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Existence and Regularity Theory for Nonlinear Elliptic Systems and Multiple Integrals in the Calculus of Variations

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by

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Multiple Integrals and Systems in Divergence Form

1.1. Notation

Throughout the lecture, we use $\mathbf{R}^n$ to denote the standard Euclidean space of n-real variables and $\mathbf{M}^{N\times n}$ to denote the Euclidean space $\mathbf{M}^{N\times n}$ of all real $N \times n$-matrices.

The norm in $\mathbf{R}^n$ or $\mathbf{M}^{N\times n}$ is all denoted by $|\cdot|$. For example, if $\xi \in \mathbf{M}^{N\times n}$ then

$$|\xi|^2 = \sum_{i=1}^{N}\sum_{\alpha=1}^{n} (\xi^i_\alpha)^2.$$

For $q \in \mathbf{R}^N$, $p \in \mathbf{R}^n$, we use $\eta = q \otimes p$ to denote the matrix with $\eta^i_\alpha = q^i\, p_\alpha$. Therefore, it is easy to see $|q \otimes p| = |p|\,|q|$.

We also denote by Ω a bounded smooth domain in $\mathbf{R}^n$ and let $\bar{\Omega}$, $\partial\Omega$ denote the closure and the boundary of Ω in $\mathbf{R}^n$, respectively. For a *Lebesgue measurable* set E in $\mathbf{R}^n$ we use $|E|$ to denote its Lebesgue n-measure. If $f\colon E \to \mathbf{R}$ is *Lebesgue integrable* and $|E| > 0$ then we define

$$⨍_E f(x)\,dx = \frac{1}{|E|}\int_E f(x)\,dx.$$

For a map u from Ω to a target space $\mathbf{R}^N$, we use $Du(x)$ to denote the Jacobian or gradient matrix of u defined by

$$(Du(x))^i_\alpha = D_\alpha u^i(x) = \partial u^i(x)/\partial x_\alpha \quad (i = 1, 2, ..., N;\ \alpha = 1, 2, ..., n).$$

This can also be viewed as a map Du from Ω to the matrix space $\mathbf{M}^{N\times n}$.

For $0 \leq k \leq \infty$, we use $C^k(\Omega;\mathbf{R}^N)$ to denote the space of all smooth maps with continuous partial derivatives up to the order k and $C^k_0(\Omega;\mathbf{R}^N)$ its subspace consisting of all such maps with compact support in Ω.

For $1 \le p < \infty$, let $W^{1,p}(\Omega; \mathbf{R}^N)$ and $W_0^{1,p}(\Omega; \mathbf{R}^N)$ be the Sobolev spaces, which are completion of $C^\infty(\Omega; \mathbf{R}^N)$ and $C_0^\infty(\Omega; \mathbf{R}^N)$, respectively, under the norm

$$\|u\|_{1,p} = \|u\|_{1,p;\Omega} = \left(\int_\Omega \big(|u(x)|^p + |Du(x)|^p\big)\, dx \right)^{1/p}.$$

If $A : \Omega \to \mathbf{M}^{N\times n}$ is a smooth map, then its divergence $\operatorname{Div} A : \Omega \to \mathbf{R}^N$ is defined by

$$(\operatorname{Div} A(x))^i = \sum_{\alpha=1}^n \partial A^i_\alpha(x)/\partial x_\alpha, \;\; i = 1, 2, ..., N.$$

If A is only locally integrable, then $\operatorname{Div} A$ is defined to be a distribution on $C_0^\infty(\Omega; \mathbf{R}^N)$ (test functions) by the pairing

$$\langle \operatorname{Div} A, \varphi \rangle = -(A, D\varphi), \quad \varphi \in C_0^\infty(\Omega; \mathbf{R}^N),$$

where

$$(A, D\varphi) = \int_\Omega A(x) \cdot D\varphi(x)\, dx \equiv \int_\Omega \sum_{i=1}^N \sum_{\alpha=1}^n A^i_\alpha(x)\, D_\alpha \varphi^i(x)\, dx.$$

In most cases, we shall also use the convention that repeated indices are to be added.

1.2. Multiple integrals in the calculus of variations

Consider the multiple integral functional

$$I(u) = \int_\Omega F(x, u(x), Du(x))\, dx, \tag{1.1}$$

where $F(x, s, \xi)$ is a given function on $\Omega \times \mathbf{R}^N \times \mathbf{M}^{N\times n}$. In the calculus of variations, such functionals $I(u)$ are usually called an **energy** functional, as is in the theory of elasticity.

Suppose $F(x, s, \xi)$ is continuous and is also smooth in s and ξ. Assume u is a nice (say, $u \in C^1(\bar\Omega; \mathbf{R}^N)$) minimizer of $I(u)$ with its own boundary data; that is, u is a map such that

$$I(u) \le I(u + t\varphi)$$

for all $t \in \mathbf{R}^1$ and $\varphi \in C_0^\infty(\Omega; \mathbf{R}^N)$. Then by taking derivative of $I(u + t\,\varphi)$ at $t = 0$ we see that u satisfies

$$\int_\Omega \big(F_{\xi^i_\alpha}(x, u, Du)\, D_\alpha \varphi^i(x) + F_{s^i}(x, u, Du)\, \varphi^i(x) \big)\, dx = 0$$

for all $\varphi \in C_0^\infty(\Omega; \mathbf{R}^N)$. (Summation notation is used here.) This equation is called the **Euler-Lagrange equation** (in weak form) or the **first variation** of I at u.

The *Euler-Lagrange* equation can be written as a differential system for map u in the distribution sense:

$$-\operatorname{Div} A(x, u, Du) + b(x, u, Du) = 0, \tag{1.2}$$

where A, b are defined by

$$A^i_\alpha(x, s, \xi) = F_{\xi^i_\alpha}(x, s, \xi), \quad b^i(x, s, \xi) = F_{s^i}(x, s, \xi). \tag{1.3}$$

Remarks. 1) If F, u are sufficiently smooth (e.g. of class C^2) then we have the *strong form* of the Euler-Lagrange equation of I at u :

$$-F_{\xi^i_\alpha\,\xi^j_\beta}(x,u,Du)\,D_\alpha D_\beta u^j - F_{\xi^i_\alpha\,s^j}(x,u,Du)\,D_\alpha u^j$$

$$-F_{\xi^i_\alpha\,x_\alpha}(x,u,Du) + F_{s^i}(x,u,Du) = 0, \quad i = 1,2,...,N,$$

which is a second-order **quasilinear system** of N coupled partial differential equations.

