}(D; \mathbf{R}^n)$, and*

$$I_p(w_k; D) + \int_\Omega |\nabla u_{j_k} - \nabla w_k| \leq \frac{k^{-2}}{2} \quad \forall k = 1, 2, \ldots .$$

Proof. The proof is standard. Let $\Omega_k = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > 1/k\}$. Let η_k be a C^∞-cutoff function satisfying $\operatorname{supp} \eta_k \subset \Omega$ and

$$0 \leq \eta_k \leq 1, \quad \eta_k\big|_{\Omega_k} = 1, \quad |\nabla \eta_k| \leq Ck.$$

By the Sobolev embedding theorem, $u_j \to u$ in $L^p(\Omega)$; thus we can choose a subsequence $\{u_{j_k}\}$ such that, for all $k = 1, 2, \ldots$,

$$(1 + \|\nabla \eta_k\|_\infty)\|u_{j_k} - u\|_{L^p(\Omega)} \leq 1/k. \tag{2.1}$$

Let $w_k = \eta_k u_{j_k} + (1 - \eta_k)u$. Note that

$$\nabla w_k = \begin{cases} \nabla u_{j_k} & \text{in } \Omega_k, \\ \eta_k \nabla u_{j_k} + (u_{j_k} - u) \otimes \nabla \eta_k + (1 - \eta_k)\nabla u & \text{in } \Omega \setminus \Omega_k, \\ \nabla u & \text{in } D \setminus \Omega, \end{cases} \tag{2.2}$$

where $a \otimes b$ stands for the rank-1 matrix $(a_i b_j)$. It is easy to see that $\{|\nabla w_k|\}$ is bounded in $L^p(D)$ and also equi-integrable in the case when $p = 1$. Hence $w_k \rightharpoonup u$ in $W^{1,p}(D; \mathbf{R}^n)$ and $\int_\Omega |\nabla u_{j_k} - \nabla w_k| \to 0$ as $k \to \infty$. Furthermore, by (1.1), we have $d_\mathcal{K}(\lambda X + Y) \leq K_0 d_\mathcal{K}(X) + K_0 + |Y|$ for $0 \leq \lambda \leq 1$ and thus

$$\begin{aligned} &I_p(w_k; \Omega \setminus \Omega_k) \\ &\quad \leq C_p\big(I_p(u_{j_k}; \Omega) + \|\nabla \eta_k\|_\infty^p \|u_{j_k} - u\|_{L^p(\Omega)}^p + \|\nabla u\|_{L^p(\Omega\setminus\Omega_k)}^p + |\Omega \setminus \Omega_k|\big), \end{aligned}$$

which, by (2.1), implies that $I_p(w_k; D) \to 0$ as $k \to \infty$. Finally, the lemma follows by choosing a subsequence of $\{w_k\}$. □

COROLLARY 2.2. *Suppose $0 \in \mathcal{K}$ and $\Gamma_1 = 0$ in (1.4). Let u_j be as given in the previous lemma, and let $p \in [\alpha, \beta]$. Then $u_j \to u$ strongly in $W^{1,p}_{\text{loc}}(\Omega; \mathbf{R}^n)$ provided that $\nabla u_j(x) \to \nabla u(x)$ a.e. in Ω.*

Proof. Let $\Omega' \subset\subset \Omega$ be given and let $k \gg 1$ so that $\Omega' \subseteq \Omega_k$. Since $p \in [\alpha, \beta]$ and $w_k - u \in W_0^{1,p}(\Omega; \mathbf{R}^n)$, by (1.5) we have

$$\int_{\Omega'} |\nabla u_{j_k} - \nabla u|^p \leq \int_\Omega |\nabla(w_k - u)|^p \leq \Gamma_0^{-1} \int_\Omega d_\mathcal{K}^p(\nabla w_k - \nabla u). \tag{2.3}$$

It is elementary to see that

$$f_k \equiv 2^p(d_{\mathcal{K}}^p(\nabla w_k) + |\nabla u|^p) - d_{\mathcal{K}}^p(\nabla w_k - \nabla u)$$

is nonnegative and tends to $2^p|\nabla u|^p$ a.e. in Ω as $k \to \infty$ if $\nabla u_j \to \nabla u$ a.e. in Ω. Thus, by Fatou's lemma, one easily deduces that

$$\limsup_{k\to\infty} \int_\Omega d_{\mathcal{K}}^p(\nabla w_k - \nabla u) = 0,$$

which by (2.3) implies that $u_j \to u$ strongly in $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbf{R}^n)$, since one can start with any subsequences of $\{u_j\}$. □

