# *On a Restricted Weak Lower Semicontinuity for Smooth Functionals on Sobolev Spaces*

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ABSTRACT. This paper is motivated by a problem suggested in Müller [11] that concerns the weak lower semicontinuity of a smooth integral functional $I(u)$ on a Sobolev space along all its weakly convergent minimizing sequences. Here we study a restricted weak lower semicontinuity of $I(u)$ along all weakly convergent *Palais-Smale sequences* (that is, sequences $\{u_k\}$ satisfying $I'(u_k) \to 0$). In view of Ekeland's variational principle, this restricted weak lower semicontinuity, replacing the usual (unrestricted) weak lower semicontinuity in the direct method of calculus of variations, is sufficient for the existence of minimizers under the standard coercivity assumption. The main purpose of the paper is to study the relationships of this restricted weak lower semicontinuity condition with the usual weak lower semicontinuity condition that is known to be equivalent to the Morrey quasiconvexity in the calculus of variations. We show that the two conditions are not equivalent in general, but are equivalent in certain interesting cases.

## 1. INTRODUCTION

In this paper, we study a problem suggested in Müller [11] that concerns a weak lower semicontinuity of a smooth integral functional $I(u)$ of the type

$$I(u) = \int_\Omega f(x, u(x), Du(x))\, dx \tag{1.1}$$



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along certain restricted weakly convergent sequences. Here $\Omega$ is a domain in $\mathbb{R}^n$, $u\colon \Omega \to \mathbb{R}^m$ is a vector-valued function with Jacobi matrix $Du(x) = (\partial u^i/\partial x_j)$ and $f(x,s,\xi)$ is a given function of point $x \in \Omega$, vector $s \in \mathbb{R}^n$ and matrix $\xi \in M^{m\times n}$, the set of all $m \times n$ matrices.

Under some structure and growth conditions, one can define the functional $I(u)$ on the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^m)$ of mappings from $\Omega$ to $\mathbb{R}^m$ and study the minimization problem of the functional $I$ on a given Dirichlet class. Such a problem can be studied by the direct method of calculus of variations [5]. An important property often linked to the direct method of calculus of variations is the so-called *weak lower semicontinuity* of the functional $I$. Recall that $I$ is (sequentially) weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ if

$$(1.2)\qquad I(u) \le \liminf_{k\to\infty} I(u_k) \quad \text{whenever } u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^m),$$

where $u_k \rightharpoonup u$ means $u_k$ weakly converges to $u$.

It has been well-known that, under certain mild conditions on $f$, the weak lower semicontinuity of integral functional $I$ defined above is equivalent to the important *quasiconvexity* condition introduced by Morrey; see, e.g., [1, 3, 5, 10, 11]. Recall that $f(x,s,\xi)$ is *quasiconvex* in $\xi$ in the Morrey sense provided that the inequality

$$f(x,s,\xi) \le \frac{1}{|\Omega|}\int_\Omega f(x,s,\xi + D\varphi(y))\,dy$$

holds for a.e. $x \in \Omega$, all $s \in \mathbb{R}^n$, $\xi \in M^{m\times n}$, and all $\varphi \in C_0^\infty(\Omega;\mathbb{R}^m)$. Note that in the case $n = 1$ (one-dimensional case) or $m = 1$ (scalar case) Morrey's quasiconvexity condition becomes the usual convexity condition.

We assume the functional $I$ defined above is continuously differentiable on $W^{1,p}(\Omega;\mathbb{R}^m)$ and bounded below. We also assume $p > 1$ and $p' = p/(p-1)$ and denote by $W^{-1,p'}(\Omega;\mathbb{R}^m)$ the dual space of $W_0^{1,p}(\Omega;\mathbb{R}^m)$. In view of an important variational principle discovered by Ekeland [6] (see also [2]), we can always obtain a minimizing sequence $\{u_k\}$ of $I$ over a Dirichlet class $\mathcal{A}_g$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ which satisfies $I'(u_k) \to 0$ in $W^{-1,p'}(\Omega;\mathbb{R}^m)$. Consequently, the weak limit (if exists) of any such minimizing sequence will be an energy minimizer provided, as has been suggested by Müller in [11], that $I(u)$ only satisfies the condition:

$$(1.3)\qquad I(u) \le \liminf_{k\to\infty} I(u_k)$$
$$\text{whenever } u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^m) \textit{ and } I'(u_k) \to 0 \text{ in } W^{-1,p'}(\Omega;\mathbb{R}^m).$$

Since a sequence $\{u_k\}$ with $I'(u_k) \to 0$ is usually called a *Palais-Smale sequence*, we call a functional $I(u)$ satisfying condition (1.3) (*PS*)-*weakly lower semicontinuous* on $W^{1,p}(\Omega;\mathbb{R}^m)$.

The main purpose of this paper is to investigate what intrinsic properties the (PS)-weak lower semicontinuity of $I(u)$ imposes on the function $f(x,s,\xi)$. Certainly, Morrey's quasiconvexity (of $f$ on $\xi$) provides a sufficient condition for the (PS)-weak lower semicontinuity. In this paper we obtain the following main results.

First, unlike the Morrey quasiconvexity, the condition on $f(x,s,\xi)$ of the (PS)-weak lower semicontinuity may depend *not only* on $\xi$ *but also* on $(x,s)$ in a nonlocal fashion. For example, by (the proof of) Proposition 4.2 and Theorem 5.1, the integral functional with a function $h(\xi)$ may not be (PS)-weak lower semicontinuous but the one with $a(x)h(\xi)$ is, for some function $a(x)$.

Second, we show that the (PS)-weak lower semicontinuity of a simple functional

$$I(u) = \int_\Omega f(Du(x))\,dx \tag{1.4}$$

defined by a function $f\colon M^{m\times n} \to \mathbb{R}^+$ may or may not be equivalent to the Morrey quasiconvexity of $f$. In the case $m = 1$, which means $u\colon \Omega \to \mathbb{R}$ is a scalar function, Theorem 5.1 shows that the (PS)-weak lower semicontinuity is equivalent to the quasiconvexity of $f$, which is simply the convexity in this case. The proof of this theorem uses a combination of a crucial convexity result for smooth functions $h\colon \mathbb{R} \to \mathbb{R}^+$ (Lemma 4.8) and a classical one-dimensional construction by layering, which produces a useful testing Palais-Smale sequence. In this layering construction, it is the dimension $m = 1$ that is critical, not the Hadamard rank-one jump condition because, in higher dimensions $m > 1$, even on the rank-one directions, such a layering construction does not produce a *Palais-Smale* sequence (see Remark 5.3) and thus cannot prove the quasiconvexity or even the rank-one convexity from the (PS)-weak lower semicontinuity. In fact, Theorem 5.8 gives an example in the one-dimensional vectorial case ($n = 1$, $m \geq 2$) that a functional (1.4) is (PS)-weak lower semicontinuous but $f$ is not (quasi)convex.

Third, we also study the impact of the coercivity on the (PS)-weak lower semicontinuity in Theorem 4.3 and Theorem 5.4. Under the given coercivity assumption, we show that the (PS)-weak lower semicontinuity is equivalent to the Morrey quasiconvexity. Without the coercivity assumption, Proposition 4.2 and Theorem 5.8 show that the two conditions may not be equivalent. The proofs of both Theorem 4.3 and Theorem 5.4 rely on the key idea that, under the coercivity assumption, the (PS)-weak lower semicontinuity implies the existence of minimizers over all Dirichlet classes, which enables one to construct suitable Palais-Smale sequences and to prove the convexity or quasiconvexity. This general existence result, which is based on Ekeland's variational principle, is also a main motivation of the paper and is presented as Theorem 3.8.

Finally, we point out that even for the simplest functional (1.4) in the general case of $m$, $n \geq 2$ without the coercivity assumption a necessary and sufficient condition on $f$ for the (PS)-weak lower semicontinuity of $I(u)$ remains open. The major difficulty in the (PS)-weak lower semicontinuity lies in that the testing sequences $\{u_k\}$ constructed by the usual techniques [1, 5, 10] do not satisfy the condition $I'(u_k) \to 0$ in $W^{-1,p'}(\Omega;\mathbb{R}^m)$. A related problem to the (PS)-weak lower semicontinuity is to characterize all the gradient Young measures [8] generated by weakly convergent Palais-Smale sequences, which is also relevant to the theory of compensated compactness [13]. Weak lower semicontinuity under certain linear differential constraints has been studied in [7]. These linear constraints $\mathcal{A}(u)$ are independent of the functional and usually have large kernel. Then the constrained lower semicontinuity of functionals may be characterized through the Jensen's inequality with the associated Young measures supported on the kernel of $\mathcal{A}$; this is the so-called $\mathcal{A}$-quasiconvexity [7].

However, the difficulty in our case is that the strong convergence $I'(u_k) \to 0$ in $W^{-1,p'}(\Omega;\mathbb{R}^m)$ cannot be realized by the Young measure of $\{Du_k\}$ in the dimension $n \geq 2$. Recently, in [14] we have successfully applied the Young measure theory in the one-dimensional case ($n = 1$) to obtain a necessary and sufficient condition of the (PS)-weak lower semicontinuity for the functional $I(u)$ of the type (1.4) for $n = 1$ and all $m \geq 1$.

## 2. NOTATION AND PRELIMINARIES

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Let $M^{m\times n}$ be the set of $m \times n$ matrices. For vectors $a, b \in \mathbb{R}^n$ and matrices $\xi, \eta \in M^{m\times n}$, we define the inner products by

$$a \cdot b = \sum_{j=1}^{n} a_i b_i, \quad \xi : \eta = \sum_{i=1}^{m}\sum_{j=1}^{n} \xi_{ij}\eta_{ij},$$

with the corresponding Euclidean norms denoted both by $|\cdot|$. For vectors $q \in \mathbb{R}^m$, $a \in \mathbb{R}^n$, we denote by $q \otimes a$ the rank-one $m \times n$ matrix $(q_i a_j)$.