2) If F, u are sufficiently smooth (e.g. of class C^2) then we have

$$\frac{d^2}{dt^2} I(u+t\varphi)\Big|_{t=0} \geq 0.$$

This implies

$$\int_\Omega F_{\xi^i_\alpha\,\xi^j_\beta}(x,u,Du)\,D_\alpha\varphi^i D_\beta\varphi^j\,dx$$

$$+\int_\Omega \left[2F_{\xi^i_\alpha\,s^j}(x,u,Du)\,\varphi^j D_\alpha\varphi^i + F_{s^i s^j}(x,u,Du)\,\varphi^i\varphi^j\right]\,dx \geq 0$$

for all $\varphi \in C_0^\infty(\Omega;\mathbf{R}^N)$. This inequality is called the **second variation** of I at minimum point u. We shall discuss some consequences of this inequality later on. □

1.3. Systems in divergence form

We consider general systems of PDE in divergence form as (1.2). Let $A : \Omega\times\mathbf{R}^N\times\mathbf{M}^{N\times n} \to \mathbf{M}^{N\times n}$ and $b : \Omega\times\mathbf{R}^N\times\mathbf{M}^{N\times n} \to \mathbf{R}^N$ be given. Consider the system of differential equations for a map $u : \Omega \to \mathbf{R}^N$

$$-\operatorname{Div} A(x,u,Du) + b(x,u,Du) = 0 \tag{1.4}$$

in the sense of distribution; this means that

$$\int_\Omega \big(A(x,u,Du)\cdot D\varphi(x)\,dx + b(x,u,Du)\cdot\varphi(x)\big)\,dx = 0$$

for all test functions $\varphi \in C_0^\infty(\Omega;\mathbf{R}^N)$.

The leading term $A(x,s,\xi)$ in system (1.4) can be classified to be

- **linear** if $A(x,s,\xi)$ is linear in both s and ξ; that is,
$$A^i_\alpha(x,s,\xi) = A^{\alpha\beta}_{ij}(x)\,\xi^j_\beta + b^\alpha_{ij}(x)\,s^j + r^i_\alpha(x);$$
- **quasilinear** if $A(x,s,\xi)$ is only linear in ξ; that is,
$$A^i_\alpha(x,s,\xi) = A^{\alpha\beta}_{ij}(x,s)\,\xi^j_\beta + P^i_\alpha(x,s);$$
- **nonlinear** for all other cases of $A(x,s,\xi)$.

The system (1.4) is said to be **linear** if $A(x,s,\xi)$, $b(x,s,\xi)$ are both linear in s and ξ.

Remark. Again, if A, u are sufficiently smooth, the strong form of (1.4) is a second-order *quasilinear system* of N coupled partial differential equations of the form

$$-A^{\alpha\beta}_{ij}(x,u,Du)\,D_\alpha D_\beta u^j + R^i(x,u,Du) = 0, \quad i = 1,2,...,N, \tag{1.5}$$

where the leading coefficients $A^{\alpha\beta}_{ij}(x,s,\xi)$ are given by

$$A^{\alpha\beta}_{ij}(x,s,\xi) = \partial A^i_\alpha(x,s,\xi)/\partial\xi^j_\beta. \tag{1.6}$$

□

1.4. Legendre ellipticity condition for systems

Definition 1.1. The system (1.4) or (1.5) is called (uniformly, strictly) **elliptic** if there exists a $\nu > 0$ such that for all (x, s, ξ)

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} A_{ij}^{\alpha\beta}(x,s,\xi)\, \eta_\alpha^i\, \eta_\beta^j \geq \nu\, |\eta|^2 \quad \forall\, \eta \in \mathbf{M}^{N\times n}, \tag{1.7}$$

where coefficients $A_{ij}^{\alpha\beta}(x,s,\xi) = \partial A_\alpha^i(x,s,\xi)/\partial \xi_\beta^j$ are defined as in (1.6). This condition is also called the (uniform, strict) **Legendre (ellipticity) condition** for the given system.

A weaker condition, obtained by setting $\eta = q \otimes p$ with $p \in \mathbf{R}^n$, $q \in \mathbf{R}^N$, is the following so-called (strong) **Legendre-Hadamard condition**:

$$\sum_{i,j} \sum_{\alpha,\beta} A_{ij}^{\alpha\beta}(x,s,\xi)\, q^i\, q^j\, p_\alpha\, p_\beta \geq \nu\, |p|^2\, |q|^2 \quad \forall\, p \in \mathbf{R}^n,\ q \in \mathbf{R}^N. \tag{1.8}$$

Note that for systems with *linear leading terms*; that is,

$$A_\alpha^i(x,s,\xi) = A_{ij}^{\alpha\beta}(x)\, \xi_\alpha^i\, \xi_\beta^j + b_{ij}^\alpha(x)\, s^j + r_\alpha^i(x),$$

the Legendre condition and Legendre-Hadamard condition become, respectively

$$A_{ij}^{\alpha\beta}(x)\, \eta_\alpha^i\, \eta_\beta^j \geq \nu\, |\eta|^2 \quad \forall\, \eta \in \mathbf{M}^{N\times n};$$

$$A_{ij}^{\alpha\beta}(x)\, q^i\, q^j\, p_\alpha\, p_\beta \geq \nu\, |p|^2\, |q|^2 \quad \forall\, p \in \mathbf{R}^n,\ q \in \mathbf{R}^N.$$

Remark. The Legendre-Hadamard condition does not imply the Legendre ellipticity condition. □

Example 1.2. Let $n = N = 2$ and define *constants* $A_{ij}^{\alpha\beta}$ by

$$A_{ij}^{\alpha\beta}\, \xi_\alpha^i\, \xi_\beta^j \equiv \det \xi + \epsilon\, |\xi|^2.$$

Since

$$A_{ij}^{\alpha\beta} p_\alpha p_\beta\, q^i q^j = \det(q \otimes p) + \epsilon\, |q \otimes p|^2 = \epsilon\, |p|^2 |q|^2,$$

the Legendre-Hadamard condition holds for all $\epsilon > 0$. But, if $0 < \epsilon < 1/2$, then there exists a matrix ξ of the form $\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$ such that $\det \xi + \epsilon|\xi|^2 < 0$; thus the Legendre ellipticity condition fails. In fact, one can check that the Legendre condition holds for this system if and only if $\epsilon > 1/2$.