3. A Variational Principle and Higher Regularity

Let D be a domain in $\mathbf{R}^m$ and let u be a map in $W^{1,p}(D; \mathbf{R}^n)$. We define a complete metric space $(\mathcal{V}, \rho)$ by $\mathcal{V} = \mathcal{V}_{u,D}$ and $\rho = \rho_D$, where

$$\mathcal{V}_{u,D} \equiv \{\, u + \zeta \mid \zeta \in W_0^{1,1}(D; \mathbf{R}^n) \,\}, \qquad \rho_D(w, v) \equiv \int_D |\nabla w - \nabla v|. \tag{3.1}$$

Then, by Fatou's lemma, the functional $I_p(v; D)$ is lower semicontinuous on $(\mathcal{V}, \rho)$ for all $p \geq 1$. The following variational principle and higher regularity result are crucial for proving our main theorem.

LEMMA 3.1. *For any $w_k \in \mathcal{V}$ with $I_p(w_k; D) \leq \inf_{v\in\mathcal{V}} I_p(v; D) + k^{-2}/2$, there exists a $b_k \in \mathcal{V}$ such that $I_p(b_k; D) \leq I_p(w_k; D)$, $\rho(b_k, w_k) \leq 1/k$, and*

$$I_p(b_k; D) < I_p(w; D) + k^{-1}\rho(w, b_k) \quad \forall w \in \mathcal{V},\ \ w \neq b_k. \tag{3.2}$$

Proof. This lemma is a special version of a general Ekeland's variational principle [3]; see [2, Thm. 4.2]. □

Note that it is a direct consequence of the coercivity condition (1.5) that if $p \in [\alpha, \beta]$ then sequences $\{w_k\}$ and $\{b_k\}$ in Lemma 3.1 are both uniformly bounded in $W^{1,p}(D; \mathbf{R}^n)$. Furthermore, we have the following theorem.

THEOREM 3.2. *There exist $\varepsilon_n > 0$ and integer N_n depending only on n, α, β, and $\mathcal{K}$ such that, if $p \in [\alpha, \beta]$, the sequence $\{b_k\}$ determined in the previous lemma must then satisfy*

$$\sup_{k\geq N_n} \int_{D'} |\nabla b_k|^{p+\varepsilon_n} < \infty \quad \forall D' \subset\subset D. \tag{3.3}$$

Proof. We first prove that the reverse Hölder inequalities

$$\begin{aligned} &⨍_{B_R} |\nabla b_k|^p \\ &\quad \leq \beta_n \Big(⨍_{B_{2R}} |\nabla b_k|^{pn/(n+p)} \Big)^{(n+p)/n} + \gamma_n \quad \forall B_{2R} = B(a, 2R) \subset\subset D \end{aligned} \tag{3.4}$$

hold for all $k \geq N_n$, where N_n, β_n, and γ_n are constants depending on n, α, β, and $\mathcal{K}$. We follow the idea used in [16]. In the following, we use $c_0, c_1, \ldots$ to denote the constants depending only on n, α, β, and $\mathcal{K}$. Let $B_{2R} = B(a, 2R) \subset\subset D$ and $0 < s < t \leq 2R$. Let $\eta \in C_0^\infty(D)$ be a cutoff function such that

$$0 \leq \eta \leq 1, \quad \eta\big|_{B_s} = 1, \quad \eta\big|_{D\setminus B_t} = 0, \quad |\nabla\eta| \leq c_0(t-s)^{-1}.$$

Let $w = \eta\nu + (1-\eta)b_k$ and $\phi = b_k - w$, where ν is a constant to be chosen later. Then $w \in \mathcal{V} = \mathcal{V}_{u,D}$, $\phi \in W_0^{1,p}(B_t; \mathbf{R}^n)$, and

$$\nabla w = (1-\eta)\nabla b_k - (b_k - \nu) \otimes \nabla\eta, \quad \nabla\phi = \eta\nabla b_k + (b_k - \nu) \otimes \nabla\eta. \tag{3.5}$$