Let $W^{1,p}(\Omega)$ be the usual Sobolev space of scalar functions on $\Omega$, and define $W^{1,p}(\Omega;\mathbb{R}^m)$ to be the space of vector functions $u : \Omega \to \mathbb{R}^m$ with each component $u^i \in W^{1,p}(\Omega)$ and we denote by $Du$ the Jacobi matrix of $u$ defined by

$$Du(x) = \left(\frac{\partial u^i}{\partial x_j}\right)_{i=1,\dots,m}^{j=1,\dots,n}.$$

Let $1 \leq p < \infty$. We make $W^{1,p}(\Omega;\mathbb{R}^m)$ a Banach space with the norm

$$\|u\|_{W^{1,p}(\Omega;\mathbb{R}^m)} = \left(\int_\Omega (|u|^p + |Du|^p)\,dx\right)^{1/p}.$$

Let $C_0^\infty(\Omega;\mathbb{R}^m)$ be the set of infinitely differentiable vector functions with compact support in $\Omega$, and let the space $W_0^{1,p}(\Omega;\mathbb{R}^m)$ be the closure of $C_0^\infty(\Omega;\mathbb{R}^m)$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. Then $W_0^{1,p}(\Omega;\mathbb{R}^m)$ is itself a Banach space and has an equivalent norm defined by $\||Du|\|_{L^p(\Omega)}$.

Let $g \in W^{1,p}(\Omega; \mathbb{R}^m)$. Define the Dirichlet class $\mathcal{A}_g = W_g^{1,p}(\Omega; \mathbb{R}^m)$ to be $g + W_0^{1,p}(\Omega; \mathbb{R}^m)$; that is,

$$\mathcal{A}_g = W_g^{1,p}(\Omega; \mathbb{R}^m) = \{u \in W^{1,p}(\Omega; \mathbb{R}^m) \mid u - g \in W_0^{1,p}(\Omega; \mathbb{R}^m)\}.$$

For $A \in M^{m\times n}$, define $g_A(x) = Ax$ and let $W_A^{1,p} = W_{g_A}^{1,p}(\Omega; \mathbb{R}^m)$ be the Dirichlet class for the linear function $g_A$.

We use $u_k \rightharpoonup u$ to denote the weak convergence in $W^{1,p}(\Omega; \mathbb{R}^m)$. Note that if $1 < p < \infty$, then any bounded sequence in $W^{1,p}(\Omega; \mathbb{R}^m)$ has a weakly convergent subsequence and if, furthermore, $\partial\Omega$ is smooth or the sequence is in $W_g^{1,p}(\Omega; \mathbb{R}^m)$, one can assume the subsequence is also convergent strongly in $L^p(\Omega; \mathbb{R}^m)$.

*Definition* **2.1**. A functional $I$ on $W^{1,p}(\Omega; \mathbb{R}^m)$ is said to be (sequentially) *weakly lower semicontinuous* on $W^{1,p}(\Omega; \mathbb{R}^m)$ provided

$$I(u) \le \liminf_{k\to\infty} I(u_k) \quad \text{whenever } u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^m). \tag{2.1}$$

*Definition* **2.2**. Let $h\colon M^{m\times n} \to \mathbb{R}$.

(i) We say that $h$ is *convex* on $M^{m\times n}$ if the inequality

$$h(\lambda\xi + (1-\lambda)\eta) \le \lambda h(\xi) + (1-\lambda)h(\eta) \tag{2.2}$$

holds for all $0 < \lambda < 1$ and $\xi, \eta \in M^{m\times n}$.

(ii) We say $h$ is *rank-one convex* if the inequality (2.2) holds only for all $\xi$, $\eta$ satisfying $\operatorname{rank}(\xi - \eta) \le 1$.

(iii) We say that $h$ is *quasiconvex* in the Morrey sense if the inequality

$$h(A) \le \frac{1}{|\Omega|}\int_\Omega h(A + D\varphi(x))\,dx \tag{2.3}$$

holds for all $A \in M^{m\times n}$ and $\varphi \in C_0^\infty(\Omega; \mathbb{R}^m)$.

For more on Morrey's quasiconvexity condition and related results, we refer to [1, 3, 5, 10–12]. Note that in the case of $n = 1$ (one-dimensional) or $m = 1$ (scalar) the convexity and quasiconvexity for the function $h\colon M^{m\times n} \to \mathbb{R}$ are equivalent.

Note also that $h$ is convex if and only if $g(t) = h(\xi + t\eta)$ is a convex function of $t$ on $\mathbb{R}$ for all $\xi, \eta \in M^{m\times n}$. For $C^1$ functions $h$, the convexity condition is equivalent to the condition

$$h(\eta) \ge h(\xi) + D_\xi h(\xi) : (\eta - \xi), \quad \forall\, \eta, \xi \in M^{m\times n}. \tag{2.4}$$

Furthermore, a $C^1$ function $h$ on $\mathbb{R}$ is convex if and only if $h'$ is nondecreasing, or equivalently, the following condition holds:

$$(h'(a) - h'(b))(a - b) \ge 0, \quad \forall\, a, b \in \mathbb{R}. \tag{2.5}$$

Let $f\colon \Omega \times \mathbb{R}^n \times M^{m\times n} \to \mathbb{R}$. We say $f$ is *Carathéodory* if $f(x,s,\xi)$ is measurable in $x \in \Omega$ for all $(s,\xi) \in \mathbb{R}^n \times M^{m\times n}$ and continuous in $(s,\xi) \in \mathbb{R}^n \times M^{m\times n}$ for almost every $x \in \Omega$. If $f(x,s,\xi)$ is measurable in $x \in \Omega$ for all $(s,\xi) \in \mathbb{R}^n \times M^{m\times n}$ and is $C^1$ in $(s,\xi) \in \mathbb{R}^n \times M^{m\times n}$ for almost every $x \in \Omega$, we shall use the following notation to denote the derivatives of $f$ on $s$ and $\xi$:

$$D_s f(x,s,\xi) = \left(\frac{\partial f}{\partial s_1},\dots,\frac{\partial f}{\partial s_n}\right),$$

$$D_\xi f(x,s,\xi) = \left(\frac{\partial f}{\partial \xi_{ij}}\right)_{i=1,\dots,m}^{j=1,\dots,n}.$$

Given a function $f(x,s,\xi)$, define the integral functional $I$ on $W^{1,p}(\Omega;\mathbb{R}^m)$ by

$$I(u) = \int_\Omega f(x,u(x),Du(x))\,dx, \quad u \in W^{1,p}(\Omega;\mathbb{R}^m).$$

The following important result has been proved by Acerbi and Fusco [1].

***Theorem 2.3.*** *Assume $f$ is Carathéodory and satisfies*

$$0 \le f(x,s,\xi) \le c_1(|\xi|^p + |s|^p) + A(x),$$

*where $c_1 > 0$ and $A \in L^1(\Omega)$. Then the functional $I$ defined above is weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ if and only if $f(x,s,\cdot)$ is quasiconvex for almost every $x \in \Omega$ and all $s \in \mathbb{R}^n$; that is, the inequality*

$$f(x,s,\xi) \le \frac{1}{|\Omega|}\int_\Omega f(x,s,\xi + D\varphi(y))\,dy$$

*holds for a.e. $x \in \Omega$, all $s \in \mathbb{R}^n$, $\xi \in M^{m\times n}$, and all $\varphi \in C_0^\infty(\Omega;\mathbb{R}^m)$.*

For smooth and bounded-below functionals on Banach space, we have the following result from the Ekeland variational principle [2, 6]. We refer to [2, 6] for the proof and more on the applications of the Ekeland variational principle.

***Theorem 2.4.*** *Let $X$ be a Banach space and $X^*$ its dual space, and let $\Phi\colon X \to \mathbb{R}$ be a $C^1$ functional which is bounded below. Then, for each $\varepsilon > 0$, there exists $u_\varepsilon \in X$ such that*

$$\Phi(u_\varepsilon) \le \inf_X \Phi + \varepsilon, \tag{2.6}$$

$$\|\Phi'(u_\varepsilon)\|_{X^*} \le \varepsilon. \tag{2.7}$$

*Therefore, there exists a minimizing sequence $\{u_k\}$ in $X$ such that*

$$\lim_{k\to\infty}\Phi(u_k) = \inf_X \Phi, \quad \lim_{k\to\infty}\|\Phi'(u_k)\|_{X^*} = 0.$$

## 3. The (PS)-weak Lower Semicontinuity

In this section, we assume $f(x,s,\xi)$ is measurable in $x \in \Omega$ for all $(s,\xi) \in \mathbb{R}^n \times M^{m\times n}$, and is $C^1$ in $(s,\xi) \in \mathbb{R}^n \times M^{m\times n}$ for almost every $x \in \Omega$. We also assume $1 < p < \infty$, and $f$ satisfies the growth conditions

$$|f(x,s,\xi)| \le c_1(|s|^p + |\xi|^p) + A(x), \tag{3.1}$$

$$|D_s f(x,s,\xi)| + |D_\xi f(x,s,\xi)| \le c_2(|s|^{p-1} + |\xi|^{p-1}) + B(x), \tag{3.2}$$

for almost every $x \in \Omega$ and for all $s \in \mathbb{R}^n$, $\xi \in M^{m\times n}$, where $c_1$, $c_2$ are positive constants and $A$, $B$ are positive functions with $A \in L^1(\Omega)$, $B \in L^{p/(p-1)}(\Omega)$.

From these assumptions, we easily obtain the following result, whose proof is left to the interested reader.

***Proposition 3.1.*** *Under the conditions above, the functional $I$ defined above is a $C^1$ functional on $W^{1,p}(\Omega;\mathbb{R}^m)$ and for each $u$ the Fréchet derivative $I'(u)$ is given by*

$$\langle I'(u), v\rangle = \int_\Omega [D_s f(x,u,Du)\cdot v + D_\xi f(x,u,Du) : Dv]\,dx,$$

*for all $v \in W^{1,p}(\Omega;\mathbb{R}^m)$.*

When minimizing the functional $I$ on a Dirichlet class $\mathcal{A}_g$, one can shift the class to the Banach space $X = W_0^{1,p}(\Omega;\mathbb{R}^m)$ since

$$\inf_{u\in\mathcal{A}_g} I(u) = \inf_{w\in X} \Phi(w), \tag{3.3}$$

where $\Phi(w) = I(w+g)$. We easily have the following result.

***Proposition 3.2.*** *Let $X = W_0^{1,p}(\Omega;\mathbb{R}^m)$. For any $g \in W^{1,p}(\Omega;\mathbb{R}^m)$, the functional $\Phi\colon X \to \mathbb{R}$ defined by $\Phi(w) = I(w+g)$ is $C^1$ and $\Phi'(w) = I'(w+g)$ as elements in $X^*$, the dual space of $X$.*

In the following we write $X^* = W^{-1,p'}(\Omega;\mathbb{R}^m)$, where $p' = p/(p-1)$. As usual, we define

$$\|I'(u)\|_{W^{-1,p'}} = \sup\{\langle I'(u), v\rangle \mid v \in W_0^{1,p}(\Omega;\mathbb{R}^m),\ \|v\|_{W_0^{1,p}} \le 1\}. \tag{3.4}$$

Note that, given a smooth functional $I$ on $X = W_0^{1,p}(\Omega;\mathbb{R}^m)$, the sequences $\{u_k\}$ in $X$ satisfying

$$|I(u_k)| \le M, \quad I'(u_k) \to 0 \text{ in } W^{-1,p'}(\Omega;\mathbb{R}^m)$$

are usually called the *Palais-Smale sequences* or (*PS*) *sequences* for the functional $I$. Therefore, for simplicity, we use the following definition.