Remark. Let $u = (v, w)$ and $(x_1, x_2) = (x, y)$. Then the system of differential equations defined by $A_{ij}^{\alpha\beta}$ given above is

$$\begin{cases} \epsilon \Delta v + w_{xy} = 0, \\ \epsilon \Delta w - v_{xy} = 0. \end{cases}$$

This system reduces to two fourth-order equations for v, w (where $\Delta f = f_{xx} + f_{yy}$):

$$\epsilon^2 \Delta^2 v - v_{xxyy} = 0, \quad \epsilon^2 \Delta^2 w + w_{xxyy} = 0.$$

We can easily see that both equations are **elliptic** if and only if $\epsilon > 1/2$. □

1.5. Convexity and rank-one convexity

We now consider the ellipticity of the *Euler-Lagrange* equation (1.2), where $A(x,s,\xi)$, $b(x,s,\xi)$ are given by (1.3) and $F(x,s,\xi)$ is C^2 in ξ. In this case, the Legendre ellipticity condition and Legendre-Hadamard condition reduce to, respectively:

$$F_{\xi^i_\alpha \xi^j_\beta}(x,s,\xi)\,\eta^i_\alpha\,\eta^j_\beta \geq \nu\,|\eta|^2 \quad \forall \eta \in \mathbf{M}^{N\times n}; \tag{1.9}$$

$$F_{\xi^i_\alpha \xi^j_\beta}(x,s,\xi)\,q^i\,q^j\,p_\alpha p_\beta \geq \nu |p|^2|q|^2 \quad \forall q \in \mathbf{R}^N,\ p \in \mathbf{R}^n. \tag{1.10}$$

Proposition 1.1. *Under the conditions* (1.9) *and* (1.10), *the following conditions hold, respectively:*

$$F(x,s,\eta) \geq F(x,s,\xi) + F_{\xi^i_\alpha}(x,s,\xi)\,(\eta^i_\alpha - \xi^i_\alpha) + \frac{\nu}{2}\,|\eta-\xi|^2 \tag{1.11}$$

and

$$F(x,s,\xi + q\otimes p) \geq F(x,s,\xi) + F_{\xi^i_\alpha}(x,s,\xi)\,p_\alpha\,q^i + \frac{\nu}{2}\,|p|^2|q|^2 \tag{1.12}$$

for all $x \in \Omega$, s, $q \in \mathbf{R}^N$, ξ, $\eta \in \mathbf{M}^{N\times n}$ *and* $p \in \mathbf{R}^n$.

Proof. Let $\zeta = \eta - \xi$ and $f(t) = F(x,s,\xi + t\zeta)$. Then, by Taylor's formula,

$$f(1) = f(0) + f'(0) + \int_0^1 (1-t)\,f''(t)\,dt.$$

Note that

$$f'(t) = F_{\xi^i_\alpha}(x,s,\xi+t\zeta)\,\zeta^i_\alpha, \quad f''(t) = F_{\xi^i_\alpha\xi^j_\beta}(x,s,\xi+t\zeta)\,\zeta^i_\alpha\zeta^j_\beta.$$

From this and the Taylor formula, inequalities (1.11) and (1.12) follow easily from (1.9) and (1.10), respectively. □

Definition 1.3. A function $F(x,s,\xi)$ is said to be **convex** in ξ if

$$F(x,s,t\xi + (1-t)\eta) \leq tF(x,s,\xi) + (1-t)F(x,s,\eta)$$

for all x, s, ξ, η and $0 \leq t \leq 1$. While $F(x,s,\xi)$ is said to be **rank-one convex** in ξ if the function $f(t) = F(x,s,\xi + t\,q\otimes p)$ is convex in $t \in \mathbf{R}^1$ for all x, s, ξ and $q \in \mathbf{R}^N$, $p \in \mathbf{R}^n$.

We easily have the following result.

Proposition 1.2. *Let* $F(x,s,\xi)$ *be* C^2 *in* ξ. *Then the* convexity *of* $F(x,s,\xi)$ *in* ξ *is equivalent to* (1.9) *with* $\nu = 0$, *while the* rank-one convexity *of* $F(x,s,\xi)$ *in* ξ *is equivalent to* (1.10) *with* $\nu = 0$.

Remarks. 1) Conditions (1.9) and (1.10) are also called the **strong convexity** and the **strong rank-one convexity** conditions of $F(x,s,\xi)$ on ξ, respectively.

2) Rank-one convexity does not imply convexity. For example, take $n = N \geq 2$, and $F(\xi) = \det \xi$. Then $F(\xi)$ is rank-one convex but not convex in ξ (why?) Later on, we will study other convexity conditions related to the energy functionals given by (1.1). □

1.6. Uniqueness of weak solutions

In this section, we prove a uniqueness of weak solutions of Euler-Lagrange equations under the hypotheses of *strong convexity* and certain *growth* conditions. For this purpose, we consider a simple case where $F(x,s,\xi) = F(x,\xi)$ satisfies, for some $1 < p < \infty$,

$$|F_{\xi^i_\alpha}(x,\xi)| \le \mu\,(\chi(x) + |\xi|^{p-1}) \quad \forall\, x \in \Omega,\ \xi \in \mathbf{M}^{N\times n}, \tag{1.13}$$

where $\mu > 0$ is a constant and $\chi \in L^{\frac{p}{p-1}}(\Omega)$ is some function. Let

$$I(u) = \int_\Omega F(x, Du(x))\,dx.$$

Theorem 1.3. *Let $F(x,\xi)$ be C^2 in ξ and satisfy the* Legendre condition (1.9). *Let $u \in W^{1,p}(\Omega;\mathbf{R}^N)$ be a weak solution of the* Euler-Lagrange *equation of I and $I(u) < \infty$. Then u must be the unique minimizer of I among the class of functions in $W^{1,p}(\Omega;\mathbf{R}^N)$ having the same boundary conditions as u.*

Proof. Since u is a weak solution of the *Euler-Lagrange* equation of I, it follows that

$$\int_\Omega F_{\xi^i_\alpha}(x, Du(x))\,D_\alpha \varphi^i\,dx = 0 \tag{1.14}$$

for all $\varphi \in C_0^\infty(\Omega;\mathbf{R}^N)$. The growth condition (1.13) implies $F_{\xi^i_\alpha}(x,Du) \in L^{\frac{p}{p-1}}(\Omega)$ and hence, by a density argument, equation (1.14) holds also for all $\varphi \in W_0^{1,p}(\Omega;\mathbf{R}^N)$. Now let $v \in W^{1,p}(\Omega;\mathbf{R}^N)$ with $v - u \in W_0^{1,p}(\Omega;\mathbf{R}^N)$. By the strong convexity condition and (1.11), we have

$$F(x,\eta) \ge F(x,\xi) + F_{\xi^i_\alpha}(x,\xi)\,(\eta^i_\alpha - \xi^i_\alpha) + \frac{\nu}{2}\,|\eta - \xi|^2, \quad \forall\, \xi,\, \eta. \tag{1.15}$$