Using the inequality $d_{\mathcal{K}}^p(X+Y) \leq 2^\beta(d_{\mathcal{K}}^p(X) + |Y|^p)$, by (1.1) and (1.4) we have

$$\begin{aligned}\int_{B_s} |\nabla b_k|^p &\leq \int_{B_t} |\nabla\phi|^p \leq \Gamma_0^{-1} \int_{B_t} d_{\mathcal{K}}^p(\nabla\phi) + \Gamma_1|B_t| \\ &\leq c_1 \int_{B_t} d_{\mathcal{K}}^p(\nabla b_k) + \frac{c_1}{(t-s)^p} \int_{B_t\setminus B_s} |b_k - \nu|^p + c_1|B_t|.\end{aligned} \tag{3.6}$$

Since $\nabla w = \nabla b_k$ in $D \setminus B_t$ and $\nabla w = 0$ in B_s, the first term in (3.6) can be estimated by (3.2) as

$$\int_{B_t} d_{\mathcal{K}}^p(\nabla b_k) \leq \int_{B_t\setminus B_s} d_{\mathcal{K}}^p(\nabla w) + d_{\mathcal{K}}^p(0)|B_s| + k^{-1} \int_{B_t} |\nabla w - \nabla b_k|. \tag{3.7}$$

We now use (3.5) and the inequality $d_{\mathcal{K}}(X) \leq |X| + d_{\mathcal{K}}(0)$ to deduce

$$\begin{aligned}&\int_{B_t\setminus B_s} d_{\mathcal{K}}^p(\nabla w) \\ &\quad\leq c_2 \int_{B_t\setminus B_s} |\nabla b_k|^p + \frac{c_2}{(t-s)^p} \int_{B_t\setminus B_s} |b_k - \nu|^p + c_2|B_t \setminus B_s|.\end{aligned} \tag{3.8}$$

Combining (3.6)–(3.8), we have

$$\begin{aligned}\int_{B_s} |\nabla b_k|^p \leq c_3 &\int_{B_t\setminus B_s} |\nabla b_k|^p + \frac{c_3}{(t-s)^p} \int_{B_{2R}} |b_k - \nu|^p \\ &+ \frac{c_3}{k} \int_{B_t} |\nabla b_k - \nabla w| + c_3|B_{2R}|.\end{aligned} \tag{3.9}$$

Note also that, since $t \leq t^p + 1$ for all $t \geq 0$ and $p \geq 1$,

$$\begin{aligned}\int_{B_t} |\nabla b_k - \nabla w| &= \int_{B_s} |\nabla b_k| + \int_{B_t\setminus B_s} |\nabla\phi| \\ &\leq \int_{B_s} |\nabla b_k|^p + c_4 \int_{B_t\setminus B_s} |\nabla b_k|^p \\ &\quad + \frac{c_4}{(t-s)^p} \int_{B_{2R}} |b_k - \nu|^p + c_4|B_{2R}|.\end{aligned} \tag{3.10}$$

We now choose $N_n = 2c_3$. Then, for $k \geq N_n$, by (3.9) and (3.10) we have

$$\int_{B_s} |\nabla b_k|^p \leq c_5 \int_{B_t\setminus B_s} |\nabla b_k|^p + \frac{c_5}{(t-s)^p} \int_{B_{2R}} |b_k - \nu|^p + c_5|B_{2R}|. \tag{3.11}$$

Filling the hole—that is, adding $c_5 \int_{B_s} |\nabla b_k|^p$ to both sides of (3.11)—we obtain

$$\int_{B_s} |\nabla b_k|^p \le \frac{c_5}{1+c_5} \int_{B_t} |\nabla b_k|^p + \frac{c_6}{(t-s)^p} \int_{B_{2R}} |b_k - \nu|^p + c_6 |B_{2R}|.$$

This inequality holds for all $0 < s < t \le 2R$. Thus, a standard iteration argument [6] yields

$$\int_{B_R} |\nabla b_k|^p \le c_7 R^{-p} \int_{B_{2R}} |b_k - \nu|^p + c_7 |B_{2R}| \tag{3.12}$$

and hence

$$\fint_{B_R} |\nabla b_k|^p \le \frac{c_8}{R^{n+p}} \int_{B_{2R}} |b_k - \nu|^p + c_8. \tag{3.13}$$

Now choose

$$\nu = \nu_R = \fint_{B_{2R}} b_k$$

and use the Sobolev–Poincaré inequality

$$\int_{B_{2R}} |b_k - \nu_R|^p \le \sigma_n \left(\int_{B_{2R}} |\nabla b_k|^{pn/(n+p)} \right)^{(n+p)/n}$$

in (3.13), and we obtain (3.4).