*Definition 3.3.* A sequence $\{u_k\}$ is said to *(PS)-weakly converge to* $u$ (with respect to $I$) in $W^{1,p}(\Omega;\mathbb{R}^m)$ and denoted by $u_k \overset{\text{ps}}{\rightharpoonup} u$ provided that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and $I'(u_k) \to 0$ in $W^{-1,p'}(\Omega;\mathbb{R}^m)$. Define the set of all (PS)-weak limits to be

$$S = \{u \in W^{1,p}(\Omega;\mathbb{R}^m) \mid \exists\, u_k \in W^{1,p}(\Omega;\mathbb{R}^m) \text{ such that } u_k \overset{\text{ps}}{\rightharpoonup} u\}. \tag{3.5}$$

Let $C = \{u \in W^{1,p}(\Omega;\mathbb{R}^m) \mid \|I'(u)\|_{W^{-1,p'}} = 0\}$. Then clearly $C \subseteq S$, and hence $S$ can be viewed as a relaxation of $C$ under the (PS)-weak convergence. However, for certain functionals $I$ the set $S$ may be empty.

*Example 3.4.* Let $f(x,\xi) = \chi_E(x)h(\xi)$, where $\chi_E$ is the characteristic function of a measurable set $E$ in $(0,1)$ with $0 < |E| < 1$ and $h(\xi) = \pi/2 + \arctan \xi$. Define

$$I(u) = \int_0^1 f(x, u'(x))\,dx, \quad u \in W^{1,2}(0,1).$$

We claim that for the functional $I$ the (PS)-weak limit set $S = \emptyset$. Suppose to the contrary $u_k \overset{\text{ps}}{\rightharpoonup} u$ in $W^{1,2}(0,1)$. Let $g_k(x) = \chi_E(x)h'(u_k'(x))$. Then, by Proposition 4.1 below, there exists a subsequence $g_{k_j} \to L$ strongly in $L^2(0,1)$ for some constant $L$. We also assume $g_{k_j}(x) \to L$ for almost every $x \in (0,1)$. Hence we must have $L = 0$ and $g_{k_j}(x) = h'(u_{k_j}'(x)) \to 0$ for almost every $x \in E$. By Egoroff's theorem, it follows that $|u_{k_j}'(x)| \to \infty$ almost uniformly on $E$, which implies $\|u_{k_j}'\|_{L^2(E)} \to \infty$, a contradiction.

*Definition 3.5.* Given any nonempty family $\mathcal{A} \subseteq W^{1,p}(\Omega;\mathbb{R}^m)$, we say that $I$ is *(PS)-weakly lower semicontinuous on* $\mathcal{A}$ provided that

$$I(u) \le \liminf_{k\to\infty} I(u_k) \quad \text{whenever } u_k,\, u \in \mathcal{A},\ u_k \overset{\text{ps}}{\rightharpoonup} u. \tag{3.6}$$

We shall technically assume this property if $\mathcal{A} \cap S = \emptyset$.

The following result shows that if $f = f(x,\xi)$ is convex in $\xi$, then the functional $I$ is in fact *(PS)-weakly continuous* on all Dirichlet classes.

***Proposition 3.6.*** *Assume $f = f(x,\xi)$ satisfies the corresponding growth conditions as in (3.1) and (3.2) above. Suppose $f(x,\xi)$ is convex in $\xi$ for almost every $x \in \Omega$. Then both $I$ and $-I$ are (PS)-weakly lower semicontinuous on all Dirichlet classes $\mathcal{A}_g$ with $g \in W^{1,p}(\Omega;\mathbb{R}^m)$. Therefore the functional $I$ is (PS)-weakly continuous on $\mathcal{A}_g$ in the sense that*

$$I(u) = \lim_{k\to\infty} I(u_k) \quad \forall\, u_k,\, u \in \mathcal{A}_g,\ u_k \overset{\text{ps}}{\rightharpoonup} u. \tag{3.7}$$

*Proof.* For any $u_k, u \in W^{1,p}(\Omega;\mathbb{R}^m)$, by the convexity of $f$, it follows from (2.4) that

$$f(x,Du_k) \geq f(x,Du) + D_\xi f(x,Du):(Du_k - Du), \tag{3.8}$$

$$f(x,Du) \geq f(x,Du_k) + D_\xi f(x,Du_k):(Du - Du_k), \tag{3.9}$$

for almost every $x \in \Omega$. If $u_k \overset{\text{ps}}{\rightharpoonup} u$ and $u - u_k \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, then integrating the inequalities above, we have

$$\liminf_{k\to\infty} I(u_k) \geq I(u) \geq \limsup_{k\to\infty} I(u_k),$$

and hence (3.7) follows. □

We show that in general the (PS)-weak lower semicontinuity on all Dirichlet classes does not imply the (PS)-weak lower semicontinuity on the whole space $W^{1,p}(\Omega;\mathbb{R}^m)$ (without the fixed boundary conditions).

***Proposition 3.7.*** *Let $\Omega$ be the unit disc in $\mathbb{R}^2$ and $I(u) = -\int_\Omega |Du|^2\,dx$ for $u\colon \Omega \to \mathbb{R}$. Then $I$ is (PS)-weakly lower semicontinuous on all Dirichlet classes of $W^{1,2}(\Omega)$ but not (PS)-weakly lower semicontinuous on $W^{1,2}(\Omega)$.*

*Proof.* By the preceding proposition, $I$ is (PS)-weakly lower semicontinuous on all Dirichlet classes of $W^{1,2}(\Omega)$. We now show it is not (PS)-weakly lower semicontinuous on $W^{1,2}(\Omega)$ (without the fixed boundary conditions). We identify $\mathbb{R}^2 \cong \mathbb{C}^1$. For $z = x_1 + ix_2 \in \Omega$ and $k = 1, 2, \ldots$, we define $u_k(x_1,x_2) = (1/\sqrt{\pi k})\Re(z^k)$. Then $u_k$ is harmonic in $\Omega$ and $\partial_{x_1}u_k - i\partial_{x_2}u_k = \sqrt{k/\pi}z^{k-1}$. Hence $|Du_k(x)| = \sqrt{k/\pi}\,|z|^{k-1}$ and thus we have $\|Du_k\|_{L^2(\Omega)} = 1$. So $u_k$ is bounded in $W^{1,2}(\Omega)$. It is easy to see $u_k \to 0$ uniformly on $\bar\Omega$ and hence $u_k \rightharpoonup 0$ in $W^{1,2}(\Omega)$. Since $u_k$ is harmonic in $\Omega$, it also follows that $Du_k \to 0$ in $W^{-1,2}(\Omega)$. Therefore, for functional $I(u) = -\int_\Omega |Du|^2\,dx$, we have $u_k \overset{\text{ps}}{\rightharpoonup} 0$, but $I(0) = 0$ and $\liminf_k I(u_k) = -1$. Hence $I$ is not (PS)-weakly lower semicontinuous on $W^{1,2}(\Omega)$. □

As we mentioned in the introduction, the (PS)-weak lower semicontinuity has been motivated by using the Ekeland variational principle in the direct method for the minimization problem. We have the following existence result.

***Theorem 3.8.*** *Assume $f$ satisfies, in addition to (3.1) and (3.2), the following coercivity condition*

$$c_0|\xi|^p - a(x) \leq f(x,s,\xi) \leq c_1(|\xi|^p + |s|^p) + A(x), \tag{3.10}$$

*where $c_0 > 0$ is a positive constant, $a \in L^1(\Omega)$ is a function. Given $g \in W^{1,p}(\Omega;\mathbb{R}^m)$, assume the functional $I$ defined above is (PS)-weakly lower semicontinuous on $\mathcal{A}_g$. Then the minimization problem $\inf_{u\in\mathcal{A}_g} I(u)$ has at least one solution $u \in \mathcal{A}_g$.*

*Proof.* The proof uses a standard direct method of the calculus of variations. Let $X = W_0^{1,p}(\Omega;\mathbb{R}^m)$. Define $\Phi\colon X \to \mathbb{R}$ by

$$\Phi(u) = I(u+g) = \int_\Omega f(x, u(x)+g(x), Du(x)+Dg(x))\,dx.$$

Then $\Phi$ is $C^1$ and bounded below on $X$, and $\Phi'(u) = I'(u+g)$ in $X^*$. By Theorem 2.4, there exists a sequence $\{u_k\}$ in $X$ such that

$$\Phi(u_k) \to \inf_X \Phi, \qquad \|\Phi'(u_k)\|_{X^*} \to 0.$$

Let $w_k = u_k + g \in \mathcal{A}_g$. Then

$$I(w_k) \to \inf_{w\in\mathcal{A}_g} I(w), \quad \|I'(w_k)\|_{W^{-1,p'}} \to 0. \tag{3.11}$$

Under the condition $c_0 > 0$ the sequence $\{w_k\}$ determined by (3.11) above is bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$ and, since $1 < p < \infty$, has a weakly convergence subsequence, relabeled $\{w_k\}$ again. Let $u$ be the weak limit. Then $u \in \mathcal{A}_g$ and $w_k \overset{\text{ps}}{\rightharpoonup} u$; hence the (PS)-weak lower semicontinuity on $\mathcal{A}_g$ implies

$$I(u) \le \lim_{k\to\infty} I(w_k) = \inf_{w\in\mathcal{A}_g} I(w).$$

Hence $I(u) = \inf_{w\in\mathcal{A}_g} I(w)$. □

*Remark **3.9**.* Under the growth assumptions (3.1) and (3.2), any minimizer $u$ of $I$ over $\mathcal{A}_g$ is a weak solution to the Dirichlet problem of the Euler-Lagrange equation of functional $I$; that is,

$$\begin{cases} -\operatorname{div} D_\xi f(x,u,Du) + D_s f(x,u,Du) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

## 4. ONE DIMENSIONAL SCALAR CASES

In this section we study the (PS)-weak lower semicontinuity in some special one dimensional scalar cases.