This implies

$$\begin{aligned} \int_\Omega F(x,Dv)\,dx \;\; &\ge \;\; \int_\Omega F(x,Du)\,dx + \int_\Omega F_{\xi^i_\alpha}(x,Du)\,D_\alpha(v^i - u^i)\,dx \\ &+ \;\; \frac{\nu}{2}\int_\Omega |Du - Dv|^2\,dx. \end{aligned}$$

Since u is a weak solution, we have

$$\int_\Omega F_{\xi^i_\alpha}(x,Du)\,D_\alpha(v^i - u^i)\,dx = 0.$$

Therefore, it follows that

$$I(v) \ge I(u) + \frac{\nu}{2}\int_\Omega |Du - Dv|^2\,dx \tag{1.16}$$

for all $v \in W^{1,p}(\Omega;\mathbf{R}^N)$ with $v - u \in W_0^{1,p}(\Omega;\mathbf{R}^N)$. This shows that u is a minimizer of I among the class of all functions $v \in W^{1,p}(\Omega;\mathbf{R}^N)$ with $v - u \in W_0^{1,p}(\Omega;\mathbf{R}^N)$. If v is another such minimizer of I we would then obtain $Du = Dv$ by (1.16) and thus $v \equiv u$. The theorem is proved. □

Existence Theory for Linear Systems

2.1. Dirichlet problem for linear systems

In this chapter, we study the solvability of Dirichelt problems of the linear elliptic systems in some Hilbert space. We study the Dirichlet problem

$$\begin{cases} -\operatorname{Div} A(x,u,Du) + b(x,u,Du) = G, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $A(x,s,\xi)$ and $b(x,s,\xi)$ are both linear in s, ξ with L^∞-coefficients; that is,

$$A^i_\alpha(x,u,Du) = A^{\alpha\beta}_{ij}(x)\, D_\beta\, u^j + b^\alpha_{ij}(x)\, u^j,$$

$$b^i(x,u,Du) = c^\alpha_{ij}(x)\, D_\alpha u^j + d_{ij}(x)\, u^j.$$

Here G is a bounded linear functional on the Hilbert space $H^1_0(\Omega;\mathbf{R}^N) = W^{1,2}_0(\Omega;\mathbf{R}^N)$ with the inner product

$$(u,v) \equiv \sum_{\alpha=1}^n \sum_{i=1}^N \int_\Omega D_\alpha u^i\, D_\alpha v^i\, dx.$$

The norm induced by this inner product is denoted by $\|\cdot\|_H$, which is equivalent to the Sobolev norm $\|\cdot\|_{1,2}$ defined earlier.

Definition 2.1. By a **weak solution** of system (2.1) we mean a function $u \in H^1_0(\Omega;\mathbf{R}^N)$ such that $B[u,v] = \langle G, v\rangle$ for all $v \in H^1_0(\Omega;\mathbf{R}^N)$, where $B[u,v]$ is the **bilinear form** on $H^1_0(\Omega;\mathbf{R}^N)$ defined by

$$B[u,v] = \int_\Omega \Big(A^{\alpha\beta}_{ij}\, D_\beta u^j D_\alpha v^i + b^\alpha_{ij}\, u^j D_\alpha v^i + c^\alpha_{ij}\, D_\alpha u^j v^i + d_{ij}\, u^j v^i \Big)\, dx. \tag{2.2}$$

Remark. By a *Poincaré-type* inequality, $\|u\|_2 \le C\, \|u\|_H$ for $u \in H^1_0(\Omega;\mathbf{R}^N)$; therefore, we have

$$|B[u,v]| \le c\, \|u\|_H\, \|v\|_H,$$

where c depends on the L^∞-norms of $A^{\alpha\beta}_{ij}$, b^α_{ij}, c^α_{ij} and d_{ij}. □

2.2. Hilbert space methods for existence

In what follows, let H be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We first recall

Theorem 2.1 (Riesz representation theorem). *For any bounded linear functional G on H there exists a unique $g_0 \in H$ such that*

$$\langle G, v\rangle = (g_0, v), \quad \forall v \in H.$$

Furthermore, $\|g_0\| = \|G\|$, the operator norm of G.

Definition 2.2. A function $\rho : H \times H \to \mathbf{R}$ is said to be

- **bilinear** if $\rho[\cdot, v]$, $\rho[u,\cdot]$ are both linear on H for any given u, $v \in H$;
- **bounded** if $|\rho[u,v]| \le c\,\|u\|\,\|v\|$ for all u, $v \in H$;
- **coercive** if $\rho[u,u] \ge \mu\|u\|^2$ for all $u \in H$ and some $\mu > 0$;
- **symmetric** if $\rho[u,v] = \rho[v,u]$ for all u, $v \in H$.

Lemma 2.2. *Let $\rho : H \times H \to \mathbf{R}$ be a bounded, coercive, symmetric bilinear form on H. Then, for any bounded linear functional G on H, there exists a unique $g_1 \in H$ such that $\langle G, v\rangle = \rho[g_1, v]$ for all $v \in H$.*

Proof. This follows from the Riesz representation theorem for one can use $(u,v)_H = \rho[u,v]$ as a new inner product on H. □

Theorem 2.3 (Lax-Milgram theorem). *Let $a : H \times H \to \mathbf{R}$ be a bounded, coercive bilinear form on H. Then, for any bounded linear functional G on H, there exists a unique $g_2 \in H$ such that $\langle G, v\rangle = \rho[g_1, v]$ for all $v \in H$.*

Proof. For each fixed $u \in H$, $\langle F, v\rangle = a[u,v]$ defines a bounded linear functional F on H; thus by Riesz's representation theorem, there exists a unique $f_0 = Tu \in H$ such that $a[u,v] = (Tu, v)$ for all $v \in H$; this defines a bounded linear operator $T : H \to H$. Let T^* be its adjoint operator; that is, $(Tu, v) = (u, T^*v)$. Define a form ρ on $H \times H$ by

$$\rho[u,v] = (TT^*u, v) = (T^*u, T^*v).$$

Then $\rho : H \times H \to \mathbf{R}$ is a *bounded, coercive, symmetric bilinear* form on H (check it). Therefore, if G is a bounded linear functional on H, by Lemma 2.2, there exists a unique $g_1 \in H$ such that

$$\langle G, v\rangle = \rho[g_1, v] = (TT^*g_1, v) = a[T^*g_1, v];$$

this proves the theorem with $g_2 = T^*g_1$. Moreover, if $|a[u,v]| \le c\,\|u\|\,\|v\|$, then $\|g_2\| \le c\,\|G\|$. □

2.3. Legendre-Hadamard condition and coercivity

In this section, we consider the following simple bilinear form

$$a[u,v] = \int_\Omega A_{ij}^{\alpha\beta}(x) D_\beta u^j D_\alpha v^i \, dx,$$

where $A_{ij}^{\alpha\beta} \in L^\infty(\Omega)$. We will use H to denote $H_0^1(\Omega; \mathbf{R}^N)$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|_H$ given above.