To continue the proof, we let $f_k = 1 + |\nabla b_k|^{pn/(n+p)}$ and $r = (n+p)/n$. Then, by (1.4), $\{f_k\}$ is bounded in $L^r(D)$ and, for all $k \ge N_n$, by (3.4) we have

$$\fint_{B_R} f_k^r \le \kappa_n \left(\fint_{B_{2R}} f_k \right)^r \quad \forall B_{2R} \subset\subset D,$$

where κ_n is a constant depending on n, α, β, and $\mathcal{K}$. Therefore, by [8, Prop. 6.1] and [6, Thm. 6.1], $\{f_k\}$ is bounded in $L^s_{\text{loc}}(D)$ for $s = r + (r-1)/10^{n+r}4^n\kappa_n^r$ and hence $\{b_k\}$ is bounded in $W^{1,p+\varepsilon}_{\text{loc}}(D)$ with

$$\varepsilon = \varepsilon(p) = \frac{p^2}{10^{n+1}4^n \kappa_n (n+p)(10\kappa_n)^{p/n}}.$$

Let $\varepsilon_n = \min\{\, \varepsilon(p) \mid \alpha \le p \le \beta \,\}$. Then it is easily seen that $\varepsilon_n > 0$, and (3.3) follows. □

4. Proof of Theorem 1.1

Let $u_0 \in W^{1,\alpha}_{\text{loc}}(D_0; \mathbf{R}^n)$ be the map given before, satisfying (1.3). Let $\{u_j\}$ be any sequence in $W^{1,\alpha}(\Omega; \mathbf{R}^n)$ that satisfies

$$u_j \rightharpoonup u_0 \ \text{ weakly in } W^{1,\alpha}(\Omega; \mathbf{R}^n) \quad \text{and} \quad \lim_{j\to\infty} I_\alpha(u_j; \Omega) = 0. \tag{4.1}$$

In the following, let $\{D_i\}$ $(i = 1, 2, \dots)$ be an arbitrary sequence of subdomains of D_0 that satisfies

$$\Omega \subset\subset D_{i+1} \subset\subset D_i \subset\subset D_0, \quad i = 1, 2, \dots .$$

We proceed in several steps.

Step 1. Let $\{w_k\}$ be the sequence determined in Lemma 2.1 from $\{u_j\}$ with $p = \alpha$, $u = u_0$, and $D = D_1$. Using D_1 and $u_0 \in W^{1,\alpha}(D; \mathbf{R}^n)$, we define space $(\mathcal{V}, \rho) = (\mathcal{V}_{u_0, D_1}, \rho_{D_1})$ as in Section 3. Then $w_k \in \mathcal{V}$ and, since $I_\alpha(w_k; D_1) \leq k^{-2}/2$, it is clear that $\inf_{\mathcal{V}} I_\alpha(v; D_1) = 0$; hence we can apply Lemma 3.1 to the sequence $\{w_k\}$. Let $\{b_k\}$ be the corresponding sequence in $\mathcal{V}$ satisfying $\rho_{D_1}(w_k, b_k) \leq 1/k$. In the following steps, we study properties of this new sequence $\{b_k\}$.

Step 2. By Theorem 3.2, $\{b_k\}$ $(k \geq N_n)$ is bounded in $W^{1,\alpha+\varepsilon_n}_{\mathrm{loc}}(D_1; \mathbf{R}^n)$, where N_n and ε_n are the absolute constants determined in Theorem 3.2. Therefore, $\{b_k\}$ converges weakly in $W^{1,\alpha+\varepsilon_n}(\Omega; \mathbf{R}^n)$. Since $\rho_{D_1}(w_k, b_k) \leq 1/k$ and $w_k \rightharpoonup u_0$ in $W^{1,\alpha}(D_1; \mathbf{R}^n)$, we deduce that $b_k \rightharpoonup u_0$ in $W^{1,\alpha+\varepsilon_n}_{\mathrm{loc}}(D_1; \mathbf{R}^n)$. This readily implies $u_0 \in W^{1,\alpha+\varepsilon_n}(D_2; \mathbf{R}^n)$. In what follows, let $\varepsilon_0 = \varepsilon_n/2$. We claim $\lim_{k\to\infty} I_{\alpha+\varepsilon_0}(b_k; \Omega) = 0$. To see this, we use the elementary estimate