We first consider the Sobolev space $H^1(0,1) = W^{1,2}(0,1)$ and functions $f(x,\xi)$ satisfying

$$0 \le f(x,\xi) \le C|\xi|^2 + A(x), \quad |f_\xi(x,\xi)| \le C|\xi| + B(x), \tag{4.1}$$

with $A \in L^1(0,1)$, $B \in L^2(0,1)$. Define

$$I(u) = \int_0^1 f(x, u'(x))\,dx, \quad \forall\, u \in H^1(0,1).$$

***Proposition 4.1.*** *If* $u_k \overset{\text{ps}}{\rightharpoonup} u$ *in* $H^1(0,1)$*, then there exists a subsequence* $\{u_{k_j}\}$ *such that* $f_\xi(x, u'_{k_j}(x)) \to L$ *strongly in* $L^2(0,1)$ *as* $j \to \infty$*, where L is a constant.*

*Proof.* Let

$$g_k(x) = f_\xi(x, u'_k(x)) \quad \text{and} \quad L_k = \int_0^1 g_k(x)\,dx.$$

Since $\{g_k\}$ is bounded in $L^2(0,1)$, we assume for a subsequence $g_{k_j} \rightharpoonup g$ in $L^2(0,1)$ as $j \to \infty$, where $g \in L^2(0,1)$. We define $v_k$ on $[0,1]$ by

$$v_k(x) = \int_0^x (g_k(t) - L_k)\,dt, \quad x \in [0,1].$$

Then it is easily seen that $v_k \in H_0^1(0,1) = W_0^{1,2}(0,1)$ and $v'_k = g_k - L_k$. Moreover, $\{v_k\}$ is bounded in $H_0^1(0,1)$ and hence

$$\langle I'(u_k), v_k\rangle = \int_0^1 f_\xi(x, u'_k(x))v'_k(x)\,dx = \int_0^1 g_k^2(x)\,dx - L_k^2 \to 0$$

as $k \to \infty$. Since $g_{k_j} \rightharpoonup g$ in $L^2(0,1)$, we have

$$L_{k_j} \to L = \int_0^1 g\,dx,$$

$$\int_0^1 g^2(x)\,dx \le \liminf_{j\to\infty} \int_0^1 g_{k_j}^2\,dx = \liminf_{j\to\infty} L_{k_j}^2 = \left(\int_0^1 g(x)\,dx\right)^2.$$

This implies $g(x) = L$ a.e. on $[0,1]$ and $g_{k_j} \to L$ strongly in $L^2(0,1)$. ◻

In contrast to the theorem of Acerbi and Fusco (Theorem 2.3), we show below by an example that the (PS)-weak lower semicontinuity of $I$ may not imply $f$ being quasiconvex in $\xi$ even for smooth functions $f(x,\xi)$ in the scalar case.

***Proposition 4.2.*** *There exists a* $C^1$ *function* $f(x,\xi)$ *satisfying condition* (4.1) *above for which the corresponding functional I is* (*PS*)*-weakly, but not* (*unrestricted*) *weakly, lower semicontinuous on* $H^1(0,1)$*.*

*Proof.* Assume $f(x,\xi) = a(x)h(\xi)$ with $a$, $h \ge 0$, both $C^1$ and satisfying the following conditions:

$$a(x) = 0 \text{ for } x \in [0,\theta], \quad a(x) > 0 \text{ for } x \in (\theta, 1], \tag{4.2}$$

$$h \ge 0, \quad (h')^{-1}(0) = \{0\}, \quad \liminf_{|\xi|\to\infty} |h'(\xi)| > 0., \tag{4.3}$$

where $\theta \in (0,1)$ is a constant. Note that the condition (4.3) implies $h(0) < h(\xi)$ for all $\xi \in \mathbb{R}$. Given any $u_k \overset{\text{ps}}{\rightharpoonup} u$ in $H^1(0,1)$, using a subsequence if necessary, we assume $\lim_{k\to\infty} I(u_k)$ exists. By Proposition 4.1 above, there exists a subsequence $\{u_{k_j}\}$ such that $f_\xi(x, u'_k) = a(x)h'(u'_{k_j}) \to L$ strongly in $L^2(0,1)$ for some constant $L$. Since $a = 0$ on $[0,\theta)$, one must have the limit $L = 0$; this also implies the whole sequence $a(x)h'(u'_k) \to 0$ strongly in $L^2(0,1)$. Therefore $h'(u'_k) \to 0$ strongly in $L^2(\theta',1)$ for any $\theta' \in (\theta,1)$. Hence, for a subsequence it follows that $h'(u'_{k_j}(x)) \to 0$ for almost every $x \in (\theta',1)$. By (4.3), we have that $u'_{k_j}(x) \to 0$ for almost every $x \in (\theta',1)$. Therefore the weak limit $u' = 0$ on $(\theta',1)$ for all $\theta' \in (\theta,1)$. This implies $u' = 0$ in $(\theta,1)$. Since $h(\xi) \geq h(0)$ for all $\xi$, we have

$$\begin{aligned}\lim_{k\to\infty} I(u_k) &= \lim_{k\to\infty} \int_\theta^1 a(x)h(u'_k(x))\,dx \\ &\geq \int_\theta^1 a(x)h(0)\,dx = I(u).\end{aligned}$$

Hence $I$ satisfies the (PS)-weak lower semicontinuity on $H^1(0,1)$. Note that the condition (4.3) does not imply that $h$ is convex. (See, e.g., condition (2.5).) Hence $I$ may not be weakly lower semicontinuous on $H^1(0,1)$ by Theorem 2.3 above. □

Despite of the result above, we shall show that the (PS)-weak lower semicontinuity is equivalent to the usual weak lower semicontinuity if $f(x,\xi)$ satisfies certain coercivity condition.

In the following, for $\beta \in \mathbb{R}$, let $W^{1,p}_\beta(0,1)$ be the Dirichlet class of functions $u$ in $W^{1,p}(0,1)$ with $u(0) = 0$, $u(1) = \beta$. Then we have the following result.

***Theorem 4.3.*** *Assume $f(x,\xi)$ and $f_\xi(x,\xi)$ are both $C^1$ on $[0,1]\times\mathbb{R}$ and satisfy, for some $p > 1$ and positive constants $c_0$, $c_1$, $c_2 > 0$,*

$$c_0|\xi|^p \leq f(x,\xi) \leq c_1(|\xi|^p + 1), \quad |f_\xi(x,\xi)| \leq c_2(|\xi|^{p-1} + 1), \tag{4.4}$$

*for all $x$ and $\xi$. If the functional $I$ defined by $f$ is (PS)-weakly lower semicontinuous on $W^{1,p}_\beta(0,1)$ for all $\beta \in \mathbb{R}$, then $f(x,\xi)$ is convex in $\xi$ for all $x \in (0,1)$.*

For the technical reason of using the following Sard's theorem [9], we have assumed that $f$ is sufficiently smooth in both $x$ and $\xi$ in the theorem.

***Lemma 4.4.*** *Let $h\colon \mathbb{R} \to \mathbb{R}$ be $C^1$ and $S = \{y \in \mathbb{R} \mid \exists x \in \mathbb{R},\ y = h(x),\ h'(x) = 0\}$. Then the Lebesgue measure $|S| = 0$ and, in particular, the set of regular values of $h$, $\mathbb{R}\setminus S$, is dense in $\mathbb{R}$.*

We proceed with several lemmas to prove Theorem 4.3. First of all, for $\beta \in \mathbb{R}$, we define $m(\beta) = \inf\{I(u) \mid u \in W^{1,p}_\beta(0,1)\}$. It follows easily from the growth condition (4.4) that

$$c_0|\beta|^p \leq m(\beta) \leq c_1(|\beta|^p + 1). \tag{4.5}$$

From Theorem 3.8 above, it follows that, if $I$ is (PS)-weakly lower semicontinuous on $W^{1,p}_\beta(0,1)$, then there exists at least one minimizer $u_\beta \in W^{1,p}_\beta(0,1)$ such that $I(u_\beta) = m(\beta)$. Hence $I'(u_\beta) = 0$ in $W^{-1,p'}(0,1)$. This implies $f_\xi(x, u'_\beta(x))$ is constant in $(0,1)$ and let $\mu(\beta)$ denote this constant. Note that $\mu(\beta)$ depends also on the minimizer $u_\beta$.

***Lemma 4.5.*** *It follows that*

$$\limsup_{\beta\to+\infty}\ \limsup_{\varepsilon\to 0^+} \frac{m(\beta+\varepsilon)-m(\beta)}{\varepsilon} = +\infty, \tag{4.6}$$

$$\liminf_{\beta\to-\infty}\ \liminf_{\varepsilon\to 0^-} \frac{m(\beta+\varepsilon)-m(\beta)}{\varepsilon} = -\infty. \tag{4.7}$$

*Proof.* We only prove Equation (4.6); the other one follows similarly. By contradiction, suppose the limit is not $+\infty$. Then there exist positive constants $\beta_0$, $\varepsilon_0$ and $M$ such that

$$\frac{m(\beta+\varepsilon)-m(\beta)}{\varepsilon} \le M, \quad \forall\, \beta \ge \beta_0,\ \varepsilon \in (0,\varepsilon_0],$$

which, in particular, implies that

$$m(\beta_0 + k\varepsilon_0) - m(\beta_0) \le Mk\varepsilon_0 \quad \forall\, k = 1,2,\dots. \tag{4.8}$$

Using (4.5), we have $m(\beta_0 + k\varepsilon_0) \ge c_0|\beta_0 + k\varepsilon_0|^p \ge \gamma k^p - C_0$ for some positive constants $\gamma$ and $C_0$, and for all $k = 1, 2, \dots$. This combined with (4.8) yields a desired contradiction, since $p > 1$. This proves (4.6). □

***Lemma 4.6.*** *For any $\beta \in \mathbb{R}$, it follows that*

$$\limsup_{\varepsilon\to 0^+} \frac{m(\beta+\varepsilon)-m(\beta)}{\varepsilon} \le \mu(\beta) \le \liminf_{\varepsilon\to 0^-} \frac{m(\beta+\varepsilon)-m(\beta)}{\varepsilon}. \tag{4.9}$$

*Proof.* For $0 < \delta < 1$ we define $w$ to be the linear function with $w(1-\delta) = 0$, $w(1) = \varepsilon$. Hence $w'(x) = \varepsilon/\delta$. Let $u_\beta$ be a minimizer for $m(\beta)$ and let $v(x) = u_\beta(x)$ on $[0, 1-\delta]$ and $v(x) = u_\beta(x) + w(x)$ on $[1-\delta, 1]$. Then $v \in W^{1,p}(0,1)$ satisfies $v(0) = 0$, $v(1) = \beta + \varepsilon$. Hence

$$m(\beta+\varepsilon) \le I(v) = I(u_\beta) + \int_{1-\delta}^{1} [f(x,v') - f(x,u'_\beta)]\,dx.$$