Theorem 2.4. *Assume that* **either** $A_{ij}^{\alpha\beta}$ *are in* $L^\infty(\Omega)$ *satisfying the* Legendre condition **or** $A_{ij}^{\alpha\beta}$ *are constants satisfying the* Legendre-Hadamard condition*. Then* $a[u,u] \geq \nu \, \|u\|_H^2$ *for all* $u \in H$.

Proof. In the first case, the conclusion follows easily from the Legendre condition. We prove the second case when $A_{ij}^{\alpha\beta}$ are constants satisfying the Legendre-Hadamard condition

$$A_{ij}^{\alpha\beta} \, q^i \, q^j \, p_\alpha p_\beta \geq \nu \, |p|^2 |q|^2, \quad \forall p \in \mathbf{R}^n, \; q \in \mathbf{R}^N.$$

We prove

$$\int_\Omega A_{ij}^{\alpha\beta} \, D_\beta u^j \, D_\alpha u^i \, dx \geq \nu \int_\Omega |Du|^2 \, dx$$

for all $u \in C_0^\infty(\Omega; \mathbf{R}^N)$. For these test functions u we extend them onto $\mathbf{R}^n$ by zero outside Ω and thus consider them as functions in $C_0^\infty(\mathbf{R}^n; \mathbf{R}^N)$. Define the *Fourier transforms* for such functions u by

$$\hat{u}(p) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-i \, p \cdot x} \, u(x) \, dx; \quad p \in \mathbf{R}^n.$$

Then, for any $u, \, v \in C_0^\infty(\mathbf{R}^n; \mathbf{R}^N)$,

$$\int_{\mathbf{R}^n} u(x) \cdot v(x) \, dx = \int_{\mathbf{R}^n} \hat{u}(p) \cdot \overline{\hat{v}(p)} \, dp,$$

$$\widehat{D_\alpha u^i}(p) = i \, p_\alpha \widehat{u^i}(p);$$

the last identity can also be written as $\widehat{Du}(p) = i \, \hat{u}(p) \otimes p$. Now, using these identities, we have

$$\int_{\mathbf{R}^n} A_{ij}^{\alpha\beta} \, D_\beta u^j(x) \, D_\alpha u^i(x) \, dx = \int_{\mathbf{R}^n} A_{ij}^{\alpha\beta} \, \widehat{D_\beta u^j}(p) \, \overline{\widehat{D_\alpha u^i}(p)} \, dp$$

$$= \int_{\mathbf{R}^n} A_{ij}^{\alpha\beta} \, p_\beta \, p_\alpha \, \widehat{u^j}(p) \, \overline{\widehat{u^i}(p)} \, dp = \operatorname{Re} \left(\int_{\mathbf{R}^n} A_{ij}^{\alpha\beta} \, p_\beta \, p_\alpha \, \widehat{u^j}(p) \, \overline{\widehat{u^i}(p)} \, dp \right).$$

Write $\hat{u}(p) = q + iy$ with $q, \, y \in \mathbf{R}^N$. Then

$$\operatorname{Re} \left(\widehat{u^j}(p) \, \overline{\widehat{u^i}(p)} \right) = q^i \, q^j + y^i \, y^j.$$

Therefore, by the Legendre-Hadamard condition,

$$\operatorname{Re} \left(A_{ij}^{\alpha\beta} \, p_\alpha \, p_\beta \, \widehat{u^j}(p) \, \overline{\widehat{u^i}(p)} \right) \geq \nu \, |p|^2 \, (|q|^2 + |y|^2) = \nu \, |p|^2 \, |\hat{u}(p)|^2.$$

Hence,

$$\begin{aligned} a[u,u] &= \int_{\mathbf{R}^n} A_{ij}^{\alpha\beta} \, D_\beta u^j(x) \, D_\alpha u^i(x) \, dx \\ &= \operatorname{Re} \left(\int_{\mathbf{R}^n} A_{ij}^{\alpha\beta} \, p_\beta \, p_\alpha \, \widehat{u^j}(p) \, \overline{\widehat{u^i}(p)} \, dp \right) \\ &\geq \nu \int_{\mathbf{R}^n} |p|^2 \, |\hat{u}(p)|^2 \, dp = \nu \int_{\mathbf{R}^n} |i\hat{u}(p) \otimes p|^2 \, dp \\ &= \nu \int_{\mathbf{R}^n} |\widehat{Du}(p)|^2 \, dp = \nu \int_{\mathbf{R}^n} |Du(x)|^2 \, dx. \end{aligned}$$

This implies $a[u,u] \geq \nu \, \|u\|_H^2$ for all $u \in H$; the proof is complete. □

Theorem 2.5. *Under the hypotheses of the previous theorem the following Dirichlet problem*

$$\begin{cases} -\operatorname{Div}(A(x,Du)) = G, \\ u|_{\partial\Omega} = \varphi \end{cases}$$

has a unique weak solution u in $H = H_0^1(\Omega;\mathbf{R}^N)$ for any bounded linear functional G on H and any function $\varphi \in W^{1,2}(\Omega;\mathbf{R}^N)$, where $A_\alpha^i(x,\xi) = A_{ij}^{\alpha\beta}(x)\,\xi_\beta^j$ as given above.