$$d_{\mathcal{K}}^{\alpha+\varepsilon_0}(X) \leq \delta(|X|^{\alpha+2\varepsilon_0} + 1) + C_\delta d_{\mathcal{K}}^{\alpha}(X) \quad \forall \delta > 0$$

with $X = \nabla b_k$, and integrate it over Ω to obtain

$$I_{\alpha+\varepsilon_0}(b_k; \Omega) \leq \delta \int_\Omega (|\nabla b_k|^{\alpha+2\varepsilon_0} + 1) + C_\delta I_\alpha(b_k; \Omega). \tag{4.2}$$

Note that, from Theorem 3.2,

$$\sup_{k \geq N_n} \int_\Omega |\nabla b_k|^{\alpha+2\varepsilon_0} \leq M < \infty, \qquad \lim_{k\to\infty} I_\alpha(b_k; D_1) = 0.$$

Let $k \to \infty$ in (4.2). We then have $\limsup_{k\to\infty} I_{\alpha+\varepsilon_0}(b_k; \Omega) \leq \delta M$ for all $\delta > 0$, which implies that

$$\lim_{k\to\infty} I_{\alpha+\varepsilon_0}(b_k; \Omega) = 0.$$

Also, by Lemmas 2.1 and 3.1, we have $\lim_{k\to\infty} \int_\Omega |\nabla u_{j_k} - \nabla b_k| = 0$.

Step 3. In Step 2, we proved that $u_0 \in W^{1,\alpha+\varepsilon_0}(D_2; \mathbf{R}^n)$ and obtained a new sequence $\{b_k^{(1)}\} = \{b_k\}$ in $W^{1,\alpha+\varepsilon_0}(\Omega; \mathbf{R}^n)$. This sequence satisfies

$$\lim_{k\to\infty} \int_\Omega |\nabla u_{j_k} - \nabla b_k^{(1)}| = 0$$

for a subsequence $\{u_{j_k}\}$ of the original sequence $\{u_j\}$ and also satisfies

$$b_k^{(1)} \rightharpoonup u_0 \text{ weakly in } W^{1,\alpha+\varepsilon_0}(\Omega; \mathbf{R}^n) \quad \text{and} \quad \lim_{k\to\infty} I_{\alpha+\varepsilon_0}(b_k^{(1)}; \Omega) = 0. \tag{4.3}$$

Hence $\{b_k^{(1)}\}$ satisfies the same type of conditions (4.1) as satisfied by $\{u_j\}$ except that now it is in a better space, $W^{1,\alpha+\varepsilon_0}(\Omega; \mathbf{R}^n)$. Therefore, if $\alpha + \varepsilon_0 \leq \beta$, we can apply Step 1 again with $\{b_k^{(1)}\}$ replacing $\{u_j\}$, D_2 replacing D_1, and $\alpha + \varepsilon_0$ replacing α. Now let integer N and number $\bar{\varepsilon} > 0$ be determined by

$$\alpha + (N-1)\varepsilon_0 \leq \beta < \alpha + N\varepsilon_0, \qquad \alpha + N\varepsilon_0 = \beta + \bar{\varepsilon}.$$

We repeat this step N times to eventually prove $u_0 \in W^{1,\beta+\bar{\varepsilon}}(D_{N+1}; \mathbf{R}^n)$ and obtain a sequence $\{b_\nu^{(N)}\}$ in $W^{1,\beta+\bar{\varepsilon}}(\Omega; \mathbf{R}^n)$ that satisfies both

$$b_\nu^{(N)} \rightharpoonup u_0 \text{ weakly in } W^{1,\beta+\bar\varepsilon}(\Omega;\mathbf{R}^n) \quad\text{and}\quad \lim_{\nu\to\infty} I_{\beta+\bar\varepsilon}(b_\nu^{(N)};\Omega)=0 \tag{4.4}$$

and also $\lim_{\nu\to\infty}\int_\Omega|\nabla u_j^{(\nu)}-\nabla b_\nu^{(N)}|=0$, where $\{u_j^{(\nu)}\}$ is a subsequence of the original sequence $\{u_j\}$.