Since $f(x,v') - f(x,u'_\beta) = f_\xi(x,u'_\beta)\varepsilon/\delta + o(\varepsilon/\delta)$ for $\varepsilon/\delta \to 0$, we have

$$m(\beta+\varepsilon) \le m(\beta) + \mu(\beta)\varepsilon + o\left(\frac{\varepsilon}{\delta}\right)\delta \le m(\beta) + \mu(\beta)\varepsilon + o(\varepsilon),$$

as $\varepsilon \to 0$. From this the lemma follows. □

The lemmas above imply

$$\limsup_{\beta\to+\infty}\mu(\beta)=+\infty,\quad \liminf_{\beta\to-\infty}\mu(\beta)=-\infty. \tag{4.10}$$

***Lemma 4.7.*** *For any constant $\theta\in\mathbb{R}$, there exists a function $q_\theta\in L^p(0,1)$ such that $f_\xi(x,q_\theta(x))=\theta$ for almost every $x\in(0,1)$.*

*Proof.* In view of (4.10) above, there exist $\beta_1<\beta_2$ such that $\mu(\beta_1)<\theta<\mu(\beta_2)$. Hence for almost every $x\in(0,1)$ we have $f_\xi(x,u'_{\beta_1}(x))<\theta<f_\xi(x,u'_{\beta_2}(x))$. Let

$$q^-(x)=\min\{u'_{\beta_1}(x),u'_{\beta_2}(x)\},$$
$$q^+(x)=\max\{u'_{\beta_1}(x),u'_{\beta_2}(x)\}.$$

Then $q^\pm\in L^p(0,1)$. By the intermediate value property of $f_\xi(x,\cdot)$, there exists $q\in(q^-(x),q^+(x))$ such that $f_\xi(x,q)=\theta$. Let $q_\theta(x)$ be the infimum of all such $q$'s. Then $f_\xi(x,q_\theta(x))=\theta$, $q_\theta(x)$ is lower semicontinuous and $q^-(x)\le q_\theta(x)\le q^+(x)$ at almost every $x\in(0,1)$ and hence $q_\theta\in L^p(0,1)$. □

The following result turns out to be quite useful in the proof of the theorem and also later; the proof is elementary and included here for the convenience of the reader.

***Lemma 4.8.*** *Let $h\colon\mathbb{R}\to\mathbb{R}$ be $C^1$ and let*

$$S=\left\{\begin{pmatrix}a\\b\end{pmatrix}\in\mathbb{R}^2\mid h'(a)=h'(b)\right\},$$
$$S_1=\left\{\begin{pmatrix}\alpha\\\beta\end{pmatrix}\in S\mid \alpha<\beta,\ h'(t)\neq h'(\alpha)\ \forall\, t\in(\alpha,\beta)\right\}.$$

*Assume $h(t)\ge 0$ for all $t\in\mathbb{R}$. Then the following statements are equivalent:*

(i) *$h$ is convex.*
(ii) *$S_1=\varnothing$.*
(iii) *$h(\lambda a+(1-\lambda)b)\le\lambda h(a)+(1-\lambda)h(b)$, $\forall\ \binom{a}{b}\in S$, $\lambda\in[0,1]$.*

*Proof.* We prove the result by showing that (i) ⇒ (iii) ⇒ (ii) ⇒ (i).

**1.** (i) ⇒ (iii): This is easy.

**2.** (iii) ⇒ (ii): Suppose to the contrary $S_1\neq\varnothing$ and let $\binom{\alpha}{\beta}\in S_1$. Using inequality (iii) with $\binom{\alpha}{\beta}\in S$ yields that $\forall\ 0<\lambda<1$ with $t_\lambda=\lambda\alpha+(1-\lambda)\beta$,

$$\frac{h(t_\lambda)-h(\alpha)}{t_\lambda-\alpha}\le\frac{h(\beta)-h(\alpha)}{\beta-\alpha}\le\frac{h(t_\lambda)-h(\beta)}{t_\lambda-\beta}. \tag{4.11}$$

Letting $\lambda \to 1^-$ and $0^+$ in (4.11) respectively yields that

$$h'(\alpha) \le \frac{h(\beta) - h(\alpha)}{\beta - \alpha} \le h'(\beta);$$

hence $h'(\alpha) = h'(\beta) = (h(\beta) - h(\alpha))/(\beta - \alpha)$. However, by the mean value property, $h'(\alpha) = (h(\beta) - h(\alpha))/(\beta - \alpha) = h'(t)$ for some $t \in (\alpha, \beta)$, which shows $\binom{\alpha}{\beta} \notin S_1$, a desired contradiction.

**3.** (ii) $\Rightarrow$ (i): Again, to the contrary, suppose $h$ is not convex. Then there exist $a < b$ such that $h'(a) > h'(b)$. We consider only the case when $h'(a) > 0$; otherwise, consider $\bar{h}(t) = h(-t)$, $\bar{a} = -b$ and $\bar{b} = -a$. We claim there exist $c < d \le a$ such that $h'(c) < h'(d)$. If not, $h'$ would be nonincreasing on $(-\infty, a]$ and hence $h$ would be concave on $(-\infty, a]$. Therefore we would have $h(t) \le h(a) + h'(a)(t - a)$ for all $t < a$. Since $h'(a) > 0$, letting $t \to -\infty$, we would have $h(t) \to -\infty$, a contradiction with $h \ge 0$. Let $c < d \le a$ be any points as above. Let $m = \max_{[c,b]} h'$. Define $\Sigma = \{t \in [c, b] \mid h'(t) = m\}$, $s^- = \min \Sigma$, and $s^+ = \max \Sigma$. Then $s^-, s^+ \in \Sigma$, and $c < s^- \le s^+ < b$. Hence $h'(c) < m$, $h'(b) < m$. We define $\alpha' < \beta'$ as follows: If $h'(c) = h'(b)$, define $\alpha' = c$, $\beta' = b$. If $h'(c) > h'(b)$, then $h'(c) \in (h'(s^+), h'(b))$ and hence by the intermediate value property of $h'$, define $\beta' \in (s^+, b)$ so that $h'(\beta') = h'(c)$, and define $\alpha' = c$. If $h'(c) < h'(b)$, then $h'(b) \in (h'(c), h'(s^-))$ and hence again by the intermediate value property of $h'$, we define $\alpha' \in (c, s^-)$ so that $h'(\alpha') = h'(b)$, and define $\beta' = b$. The points $\alpha' < \beta'$ defined this way will satisfy $\alpha' < s^- \le s^+ < \beta'$ and $h'(\alpha') = h'(\beta) < h'(s^-)$. Let $G = \{t \in (\alpha', \beta') \mid h'(t) > h'(\alpha')\}$. Then $G$ is an open set and $s^- \in G$. Let $(\alpha, \beta)$ be the component of $G$ containing $s^-$. Then it follows that $\binom{\alpha}{\beta} \in S_1$, a contradiction with (ii); hence $h$ is convex. $\square$

*Remark 4.9.* It can be shown that conditions (i) and (iii) in the lemma above are not equivalent for functions $h\colon \mathbb{R}^m \to \mathbb{R}$ if $m \ge 2$ (see Remark 5.9 later).

*Proof of Theorem 4.3.* Given any $x_0 \in (0, 1)$, we prove $f(x_0, \cdot)$ is convex. By Lemma 4.8, it suffices to show that there exist no numbers $\xi_1 < \xi_2$ such that

$$(4.12) \qquad f_\xi(x_0, \xi_1) = f_\xi(x_0, \xi_2), \quad f_\xi(x_0, t) \ne f_\xi(x_0, \xi_1), \quad \forall\, t \in (\xi_1, \xi_2).$$

We prove this by contradiction. Suppose $\xi_1 < \xi_2$ satisfy (4.12). We will derive a contradiction by showing such $\xi_i$'s must satisfy

$$(4.13) \qquad f(x_0, \lambda\xi_1 + (1 - \lambda)\xi_2) \le \lambda f(x_0, \xi_1) + (1 - \lambda) f(x_0, \xi_2)$$

for all $\lambda \in (0, 1)$, which gives a desired contradiction as in the step 2 of the proof of Lemma 4.8.

To this end, assume $f_\xi(x_0,\xi_1) = f_\xi(x_0,\xi_2) = \theta_0$. Without loss of generality, assume $f_\xi(x_0,t) > f_\xi(x_0,\xi_1)$ for all $t \in (\xi_1,\xi_2)$. Let

$$\max_{[\xi_1,\xi_2]} f_\xi(x_0,\cdot) = f_\xi(x_0,\bar{\xi}) = \bar{\theta} > \theta_0.$$

To proceed, we need the following lemma, which is the only place we use the smooth assumption of $f_\xi(x,\xi)$ on $(x,\xi)$.

***Lemma 4.10.*** *There exist a sequence $\theta_n \in (\theta_0,\bar{\theta})$ with $\theta_n \to \theta_0$ as $n \to \infty$, a closed interval $J_n = [a_n,b_n] \subset (0,1)$ containing $x_0$, and two continuous functions $q_n^\pm \colon J_n \to (\xi_1,\xi_2)$ such that $q_n^-(x) < q_n^+(x)$ and $f_\xi(x,q_n^\pm(x)) = \theta_n$ for all $x \in J_n$. Moreover, $q_n^{-,+}(x_0) \to \xi_{1,2}$ as $n \to \infty$.*

*Proof.* The proof is based on a use of Sard's theorem. By Lemma 4.4 above with $h(\xi) = f_\xi(x_0,\xi)$, the set of regular values of $f_\xi(x_0,\cdot)$ is dense. Hence there exists a sequence of regular values $\theta_n$ of $f_\xi(x_0,\cdot)$ in $(\theta_0,\bar{\theta})$ such that $\theta_n \to \theta_0$ as $n \to \infty$. Since $f_\xi(x_0,\xi_{1,2}) = \theta_0 < \theta_n < \bar{\theta} = f_\xi(x_0,\bar{\xi})$, by intermediate value property, there exist $\xi_n^- \in (\xi_1,\bar{\xi})$ and $\xi_n^+ \in (\bar{\xi},\xi_2)$ such that $f_\xi(x_0,\xi_n^\pm) = \theta_n$. The assumption (4.12) implies $\xi_n^- \to \xi_1$ and $\xi_n^+ \to \xi_2$ as $n \to \infty$. Since $\theta_n$ is a regular value of $f_\xi(x_0,\cdot)$, it follows that $f_{\xi\xi}(x_0,\xi_n^\pm) \neq 0$. By the implicit function theorem, we have an interval $J_n = [a_n,b_n] \subset (0,1)$ containing $x_0$ and two differentiable functions $q_n^\pm \colon J_n \to (\xi_1,\xi_2)$ such that

$$q_n^\pm(x_0) = \xi_n^\pm, \quad f_\xi(x,q_n^\pm(x)) = \theta_n, \quad \forall\, x \in J_n. \tag{4.14}$$