Proof. We know that every bounded linear functional G on $H = H_0^1(\Omega;\mathbf{R}^N)$ is of the form $G = g + \sum_{\alpha=1}^n D_\alpha f_\alpha$ as a distribution, where $g,\ f_\alpha \in L^2(\Omega;\mathbf{R}^N)$. In order to solve for u we let $u = v + \varphi$; then we need to solve for $v \in H$ which satisfies

$$\begin{cases} -\operatorname{Div}(A(x,Dv)) = G + \operatorname{Div}(A(x,D\varphi)), \\ v|_{\partial\Omega} = 0. \end{cases} \tag{2.3}$$

Note that $\tilde G = G + \operatorname{Div}(A(x,D\varphi))$ is a bounded linear functional on H. Since under the hypotheses of the theorem the bilinear form $a[u,v]$ is *bounded* and *coercive*, the existence of a unique solution v of the problem (2.3) follows by virtue of the Lax-Milgram theorem. We have thus proved the theorem. □

Theorem 2.6. *If $A_{ij}^{\alpha\beta} \in L^\infty(\Omega)$ and the* **coercivity condition**

$$\int_\Omega A_{ij}^{\alpha\beta}(x)\, D_\beta u^j\, D_\alpha u^i\, dx \ge \nu \int_\Omega |Du|^2\, dx \tag{2.4}$$

holds for all $u \in H_0^1(\Omega;\mathbf{R}^N)$ then the Legendre-Hadamard condition *holds for almost every $x \in \Omega$:*

$$A_{ij}^{\alpha\beta}(x)\, q^i\, q^j\, p_\alpha\, p_\beta \ge \nu\, |p|^2\, |q|^2, \quad \forall p \in \mathbf{R}^n,\ q \in \mathbf{R}^N.$$

Proof. Let $\rho(t)$ be the 2-periodic "sawtooth" function equaling t on interval $[0,1]$ and $2-t$ on interval $[1,2]$. Thus $\rho'(t) = \pm 1$ for a.e. $t \in \mathbf{R}$. For any $\zeta \in C_0^\infty(\Omega)$, $p \in \mathbf{R}^n$ and $q \in \mathbf{R}^N$, define

$$u_\epsilon(x) = \epsilon\,\zeta(x)\,\rho(p\cdot x/\epsilon)\,q, \quad \epsilon > 0.$$

It is easy to see $u_\epsilon \in H_0^1(\Omega;\mathbf{R}^N)$ and

$$D_\alpha u_\epsilon^i(x) = \epsilon\, D_\alpha\zeta(x)\,\rho(p\cdot x/\epsilon)\,q^i + \zeta(x)\,\rho'(p\cdot x/\epsilon)\,p_\alpha\, q^i.$$

Inserting them into the coercivity condition and letting $\epsilon \to 0$ we have

$$\int_\Omega A_{ij}^{\alpha\beta}(x)\, q^i\, q^j\, p_\alpha\, p_\beta\, \zeta^2(x)\, dx \ge \nu \int_\Omega |p|^2\, |q|^2\, \zeta^2(x)\, dx.$$

This is true for all $\zeta \in C_0^\infty(\Omega)$; thus we have for a.e. $x \in \Omega$,

$$A_{ij}^{\alpha\beta}(x)\, q^i\, q^j\, p_\alpha\, p_\beta \ge \nu\, |p|^2\, |q|^2, \quad \forall p \in \mathbf{R}^n,\ q \in \mathbf{R}^N.$$

The proof is complete. □

Remark. The reverse of the theorem is not true; that is, the Legendre-Hadamard condition does not imply the coercivity condition (2.4), even for $A_{ij}^{\alpha\beta} \in C^\infty(\Omega)$. (Le Dret '87) □

2.4. Gårding's inequality and existence results

In this section, we consider the general bilinear form $B[u,v]$ defined earlier by (2.2). We prove the following result known as **Gårding's inequality.**

Theorem 2.7. *Let $B[u,v]$ be defined by (2.2). Assume*

1) $A_{ij}^{\alpha\beta} \in C(\bar\Omega)$,

2) the Legendre-Hadamard condition *holds for all $x \in \Omega$; that is,*

$$A_{ij}^{\alpha\beta}(x)\, q^i\, q^j\, p_\alpha\, p_\beta \geq \nu\, |p|^2\, |q|^2, \quad \forall p \in \mathbf{R}^n,\; q \in \mathbf{R}^N.$$

3) $b_{ij}^\alpha,\; c_{ij}^\alpha,\; d_{ij} \in L^\infty(\Omega)$.

Then, there exist constants $\lambda_0 > 0$ and λ_1 such that

$$B[u,u] \geq \lambda_0\, \|u\|_H^2 - \lambda_1\, \|u\|_{L^2}^2, \quad \forall\, u \in H_0^1(\Omega;\mathbf{R}^N).$$

Proof. By uniform continuity, we can choose a small $\epsilon > 0$ such that

$$|A_{ij}^{\alpha\beta}(x) - A_{ij}^{\alpha\beta}(y)| \leq \frac{\nu}{2}, \quad \forall\, x,\, y \in \bar\Omega,\; |x-y| \leq \epsilon.$$

We claim

$$\int_\Omega A_{ij}^{\alpha\beta}(x)\, D_\alpha u^i\, D_\beta u^j\, dx \geq \frac{\nu}{2} \int_\Omega |Du(x)|^2\, dx \tag{2.5}$$

for all $u \in C_0^\infty(\Omega;\mathbf{R}^N)$ with the diameter of the support $\operatorname{diam}(\operatorname{supp} u) \leq \epsilon$. To see this, we choose any point $x_0 \in \operatorname{supp} u$. Then

$$\begin{aligned}\int_\Omega A_{ij}^{\alpha\beta}(x)\, D_\alpha u^i\, D_\beta u^j\, dx &= \int_\Omega A_{ij}^{\alpha\beta}(x_0)\, D_\alpha u^i\, D_\beta u^j\, dx \\ &\quad + \int_{\operatorname{supp} u} \big(A_{ij}^{\alpha\beta}(x) - A_{ij}^{\alpha\beta}(x_0)\big)\, D_\alpha u^i\, D_\beta u^j\, dx \\ &\geq \nu \int_\Omega |Du(x)|^2\, dx - \frac{\nu}{2} \int_\Omega |Du(x)|^2\, dx,\end{aligned}$$

which proves (2.5). We now cover $\bar\Omega$ with finitely many open balls $\{B_{\epsilon/4}(x^k)\}$ with $x^k \in \Omega$ and $k = 1, 2, ..., K$. For each k, let $\zeta_k \in C_0^\infty(B_{\epsilon/2}(x^k))$ with $\zeta_k(x) = 1$ for $x \in B_{\epsilon/4}(x^k)$. Since for any $x \in \bar\Omega$ we have at least one k such that $x \in B_{\epsilon/4}(x^k)$ and thus $\zeta_k(x) = 1$, we may therefore define

$$\varphi_k(x) = \frac{\zeta_k(x)}{\big(\sum_{j=1}^K \zeta_j^2(x)\big)^{1/2}}, \quad k = 1, 2, ..., K.$$