Step 4. Finally, let $\{v_k\}$ be the sequence in $W^{1,\beta+\bar\varepsilon}(D_{N+1};\mathbf{R}^n)$ determined from $\{b_\nu^{(N)}\}$ in the same way that $\{w_k\}$ was from $\{u_j\}$ in Lemma 2.1. We extend $\{v_k\}$ to $D_0\setminus D_{N+1}$ by u_0. Then, this sequence satisfies all the requirements of Theorem 1.1 and thus proves the theorem. □

5. Strong Convergence of Weakly Almost Conformal Mappings

We now consider the so-called *conformal set*

$$\mathcal{K}=C_n=\{\lambda R \mid \lambda\geq 0,\ R\in \mathrm{SO}(n)\}, \tag{5.1}$$

where $\mathrm{SO}(n)$ is the set of all orthogonal matrices of determinant 1. Since $\mathcal{K}=C_n$ is a closed cone, condition (1.1) is satisfied.

Recall that a map φ on the extended space $\mathbf{R}^n\cup\{\infty\}$ is a *Möbius transformation* if it is a composition of finitely many similarities and inversions with respect to the sphere [12; 13]. A sequence $\{u_j\}$ in $W^{1,p}(\Omega;\mathbf{R}^n)$ is said to be (*weakly* if $p<n$) *almost conformal* if

$$\lim_{j\to\infty}\int_\Omega d^p_{C_n}(\nabla u_j(x))\,dx=0. \tag{5.2}$$

As an application of Theorem 1.1, we prove in this section the following strong convergence result concerning the weakly almost conformal sequences. See also [11; 14; 16].

THEOREM 5.1. *There exists a number $p<n$ such that any sequence $\{u_j\}$ converging weakly in $W^{1,p}(\Omega;\mathbf{R}^n)$ and satisfying* (5.2) *must converge strongly to a Möbius transformation in both $W^{1,1}(\Omega;\mathbf{R}^n)$ and $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbf{R}^n)$.*

REMARKS. (1) It follows from Yan [14] that any number p that validates Theorem 5.1 must be at least the half-dimension, that is, $p\geq n/2$. Note also that the strong convergence in $W^{1,p}(\Omega;\mathbf{R}^n)$ for $p\geq n$ follows easily from a theorem of Evans and Gariepy [4, Thm. 1]. See also the proof of Proposition 5.5.

(2) If the dimension $n=2l$ is even, Müller, Šverák, and Yan [11] proved that the smallest such p is precisely the half-dimension $l=n/2$. A key ingredient, as observed by Iwaniec and Martin [9], is that in this case the conformality of a map can be characterized by a nonlinear Cauchy–Riemann equation involving the determinants of $l\times l$ sub-Jacobians, which enables us to use standard elliptic estimates and the compensated compactness method; see [11] for details.

(3) Because the equations (the so-called Beltrami systems) governing the conformal mappings in the case of odd dimensions are essentially nonlinear, the methods used in [11] do not apply and in this case the problem regarding the smallest p validating Theorem 5.1 remains unsolved; see also [7; 9; 16].

In order to prove Theorem 5.1, we will need the following lemmas.

LEMMA 5.2. *There exist $p_0 < n$ and $\Gamma > 0$ such that, for every ball $B \subset \mathbf{R}^n$,*

$$\int_B d_{C_n}^p(\nabla\phi) \geq \Gamma \int_B |\nabla\phi|^p \quad \forall p \in [p_0, n], \quad \phi \in C_0^\infty(B; \mathbf{R}^n).$$

Proof. This has been proved in Yan [14, Thm. 1.3] using the estimates of very weak solutions of p-harmonic equations established in Iwaniec [7]; see also [10] and [16, Cor. 3.3]. We only remark here that $p_0 \geq n/2$, from [14, Thm. 1.4]. □

LEMMA 5.3. *There is a $p_1 < n$ such that, for any sequence $\{u_j\}$ as given in Theorem 5.1 with $p \in [p_1, n]$, the weak limit $\bar{u}$ must be a restriction of an orientation-preserving Möbius transformation onto Ω.*