Then the functions $q_n^\pm(x)$ satisfy the requirements of the lemma. ∎

We continue the proof of the theorem. Let $\theta_n \in (\theta_0,\bar{\theta})$, $J_n = [a_n,b_n]$ and $q_n^\pm \colon J_n \to (\xi_1,\xi_2)$ be given as in the lemma above. Let $J = [a,b] \subset J_n$ be any interval containing $x_0$. Let $q_n \in L^p(0,1)$ be the function $q_\theta$ determined by Lemma 5.2, with $\theta = \theta_n$. In what follows, we fix $n$. For each $k = 1, 2, \ldots,$ we define function $u_k(x)$ by $u_k(x) = \int_0^x w_k(t)\,dt$, where $w_k(t)$ is defined as follows:

$$w_k(t) = \begin{cases} q_n(t), & t \in [0,1] \setminus [a,b], \\ q_n^-(t), & t \in \displaystyle\bigcup_{j=1}^{k} \Big(a + \frac{j-1}{k}(b-a), a + \frac{j-1+\lambda}{k}(b-a)\Big), \\ q_n^+(t), & t \in \displaystyle\bigcup_{j=1}^{k} \Big(a + \frac{j-1+\lambda}{k}(b-a), a + \frac{j}{k}(b-a)\Big). \end{cases} \tag{4.15}$$

It is easily seen that $u_k \in W^{1,p}(0,1)$ and $\{u_k\}$ is bounded in $W^{1,p}(0,1)$.

***Lemma 4.11.*** *For all continuous functions $\Phi(x,\xi)$, it follows that*

$$\lim_{k\to\infty}\int_a^b \Phi(x,u_k'(x))\,dx = \int_a^b [\lambda\Phi(x,q_n^-(x)) + (1-\lambda)\Phi(x,q_n^+(x))]\,dx.$$

*Proof.* It is easy to see

$$\int_a^b \Phi(x,u_k'(x))\,dx = \sum_{j=1}^k \int_{a+(j-1)(b-a)/k}^{a+(j-1+\lambda)(b-a)/k} \Phi(x,q_n^-(x))\,dx + \sum_{j=1}^k \int_{a+(j-1+\lambda)(b-a)/k}^{a+j(b-a)/k} \Phi(x,q_n^+(x))\,dx \tag{4.16}$$

$$= \lambda\sum_{j=1}^k \Phi(c_j,q_n^-(c_j))\frac{b-a}{k} + (1-\lambda)\sum_{j=1}^k \Phi(d_j,q_n^+(d_j))\frac{b-a}{k}, \tag{4.17}$$

where

$$a+\frac{j-1}{k}(b-a) \le c_j \le a+\frac{j-1+\lambda}{k}(b-a) \le d_j \le a+\frac{j}{k}(b-a)$$

are some points. Hence the sums in (4.16) and (4.17) are Riemann sums; therefore, as $k\to\infty$, the lemma follows. ◻

Let $\bar u \in W^{1,p}(0,1)$ be defined by $\bar u(x) = \int_0^x \bar w(t)\,dt$, where

$$\bar w(t) = \begin{cases} q_n(t), & t\in[0,1]\setminus[a,b],\\ \lambda q_n^-(t) + (1-\lambda)q_n^+(t), & t\in[a,b]. \end{cases}$$

From the lemma above, it easily follows that $u_k \rightharpoonup \bar u$ in $W^{1,p}(0,1)$. In particular, $\varepsilon_k = \bar u(1) - u_k(1) \to 0$ as $k\to\infty$. By the definition of $u_k$ it follows easily that $f_\xi(x,u_k'(x)) = \theta_n$ for almost every $x\in(0,1)$; hence $I'(u_k) = 0$ in $W^{-1,p'}(0,1)$. We now modify $u_k$ to a function $\tilde u_k \in W_\beta^{1,p}(0,1)$ with $\beta = \bar u(1)$. For $0<\delta<1-b$ to be selected later, we define $\tilde u_k(x) = u_k(x)$ for $x\in[0,1-\delta]$, and $\tilde u_k(x) = u_k(x) + v_k(x)$ for $x\in[1-\delta,1]$, where $v_k$ is a linear function on $[1-\delta,1]$ with $v_k(1-\delta)=0$, $v_k(1)=\varepsilon_k$. Hence $\tilde u_k \in W^{1,p}(0,1)$ with $\tilde u_k(0) = 0$, $\tilde u_k(1) = \bar u(1) = \beta$. Note that $v_k'(x) = \varepsilon_k/\delta$. Hence we select $\delta = \delta_k = |\varepsilon_k|^{1/2}$ for all sufficiently large $k$. For this choice of $\delta$, it is easily shown that the function $\tilde u_k \in W_\beta^{1,p}(0,1)$ satisfies $u_k - \tilde u_k \to 0$ in $W^{1,p}(0,1)$, and hence it follows that $I'(\tilde u_k) \to 0$ in $W^{-1,p'}(0,1)$ and $I(u_k) - I(\tilde u_k) \to 0$ as $k\to\infty$. In particular, $\tilde u_k \overset{\text{ps}}{\rightharpoonup} \bar u$ in $W_\beta^{1,p}(0,1)$. Therefore, by the (PS)-weak lower semicontinuity of $I$ on

$W_{\beta}^{1,p}(0,1)$, we have $I(\bar{u}) \le \liminf_k I(\tilde{u}_k) = \liminf_k I(u_k)$. Using Lemma 4.11, after easy computations, this implies

$$\int_a^b f(x, \lambda q_n^-(x) + (1-\lambda) q_n^+(x))\,dx \le \int_a^b [\lambda f(x, q_n^-(x)) + (1-\lambda) f(x, q_n^+(x))]\,dx.$$

This holds for all intervals $[a,b] \subset J_n$ containing $x_0$ and hence, letting $[a,b]$ shrink to $\{x_0\}$, we have

$$f(x_0, \lambda q_n^-(x_0) + (1-\lambda) q_n^+(x_0)) \le \lambda f(x_0, q_n^-(x_0)) + (1-\lambda) f(x_0, q_n^+(x_0)).$$

Finally letting $n \to \infty$, by Lemma 4.10, we have

$$f(x_0, \lambda\xi_1 + (1-\lambda)\xi_2) \le \lambda f(x_0, \xi_1) + (1-\lambda) f(x_0, \xi_2),$$

as desired by (4.13).

The proof of the theorem is now completed. □

## 5. SPECIAL CASES WITH $f = f(\xi)$

In this section, we study some special cases with function $f = f(\xi)$, where $f\colon M^{m\times n} \to \mathbb{R}$ is a $C^1$ function satisfying the following growth conditions:

$$c_0|\xi|^p \le f(\xi) \le c_1(|\xi|^p + 1), \tag{5.1}$$

$$|D_\xi f(\xi)| \le c_2(|\xi|^{p-1} + 1), \tag{5.2}$$

where $1 < p < \infty$ and $c_0 \ge 0$, $c_1 > 0$, $c_2 > 0$ are constants. In this case, we shall also use the simplified notation $D_\xi f(\xi) = Df(\xi) = f'(\xi)$. As before, let $I$ be the functional associated with $f$:

$$I(u) = \int_\Omega f(Du(x))\,dx, \quad u \in W^{1,p}(\Omega; \mathbb{R}^m).$$

We first have the following result when $m = 1$ (the scalar case), with $c_0 = 0$ in (5.1), which is in contrast to Proposition 4.2 above.

***Theorem 5.1.*** *Let $m = 1$ and let $f\colon \mathbb{R}^n \to \mathbb{R}$ satisfy the conditions (5.1) and (5.2) above, with $c_0 = 0$. Then the functional $I$ is (PS)-weakly lower semicontinuous on the Dirichlet classes $W_A^{1,p}$ for all $A \in \mathbb{R}^n$ if and only if $f$ is convex on $\mathbb{R}^n$.*

*Proof.* By Theorem 2.3, we only need to show the necessary part of the theorem. Thus assume $I$ is (PS)-weakly lower semicontinuous on the Dirichlet classes $W_A^{1,p}$ for all $A \in \mathbb{R}^n$. We prove that $f$ is convex on $\mathbb{R}^n$. To this end, let $\xi, \eta \in \mathbb{R}^n$

and $|\eta| = 1$ be given, and let $h(t) = f(\xi + t\eta)$. We show that $h$ is a convex function of $t \in \mathbb{R}$; this implies $f$ is convex on $\mathbb{R}^n$. By virtue of Lemma 4.8 above, to show $h$ is convex, it suffices to establish the inequality (iii) in that lemma for all $a, b \in \mathbb{R}$ with $a < b$ and $h'(a) = h'(b)$. Note that $h'(t) = f'(\xi + t\eta) \cdot \eta$. Given such $a, b$, let $\alpha = \xi + a\eta$, $\beta = \xi + b\eta$. Then $h(a) = f(\alpha)$, $h(b) = f(\beta)$ and hence

$$h'(a) - h'(b) = (f'(\alpha) - f'(\beta)) \cdot \eta = 0. \tag{5.3}$$

Given any $\lambda \in (0, 1)$, let $\theta(t)$ be the periodic function on $\mathbb{R}$ of period 1 satisfying $\theta = 0$ on $[0, \lambda)$ and $\theta = 1$ on $[\lambda, 1)$. Let $\rho(t)$ be the Lipschitz function on $\mathbb{R}$ with $\rho(0) = 0$ and $\rho'(t) = \theta(t)$ for almost every $t \in \mathbb{R}$. For $k = 1, 2, \ldots,$ we define functions

$$u_k(x) = \alpha x + \frac{b-a}{k}\rho(kx \cdot \eta), \quad x \in \mathbb{R}^n. \tag{5.4}$$