Then $\sum_{k=1}^K \varphi_k^2(x) = 1$ for all $x \in \Omega$. (This is a special case of *partition of unity.*) We have thus

$$A_{ij}^{\alpha\beta}(x)\, D_\alpha u^i\, D_\beta u^j = \sum_{k=1}^K \Big(A_{ij}^{\alpha\beta}(x)\, \varphi_k^2\, D_\alpha u^i\, D_\beta u^j\Big) \tag{2.6}$$

and each term (no summation on k)

$$\begin{aligned} A_{ij}^{\alpha\beta}(x)\, \varphi_k^2\, D_\alpha u^i\, D_\beta u^j &= A_{ij}^{\alpha\beta}(x)\, D_\alpha(\varphi_k\, u^i)\, D_\beta(\varphi_k\, u^j) \\ -A_{ij}^{\alpha\beta}(x)\, \big(\varphi_k\, D_\beta\varphi_k\, u^j\, D_\alpha u^i &+ \varphi_k\, D_\alpha\varphi_k\, u^i\, D_\beta u^j + D_\alpha\varphi_k\, D_\beta\varphi_k\, u^i\, u^j\big)\,. \end{aligned}$$

Since $\varphi_k u \in C_0^\infty(\Omega \cap B_{\epsilon/2}(x^k); \mathbf{R}^N)$ and $\operatorname{diam}(\Omega \cap B_{\epsilon/2}(x^k)) \leq \epsilon$, we have by (2.5)

$$\int_\Omega A_{ij}^{\alpha\beta}(x)\, D_\alpha(\varphi_k\, u^i)\, D_\beta(\varphi_k\, u^j)\, dx \geq \frac{\nu}{2} \int_\Omega |D(\varphi_k\, u)|^2\, dx.$$

Note also that

$$|D(\varphi_k\, u)|^2 = \varphi_k^2\, |Du|^2 + |D\varphi_k|^2\, |u|^2 + 2\, \varphi_k\, D_\alpha\varphi_k\, u^i\, D_\alpha u^i.$$

Therefore, we have by (2.6) and the fact that $\sum_{k=1}^K \phi_k^2 = 1$ on Ω,

$$\begin{aligned}&\int_\Omega A_{ij}^{\alpha\beta}(x)\, D_\alpha u^i\, D_\beta u^j\, dx \\ &\geq \frac{\nu}{2} \int_\Omega |Du|^2\, dx - C_1\, \|u\|_{L^2}\, \|Du\|_{L^2} - C_2\, \|u\|_{L^2}^2.\end{aligned}$$

The terms in $B[u,u]$ involving b_{ij}^α, c_{ij}^α and d_{ij} can be estimated by $\|u\|_{L^2}\, \|Du\|_{L^2}$ and $\|u\|_{L^2}^2$. Finally, by all of these estimates and the inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon}\, b^2$$

we have $B[u,u] \geq \lambda_0\, \|u\|_H^2 - \lambda_1\, \|u\|_{L^2}^2$ for all $u \in H_0^1(\Omega; \mathbf{R}^N)$. This completes the proof. □

Note that the bilinear form $\tilde{B}[u,v] = B[u,v] + \lambda\, (u,v)_{L^2}$ is a bounded, coercive, bilinear form on $H = H_0^1(\Omega; \mathbf{R}^N)$ for all $\lambda > \lambda_1$; thus, by the Lax-Milgram theorem, we easily obtain the following existence result.

Theorem 2.8. *Under the hypotheses of the previous theorem, the Dirichlet problem*

$$\begin{cases} -\operatorname{Div}(A(x,u,Du)) + b(x,u,Du) + \lambda u = G, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{2.7}$$

has a unique weak solution u in $H_0^1(\Omega; \mathbf{R}^N)$ for any bounded linear functional G on H, where $A(x,u,Du)$, $b(x,u,Du)$ are the linear operators as given in the previous theorem and λ_1 is the constant in the theorem. Moreover, the solution u satisfies $\|u\|_H \leq C\, \|G\|$ with a constant C depending on λ and the L^∞-norms of the coefficients of $A(x,s,\xi)$ and $b(x,s,\xi)$.

Finally, we have the following existence theorem, which follows from the *Fredholm alternative theorem* in Hilbert spaces.

Theorem 2.9. *There exists an at most countable set $\sigma \subset \mathbf{R}$ such that Dirichlet problem (2.7) has a unique weak solution u in $H_0^1(\Omega; \mathbf{R}^N)$ for any bounded linear functional G on H if and only if $\lambda \notin \sigma$. Moreover, if σ is infinite then σ is a nonincreasing sequence $\{\sigma_k\}$, $k = 1, 2, \ldots$ with $\sigma_k \to -\infty$ as $k \to \infty$. The values $\{-\sigma_k\}$ are called the* **eigenvalues** *of the linear operator $Lu \equiv -\operatorname{Div}(A(x,u,Du)) + b(x,u,Du)$ with the zero Dirichlet boundary condition.*

Proof. Let $\tilde{\lambda} > \lambda_1$ be a fixed number. Then, by the theorem above, there exists an inverse $(L + \tilde{\lambda})^{-1} \colon H^* \to H$, where $H = H_0^1(\Omega; \mathbf{R}^N)$. Let $P \colon H \to H^*$ be the map defined by

$$\langle Pu, v \rangle = \int_\Omega u \cdot v\, dx, \quad \forall\, u,\, v \in H.$$

Then Dirichlet problem (2.7) is equivalent to the following equation in H

$$[I + (\lambda - \tilde{\lambda})(L + \tilde{\lambda})^{-1} P]\, u = (L + \tilde{\lambda})^{-1} G,$$

where $G \in H^*$ is any bounded linear functional on H. This equation has a unique solution $u \in H$ if and only if

$$\text{Range}\,[I + (\lambda - \tilde{\lambda})T] = H, \quad \text{Ker}\,[I + (\lambda - \tilde{\lambda})T] = \{0\}, \tag{2.8}$$

where $T = (L + \tilde{\lambda})^{-1}P\colon H \to H$ is a bounded linear operator on H. We have the following