Proof. By [16, Cor. C], we know $p_1 < n$ can be chosen such that any weak limit $\bar{u}$ as given in the lemma must be a weakly conformal map; that is, $\bar{u}$ satisfies

$$\nabla\bar{u}(x) \in C_n \quad \text{a.e. } x \in \Omega.$$

We can choose $p_1 < n$ even closer to n so that the generalized Liouville's theorem in Iwaniec [7, Thm. 3] will assert that $\bar{u}$ must be a restriction of an orientation-preserving Möbius transformation onto Ω. The proof is thus complete. □

LEMMA 5.4. *If $p \geq n/2$ and $\partial\Omega$ is sufficiently smooth, then any Möbius transformation that belongs to $W^{1,p}(\Omega; \mathbf{R}^n)$ must be a C^∞-diffeomorphism of a neighborhood D of $\bar{\Omega}$ into $\mathbf{R}^n$.*

Proof. Let δ and μ be defined by

$$\delta(x) \equiv Ax + b, \qquad \mu(x) \equiv a + r^2|x-a|^{-2}(x-a). \tag{5.3}$$

A *similarity* is a transformation δ with $A = \lambda P$ for some $\lambda \in \mathbf{R}$ and orthogonal matrix P. By a representation result in [13, p. 75], a Möbius transformation φ is either a similarity or a transformation representable as $\varphi = \delta \circ \mu$, with δ, μ defined by (5.3) and A orthogonal.

Let φ be a Möbius transformation and let $\varphi \in W^{1,p}(\Omega; \mathbf{R}^n)$. If φ is a similarity then it extends to the whole $\mathbf{R}^n$. Suppose now that φ is given by $\varphi = \delta \circ \mu$, where δ and μ are defined by (5.3) and A is orthogonal. Then

$$\nabla\varphi(x) = r^2|x-a|^{-2}A\left(I - 2\frac{x-a}{|x-a|} \otimes \frac{x-a}{|x-a|}\right).$$

Thus

$$|\nabla\varphi(x)| = \sqrt{n}\,r^2|x-a|^{-2}.$$

Suppose $\partial\Omega$ is sufficiently smooth; then $\varphi \in W^{1,p}(\Omega; \mathbf{R}^n)$ only for $1 \leq p < n/2$ if $a \in \bar{\Omega}$. Thus, if $p \geq n/2$ and $\varphi \in W^{1,p}(\Omega; \mathbf{R}^n)$, we have $a \notin \bar{\Omega}$ and hence $\varphi \in C^\infty(D; \mathbf{R}^n)$ for some domain D containing $\bar{\Omega}$. The lemma is proved. □

In the following, we let $p_* = \max\{p_0, p_1\}$. Then $n/2 \leq p_* < n$. Let $p \in [p_*, n]$, let $\{u_j\}$ be a sequence in $W^{1,p}(\Omega; \mathbf{R}^n)$ as given in the theorem, and let $\bar{u}$ be the weak limit. It follows from Lemmas 5.2–5.4 that $\bar{u}$ extends to a C^∞-diffeomorphism in a neighborhood D of $\bar{\Omega}$ as an orientation-preserving Möbius transformation. Thus $\nabla\bar{u}(x) \in C_n$ for all $x \in D$. Consequently, all conditions of Theorem 1.1 are satisfied with $\alpha = p_*$, $\beta = n$, $D_0 = D$, and $u_0 = \bar{u}$. Therefore, by Theorem 1.1, we obtain a sequence $\{v_k\}$ in $W^{1,n+\bar{\varepsilon}}_{\mathrm{loc}}(D; \mathbf{R}^n)$ that satisfies $v_k = \bar{u}$ in $D \setminus \Omega$ and $\int_\Omega |\nabla v_k - \nabla u_{j_k}| \to 0$ as $k \to \infty$ for some subsequence $\{u_{j_k}\}$; moreover,

$$v_k \rightharpoonup \bar{u} \text{ in } W^{1,n+\bar{\varepsilon}}_{\mathrm{loc}}(D; \mathbf{R}^n) \quad \text{and} \quad \lim_{k\to\infty} \int_D d^{n+\bar{\varepsilon}}_{C_n}(\nabla v_k) = 0. \tag{5.4}$$

PROPOSITION 5.5. *We have $v_k \to \bar{u}$ strongly in $W^{1,n}(\Omega; \mathbf{R}^n)$.*

Proof. Since $\det \nabla u$ is a null Lagrangian [1], by (5.4) we have

$$\lim_{k\to\infty} \int_\Omega \det \nabla v_k = \int_\Omega \det \nabla \bar{u}. \tag{5.5}$$