Then $Du_k(x) = \alpha + (b - a)\theta(kx \cdot \eta)\eta$ and hence

$$Du_k(x) = \begin{cases} \alpha & \text{if } x \cdot \eta \in \bigcup_{j=-\infty}^{\infty} \left(\frac{j}{k}, \frac{j+\lambda}{k}\right), \\ \beta & \text{if } x \cdot \eta \in \bigcup_{j=-\infty}^{\infty} \left(\frac{j+\lambda}{k}, \frac{j+1}{k}\right). \end{cases} \tag{5.5}$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an orthonormal basis of $\mathbb{R}^n$, with $\eta_1 = \eta$. For each $x \in \mathbb{R}^n$, we write $x = \sum_{i=1}^n t_i \eta_i$ and define

$$A_k^j = \left\{x \in \mathbb{R}^n \mid t_1 \in \left(\frac{j}{k}, \frac{j+\lambda}{k}\right)\right\},$$

$$B_k^j = \left\{x \in \mathbb{R}^n \mid t_1 \in \left(\frac{j+\lambda}{k}, \frac{j+1}{k}\right)\right\}.$$

Let $\Omega_\alpha^k = \Omega \cap (\bigcup_j A_k^j)$, $\Omega_\beta^k = \Omega \cap (\bigcup_j B_k^j)$. Then one can easily show that

$$\lim_{k\to\infty} |\Omega_\alpha^k| = \lambda|\Omega|, \quad \lim_{k\to\infty} |\Omega_\beta^k| = (1 - \lambda)|\Omega|. \tag{5.6}$$

For any $1 \leq p < \infty$, the sequence $\{u_k\}$ defined by (5.4) above satisfies $u_k \rightharpoonup \bar{u}$ in $W^{1,p}(\Omega)$ as $k \to \infty$, where $\bar{u}(x) = [\lambda\alpha + (1 - \lambda)\beta]x$. In fact, one can show that $u_k \to \bar{u}$ uniformly on $\bar{\Omega}$. We leave the proof of these facts to the interested reader.

***Lemma 5.2.*** *$I'(u_k) = 0$ in $W^{-1,p'}(\Omega)$.*

*Proof.* Given any $v \in W_0^{1,p}(\Omega)$, we extend $v$ to be zero outside $\Omega$. Let $Q_N$ be the cube

$$Q_N = \Big\{x \in \mathbb{R}^n \mid x = \sum_{i=1}^{n} t_i \eta_i,\ |t_i| < N,\ \forall\, i = 1, 2, \dots, n\Big\}.$$

Assume $N$ is large enough so that $\bar{\Omega} \subset Q_N$. Then

$$\begin{aligned}\int_\Omega f'(Du_k) \cdot Dv &= \int_{Q_N} f'(Du_k) \cdot Dv \\ &= \sum_{j=-kN}^{kN-1} \int_{Q_k^j} f'(Du_k) \cdot Dv,\end{aligned}$$

where $Q_k^j = \{x \in Q_N \mid x \cdot \eta \in (j/k, (j+1)/k)\}$. We write $Q_k^j = A^j \cup B^j \cup \Gamma^j$, where $A^j = Q_k^j \cap A_k^j$, $B^j = Q_k^j \cap B_k^j$ and $\Gamma^j = \{x \in Q_k^j \mid x \cdot \eta = (j+\lambda)/k\}$. We also define $F^j = \{x \in Q_N \mid x \cdot \eta = j/k\}$. Note that $Du_k = \alpha$ on $A^j$ and $Du_k = \beta$ on $B^j$ and hence, by the divergence theorem and (5.3) as well, we have

$$\begin{aligned}\int_{Q_k^j} f'(Du_k) \cdot Dv &= \int_{A^j} f'(Du_k) \cdot Dv + \int_{B^j} f'(Du_k) \cdot Dv \\ &= f'(\alpha) \cdot \int_{A^j} Dv + f'(\beta) \cdot \int_{B^j} Dv \\ &= f'(\alpha) \cdot \Big(\int_{F^j} v\, dS\Big)(-\eta) + f'(\alpha) \cdot \Big(\int_{\Gamma^j} v\, dS\Big)\eta \\ &\qquad + f'(\beta) \cdot \Big(\textstyle\int_{\Gamma^j} v\, dS\Big)(-\eta) + f'(\beta) \cdot \Big(\textstyle\int_{F^{j+1}} v\, dS\Big)\eta \\ &= f'(\alpha) \cdot \eta \Big(\int_{F^{j+1}} v\, dS - \int_{F^j} v\, dS\Big).\end{aligned}$$

Hence, since $F^{\pm kN}$ lies in $\mathbb{R}^n \setminus \bar{\Omega}$, where $v = 0$, it follows that

$$\int_\Omega f'(Du_k(x)) \cdot Dv(x)\, dx = f'(\alpha) \cdot \eta \Big(\int_{F^{kN}} v\, dS - \int_{F^{-kN}} v\, dS\Big) = 0.$$

This proves $I'(u_k) = 0$ in $W^{-1,p'}(\Omega)$. ∎

To continue the proof of the theorem, we now modify the sequence $\{u_k\}$ above into a sequence in $W_A^{1,p}(\Omega)$, where $A = \lambda\alpha + (1-\lambda)\beta$. For all sufficiently large $j$, say $j \geq j_0$, consider nonempty open sets

$$\Omega_j = \Big\{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \frac{1}{j}\Big\}.$$

Note that the measure $|\Omega \setminus \Omega_j| \to 0$ as $j \to \infty$. Let $\varphi_j \in C_0^\infty(\Omega)$ be the cut-off functions such that $\varphi_j = 1$ on $\Omega_j$ and $0 \le \varphi_j \le 1$ in $\Omega$. Since $u_k \to \bar{u}$ uniformly on $\bar{\Omega}$, we have that, for each $j \ge j_0$, there exists $k_j > j$ satisfying

$$\|(u_{k_j} - \bar{u})D\varphi_j\|_{L^p(\Omega)} < \frac{1}{j}. \tag{5.7}$$

Let $\tilde{u}_j = \varphi_j u_{k_j} + (1-\varphi_j)\bar{u}$. Then $\tilde{u}_j \in W^{1,p}_{\bar{u}}(\Omega) = W^{1,p}_A(\Omega)$ and $D\tilde{u}_j = \varphi_j Du_{k_j} + (1-\varphi_j)D\bar{u} + (u_{k_j} - \bar{u})D\varphi_j$. Hence, by (5.7) and also since $Du_{k_j}$, $D\bar{u}$ are bounded; it follows that

$$\lim_{j\to\infty} \|D\tilde{u}_j\|_{L^p(\Omega\setminus\Omega_j)} = 0. \tag{5.8}$$

Therefore $\tilde{u}_j \rightharpoonup \bar{u}$ in $W^{1,p}(\Omega)$ as $j \to \infty$. Since $\tilde{u}_j = u_{k_j}$ on $\Omega_j$, by (5.8) and the growth conditions (5.1)–(5.2), it easily follows that

$$\lim_{j\to\infty} \|I'(\tilde{u}_j) - I'(u_{k_j})\|_{W^{-1,p'}(\Omega)} = 0,$$

$$\lim_{j\to\infty} |I(\tilde{u}_j) - I(u_{k_j})| = 0.$$

Hence $\tilde{u}_j, \bar{u} \in W^{1,p}_A(\Omega)$, and $\tilde{u}_j \overset{\text{ps}}{\rightharpoonup} \bar{u}$ since $I'(u_{k_j}) = 0$. By the (PS)-weak lower semicontinuity of $I$ on $W^{1,p}_A(\Omega)$, we have $I(\bar{u}) \le \liminf_j I(\tilde{u}_j) = \liminf_j I(u_{k_j})$. Using (5.6), we easily see that this implies

$$h(\lambda a + (1-\lambda)b) \le \lambda h(a) + (1-\lambda)h(b).$$

Hence by Lemma 4.8 above, $h(t) = f(\xi + t\eta)$ is convex for all $\xi$, $\eta$ with $|\eta| = 1$. This proves $f$ is convex on $\mathbb{R}^n$. ◻

*Remark* ***5.3***. As we discussed in the introduction, the dimension $m = 1$ is critical in the proof. In the case $m \ge 2$, one would try to use a similar construction with a rank-one matrix $\eta = p \otimes q$ and to define

$$u_k(x) = \alpha x + \frac{b-a}{k}\rho(kx \cdot q)p.$$

But this sequence $\{u_k\}$ does not verify Lemma 5.2 and hence is not a Palais-Smale sequence. Thus the method fails to show in this case $f$ is even rank-one convex. In fact, Theorem 5.8 below shows, for $m \ge 2$, $f$ may not be rank-one convex.

We now study the general case with $m \ge 2$. Under the coercivity condition that $c_0 > 0$ in (5.1), we have the following result, which was announced a few years ago in [15], but has never been published; we include it here for the convenience of the reader.

***Theorem 5.4.*** *Let $n, m \geq 1$ and let $f\colon M^{m\times n} \to \mathbb{R}$ satisfy the conditions* (5.1) *and* (5.2) *above with $c_0 > 0$. Then the following statements are equivalent:*

(i) *$I$ is weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$.*
(ii) *$I$ is (PS)-weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$.*
(iii) *$I$ is (PS)-weakly lower semicontinuous on all $W^{1,p}_A(\Omega;\mathbb{R}^m)$.*
(iv) *$f$ is quasiconvex.*
(v) *$I(g_A) \leq \liminf_{k\to\infty} I(u_k)$ whenever $A \in M^{m\times n}$, $u_k \in W^{1,p}_A$ and $I'(u_k) \to 0$ in $W^{-1,p'}(\Omega;\mathbb{R}^m)$.*

*Proof.* By the theorem of Acerbi-Fusco (Theorem 2.3), (i) $\iff$ (iv) even when $c_0 = 0$. Moreover, by the definition of quasiconvexity and using approximation, if $f$ is quasiconvex and only satisfies (5.1) with $c_0 \in \mathbb{R}$, then it readily follows that $I(g_A) \leq I(u)$ for all $u \in W^{1,p}_A(\Omega;\mathbb{R}^m)$; hence (iv) $\Rightarrow$ (v). It is also obvious that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) in general cases.