Consider the function

$$G(X) = |X|^n - n^{n/2} \det X. \tag{5.6}$$

Note that, by Hadamard's inequality, $G(X) \geq 0$ and $G(X) = 0$ if and only if $X \in C_n$. By homogeneity, $G(X) \leq \tau |X|^n + C_\tau d^n_{C_n}(X)$ for all $\tau > 0$. Thus, by (5.4), we have

$$\lim_{k\to\infty} \int_D G(\nabla v_k) = 0 = \int_D G(\nabla \bar{u}); \tag{5.7}$$

combined with (5.5), this yields $\int_\Omega |\nabla v_k|^n \to \int_\Omega |\nabla \bar{u}|^n$ and hence $\nabla v_k \to \nabla \bar{u}$ strongly in $L^n(\Omega; \mathcal{M}^{n\times n})$. The proof is complete. □

Note that the strong convergence as asserted in the proposition also follows from (5.4) and (5.7) by the result [4, Thm. 1], since $G(X)$ is uniformly strictly $W^{1,n}$-quasiconvex in the sense defined by [4].

Proof of Theorem 5.1. By Proposition 5.5, the subsequence $\{u_{j_k}\}$ determined as above converges strongly to $\bar{u}$ in $W^{1,1}(\Omega; \mathbf{R}^n)$, so $\nabla u_{j_k}(x) \to \nabla \bar{u}(x)$ for a.e. $x \in \Omega$. Therefore, by Corollary 2.2 and Lemma 5.2, $u_{j_k} \to \bar{u}$ strongly in $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbf{R}^n)$. Since we can start with arbitrary subsequences of $\{u_j\}$, we see that the original sequence $\{u_j\}$ converges strongly to $\bar{u}$ in both $W^{1,1}(\Omega; \mathbf{R}^n)$ and $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbf{R}^n)$. The proof of Theorem 5.1 is now complete. □

References

[1] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity,* Arch. Rational Mech. Anal. 63 (1977), 337–403.

[2] D. G. De Figueiredo, *Lectures on the Ekeland variational principle with applications and detours,* Tata Institute lecture. Springer, New York, 1989.

[3] I. Ekeland, *On the variational principle,* J. Math. Anal. Appl. 47 (1974), 324–353.

[4] L. C. Evans and R. F. Gariepy, *Some remarks concerning quasiconvexity and strong convergence,* Proc. Roy. Soc. Edinburgh Sect. A 106 (1987), 53–61.

[5] F. W. Gehring, *The L^p-integrability of the partial derivatives of a quasiconformal mapping,* Acta Math. 130 (1973), 265–277.
[6] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems,* Princeton Univ. Press, Princeton, NJ, 1983.
[7] T. Iwaniec, *p-Harmonic tensors and quasiregular mappings,* Ann. of Math. (2) 136 (1992), 589–624.
[8] ———, *The Gehring lemma,* preprint.
[9] T. Iwaniec and G. Martin, *Quasiregular mappings in even dimensions,* Acta Math. 170 (1993), 29–81.
[10] T. Iwaniec and C. Sbordone, *Weak minima of variational integrals,* J. Reine Angew. Math. 454 (1994), 143–161.
[11] S. Müller, V. Šverák, and B. Yan, *Sharp stability results for almost conformal maps in even dimensions,* J. Geom. Anal. (to appear).
[12] Y. G. Reshetnyak, *Space mappings with bounded distortion,* Transl. Math. Monographs, 73, Amer. Math. Soc., Providence, RI, 1989.
[13] ———, *Stability theorems in geometry and analysis,* Kluwer, Dordrecht, 1994.
[14] B. Yan, *Remarks about $W^{1,p}$-stability of the conformal set in higher dimensions,* Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1996), 691–705.
[15] ———, *On rank-one convex and polyconvex conformal energy functions with slow growth,* Proc. Roy. Soc. Edinburgh Sect. A 127 (1997), 651–663.
[16] B. Yan and Z. Zhou, *Stability of weakly almost conformal mappings,* Proc. Amer. Math. Soc. (to appear).

B. Yan
Department of Mathematics
Michigan State University
East Lansing, MI 48824

yan@math.msu.edu

Z. Zhou
Department of Mathematics
Michigan State University
East Lansing, MI 48824

zfzhou@math.msu.edu