Therefore, to prove the theorem, it suffices to show that (iii) $\Rightarrow$ (iv) and that (v) $\Rightarrow$ (iv) (in fact if $c_0 \geq 0$). We prove them as separate results in two lemmas below. ◻

***Lemma 5.5.*** *Under the assumptions* (5.1) *with $c_0 \geq 0$ and* (5.2)*,* (v) $\Rightarrow$ (iv).

*Proof.* The proof uses the Ekeland variational principle as given by Theorem 2.4. Given $A \in M^{m\times n}$, define a functional $\Phi$ on $X = W^{1,p}_0(\Omega;\mathbb{R}^m)$ by

$$\Phi(v) = \int_\Omega f(A + Dv(x))\,dx = I(g_A + v).$$

Then $\Phi\colon X \to \mathbb{R}$ is $C^1$ and bounded below, and $\Phi'(v) = I'(g_A + v)$. By Theorem 2.4, there exists a sequence $\{v_k\}$ in $X$ such that

$$\Phi(v_k) \to \inf_{v\in X} \Phi(v), \quad \|\Phi'(v_k)\|_{X^*} \to 0.$$

Let $u_k = g_A + v_k \in W^{1,p}_A(\Omega;\mathbb{R}^m)$. Then

$$I(u_k) = \Phi(v_k) \to \inf_X \Phi,$$
$$\|I'(u_k)\|_{X^*} = \|\Phi'(v_k)\|_{X^*} \to 0.$$

Therefore, by (v), it follows that

$$f(A)|\Omega| = I(g_A) \leq \liminf_{k\to\infty} I(u_k) = \inf_{v\in X} \Phi(v) = \inf_{v\in X} I(g_A + v).$$

This implies

$$f(A)|\Omega| \leq \int_\Omega f(A + Dv(x))\,dx, \quad \forall\, v \in W^{1,p}_0(\Omega;\mathbb{R}^m),$$

and thus $f$ is quasiconvex. ◻

***Lemma 5.6.*** *Under the assumptions* (5.1) *with* $c_0 > 0$ *and* (5.2), (iii) $\Rightarrow$ (iv).

*Proof.* Given $A \in M^{m\times n}$, by Theorem 3.8 (note that $c_0 > 0$ is needed here), there exists $\bar{u} \in W_A^{1,p}(\Omega;\mathbb{R}^m)$ which is a minimizer of $I(u)$ on $W_A^{1,p}(\Omega;\mathbb{R}^m)$. We now apply the standard technique of Vitali covering [5] to construct a sequence $\{u_k\}$ in $W_A^{1,p}(\Omega;\mathbb{R}^m)$ satisfying

$$I(u_k) = I(\bar{u}) = \inf_{u\in W_A^{1,p}} I(u), \quad u_k \rightharpoonup g_A \text{ in } W^{1,p}(\Omega;\mathbb{R}^m) \text{ as } k\to\infty; \tag{5.9}$$

$$I'(u_k) = 0 \quad \text{in } W^{-1,p'}(\Omega;\mathbb{R}^m). \tag{5.10}$$

Note that (5.10) will follow from (5.9) since $u_k \in W_A^{1,p}(\Omega;\mathbb{R}^m)$ is also a minimizer of $I(u)$ on $W_A^{1,p}(\Omega;\mathbb{R}^m)$. Once we have constructed such a sequence $\{u_k\}$, which certainly satisfies $u_k \overset{\text{ps}}{\rightharpoonup} g_A$, the (PS)-weak lower semicontinuity condition (iii) will imply

$$I(g_A) \le \liminf_{k\to\infty} I(u_k) = I(\bar{u}) = \inf_{u\in W_A^{1,p}} I(u),$$

for all $A \in M^{m\times n}$, which is exactly the quasiconvexity condition of $f$, and hence the result follows. Assume, without loss of generality, $0 \in \Omega$, and then we use the Vitali covering theorem to decompose $\Omega$ as follows:

$$\Omega = \bigcup_{j=1}^{\infty} \bar{\Omega}_j \cup N; \quad \bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset \quad (i \ne j),$$

where $\Omega_j = a_j + \varepsilon_j\Omega \Subset \Omega$ with $a_j \in \Omega$, $0 < \varepsilon_j < 1/k$, and $|N| = 0$. Let $\bar{u} = g_A + \bar{v}$, where $\bar{v} \in W_0^{1,p}(\Omega;\mathbb{R}^m)$. We define

$$u_k(x) = \begin{cases} Ax + \varepsilon_j \bar{v}\left(\dfrac{x-a_j}{\varepsilon_j}\right) & \text{if } x \in \Omega_j, \\ Ax & \text{otherwise.} \end{cases} \tag{5.11}$$

Then one can easily check that $u_k$ belongs to $W_A^{1,p}(\Omega;\mathbb{R}^m)$ and satisfies

$$\int_\Omega \psi(Du_k(x))\,dx = \int_\Omega \psi(D\bar{u}(x))\,dx,$$

for all continuous functions $\psi\colon M^{m\times n} \to \mathbb{R}$ satisfying $|\psi(\xi)| \le C(|\xi|^p + 1)$. Certainly this implies (5.9). Furthermore, it is easy to see

$$\|u_k - g_A\|_{L^p(\Omega)} \le \frac{1}{k}\|\bar{u} - g_A\|_{L^p(\Omega)}.$$

Hence $u_k \rightharpoonup g_A$ as $k \to \infty$. As mentioned above, Condition (5.10) follows from (5.9). This completes the construction of $\{u_k\}$ and thus the proof of the lemma. ◻

*Remark 5.7.* For any $u \in W_A^{1,p}(\Omega;\mathbb{R}^m)$, we write $u = g_A + v$ with $v \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ and define a sequence $u_k \in W_A^{1,p}(\Omega;\mathbb{R}^m)$ as in (5.11) above, with $\bar{v} = v$. Then, if $u$ is not a minimizer of $I$ over $W_A^{1,p}(\Omega;\mathbb{R}^m)$, one only has $I'(u_k) \rightharpoonup 0$, but not strongly, in $W^{-1,p'}(\Omega;\mathbb{R}^m)$, as $k \to \infty$, even when $I'(u) = 0$; hence the (PS)-weak lower semicontinuity cannot be applied to this sequence.

Finally, we show that, without the coercivity assumption $c_0 > 0$ in (5.1), the results of Theorem 5.4 may fail in general; we give an example in the case $n = 1$, $m = 2$.

**Theorem 5.8.** *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a $C^1$ function satisfying the conditions (5.1) and (5.2) with $c_0 = 0$. Suppose the derivative map $Df = f'\colon \mathbb{R}^2 \to \mathbb{R}^2$ is one-to-one. Then the functional $I(u) = \int_0^1 f(u'(x))\,dx$ is (PS)-weakly lower semicontinuous on the Sobolev space $X = W^{1,p}((0,1);\mathbb{R}^2)$.*

*Proof.* Let $u \in X = W^{1,p}((0,1);\mathbb{R}^2)$. Then

$$\langle I'(u), v\rangle = \int_0^1 [f_{\xi_1}(u'(x))(v^1)' + f_{\xi_2}(u'(x))(v^2)']\,dx, \quad \forall\, v = (v^1, v^2) \in X,$$

and hence it can be shown that

$$\|I'(u)\|_{W^{-1,p'}} \cong \|f_{\xi_1}(u') - C_1(u)\|_{L^{p'}(0,1)} + \|f_{\xi_2}(u') - C_2(u)\|_{L^{p'}(0,1)},$$

where $C_1(u)$, $C_2(u)$ are two constants depending *boundedly* on $u \in X$. Assume $u_k \overset{\text{ps}}{\rightharpoonup} u$ in $X$. We also assume that $\lim_{k\to\infty} I(u_k)$ exists. Then there exists a subsequence $\{u_{k_j}\}$ such that

$$\|f_{\xi_1}(u'_{k_j}) - C_1\|_{L^{p'}(0,1)} + \|f_{\xi_2}(u'_{k_j}) - C_2\|_{L^{p'}(0,1)} \to 0$$

as $j \to \infty$, where $C_1$, $C_2$ are some constants; from this, we also assume there exists a measurable set $E \subset (0,1)$ such that

$$|E| = 1, \quad \lim_{j\to\infty} f_{\xi_\nu}(u'_{k_j}(x)) = C_\nu \quad \forall\, x \in E (\nu = 1, 2). \tag{5.12}$$

Note that for all $M > 0$ the measure $|\{x \in E \mid |u'_k(x)| > M\}| \le C/M^p$ for all $k$, where $C$ is a constant; hence there exists a sufficiently large $M > 0$ such that the measure $|E_j| > \frac{1}{2}$, where $E_j = \{x \in E \mid |u'_{k_j}(x)| \le M\}$. It is then an easy exercise that there exists a subsequence $\{E_{j_s}\}$, with $j_s \to \infty$ as $s \to \infty$, of sets $\{E_j\}$ such that the set $E_\infty = \bigcap_{s=1}^{\infty} E_{j_s}$ is non-empty; we leave the proof to the interested reader. Therefore there exists at least one $x_0 \in E_\infty$, for which $|u'_{k_{j_s}}(x_0)| \le M$ for all $s = 1, 2, \ldots$. By taking a further subsequence, we have $u'_{k_{j_s}}(x_0) \to \alpha \in \mathbb{R}^2$ along a subsequence of $s \to \infty$. Therefore, by (5.12), it follows that $f'(\alpha) = (C_1, C_2)$. Since the map $f'$ is one-to-one, from (5.12) it follows that $u'_{k_j}(x) \to \alpha$ as $j \to \infty$ for all $x \in E$. This implies that $u'(x) = \alpha$ and $f(u'_{k_j}(x)) \to f(\alpha)$ as $j \to \infty$ for all $x \in E$. Hence, by Fatou's lemma,

$$I(u) = f(\alpha) \le \liminf_{j\to\infty} I(u_{k_j}) = \lim_{k\to\infty} I(u_k),$$

which proves the (PS)-weak lower semicontinuity of $I$ on $X$. $\square$

*Remark* ***5.9***. Note that if $f(\xi_1, \xi_2) = \varphi(\xi_1 - \xi_2^2)$ on $\mathbb{R}^2$, where $\varphi \ge 0$ is any $C^1$ function with a strictly increasing derivative $\varphi' > 0$ on $\mathbb{R}$, then $f' \colon \mathbb{R}^2 \to \mathbb{R}^2$ is one-to-one, but $f$ is not convex on $\mathbb{R}^2$. An example of such a $\varphi$ is given by

$$\varphi(t) = \begin{cases} e^t, & t \le 0; \\ t^2 + t + 1, & t > 0. \end{cases}$$

Note that the corresponding function $f(\xi) = f(\xi_1, \xi_2) = \varphi(\xi_1 - \xi_2^2)$ then also satisfies the conditions (5.1) and (5.2) with $c_0 = 0$ and $p = 4$. For such a function $h = f$, the condition (iii) in Lemma 4.8 above holds automatically, but $h$ is not convex; this shows that conditions (i) and (iii) in Lemma 4.8 are not equivalent in general for functions $h \colon \mathbb{R}^2 \to \mathbb{R}$.

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