Lemma 2.10. *$T\colon H \to H$ is a* **compact** *operator.*

Proof. It is sufficient to prove that $\{Tu_k\}$ has a convergent subsequence for any bounded sequence $\{u_k\}$ in H. Note that

$$\|Pu\| \leq \|u\|_{L^2}.$$

By Sobolev's embedding theorem we have a subsequence $\{u_{k_j}\}$ such that $u_{k_j} \to \bar{u}$ in $L^2(\Omega; \mathbf{R}^N)$ as $j \to \infty$. Therefore,

$$\|Tu_{k_j} - Tu_{k_h}\| \leq \|(L + \tilde{\lambda})^{-1}\| \cdot \|u_{k_j} - u_{k_h}\|_{L^2} \to 0 \quad \text{as } j,\, h \to \infty$$

and hence $\{Tu_{k_j}\}$ converges in H; thus T is compact. □

From this lemma, by Fredholm's alternative theorem, the conclusion (2.8) is true for all λ except for an at most countable set σ whose only possible limit point is $-\infty$. The theorem is proved. □

Chapter 3

Direct Methods in the Calculus of Variations

A variational problem involves finding minimizers or general critical points of a given functional. For systems in variational form, we can find a solution as a minimizer of a related multiple integral functional. Usually, the existence of such a minimizer is proved by an abstract idea of an existence theorem of *Weierstrass*. This method has been known as a *direct method* in the calculus of variations, which dates back to the well-known Dirichlet principle, where a harmonic function with given boundary data is found by minimizing the energy functional in the class of functions with the same given boundary data. In this chapter, we deal with some important issues related to the direct methods in the calculus of variations.

3.1. Abstract theorems on existence of minimizers

Let us first recall a few topological facts. Let X be a topological space, $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ and $\mathcal{A}$ a subset of X.

Definition 3.1. We say Φ is **sequentially lower semicontinuous** (write **s.l.s.c.**) on X if for any $\bar{x} \in X$ and every sequence $\{x_j\}$ converging to $\bar{x}$

$$\Phi(\bar{x}) \leq \liminf_{j\to\infty} \Phi(x_j).$$

We say $\mathcal{A}$ is **sequentially compact** if from every sequence of points in $\mathcal{A}$ one can select a subsequence converging to a point still in $\mathcal{A}$.

The following theorem plays a fundamental role in the direct methods we shall discuss later on.

Theorem 3.1 (Weierstrass)**.** *Let* $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ *be* s.l.s.c. *on* X *and let* $\mathcal{A}$ *be a* sequentially compact *subset of* X*. Then the infimum of* Φ *over* $\mathcal{A}$ *is attained at some point in* $\mathcal{A}$.

Proof. In order to see how (the idea of) this simple existence theorem can be used later for many variational problems, we present a short proof of this result. To do so, assume

$$m = \inf_{x\in\mathcal{A}} \Phi(x).$$

Then there exists a *minimizing sequence* $\{x_k\}$ in $\mathcal{A}$ such that

$$\lim_{k\to\infty} \Phi(x_k) = m.$$

Now by the sequential compactness of $\mathcal{A}$ we can select a subsequence $\{x_{k_j}\}$ converging to a point x_0 in $\mathcal{A}$. Then, the sequential lower semicontinuity of Φ will imply

$$\Phi(x_0) \leq \liminf_{j\to\infty} \Phi(x_{k_j}) = m.$$

Since $x_0 \in \mathcal{A}$, by the definition of m, it follows that

$$\Phi(x_0) = m = \inf_{x\in\mathcal{A}} \Phi(x).$$

Hence Φ has a minimizer x_0 over $\mathcal{A}$. From the proof, we also see that $m = \Phi(x_0) > -\infty$; thus Φ must be *bounded below* over the set $\mathcal{A}$. $\square$

In many variational problems, where we are trying to find a minimizer of a function Φ on a set $\mathcal{A}$. Generally, $\mathcal{A}$ is not equipped *a priori* with a topology. So our minimization problem can be seen as a problem of introducing a topology for which both $\mathcal{A}$ is a *sequentially compact* set and Φ is a *s.l.s.c.* function. Note that in order to grant that Φ be *s.l.s.c.* we need a rich topology, while for the sequential compactness the topology need not be too rich.

We shall see that this compromise can be reached satisfactorily for a large class of multiple integral functionals working in the Sobolev spaces $W^{1,p}(\Omega; \mathbf{R}^N)$.

We consider a very useful case where $\mathcal{A}$ is a subset of a given Banach space X. Then, besides the norm-topology of X we have the *weak topology* on X. Under this weak topology, a sequence $\{x_k\}$ is to converge to a point $\bar{x}$ as $k \to \infty$ provided that

$$\lim_{k\to\infty} \langle L, x_k\rangle = \langle L, \bar{x}\rangle$$

for all bounded linear functionals $L\colon X \to \mathbf{R}$ on X; the set of all these bounded linear functionals is called the **dual space** of X and denoted by X^*. If X is a *reflexive* Banach space, i.e., $(X^*)^* \cong X$, then every bounded closed subset of X is sequentially compact in the weak topology of X (also called **sequentially weakly compact**); this is Banach-Aloglu theorem.

Theorem 3.2. *Let X be a reflexive Banach space and $\mathcal{A}$ be a subset of X which is closed in the weak topology of X. Suppose $\Phi : \mathcal{A} \to \mathbf{R} \cup \{+\infty\}$ is*

- **s.l.s.c.** *in the weak topology of X* (we also write Φ is **w.s.l.s.c.**),
- **bounded below**, *and*
- **norm-coercive** *in the sense that $\Phi(x_j) \to \infty$ as $\|x_j\| \to \infty$.*

Then Φ attains its minimum at some point in $\mathcal{A}$.

Proof. Take a minimizing sequence $\{x_j\}$ in $\mathcal{A}$ such that

$$\lim_{j\to\infty} \Phi(x_j) = m = \inf_{x\in\mathcal{A}} \Phi(x).$$

Since Φ is bounded below, it follows that $-\infty < m \leq \infty$. If $m = \infty$ then there is nothing to prove since $\Phi \equiv +\infty$. Now assume $m < \infty$. Then the norm-coercivity of Φ implies that sequence $\{\|x_j\|\}$ is bounded. Therefore, by the self-reflexivity of X, there exists a subsequence $\{x_{j_k}\}$ which weakly converges to some point $\bar{x} \in X$. Since $\mathcal{A}$ is weakly closed, we have $\bar{x} \in \mathcal{A}$. Finally the lower semicontinuity of Φ implies

$$\Phi(\bar{x}) \leq \liminf_{k\to\infty} \Phi(x_{j_k}) = m$$

and the membership $\bar{x} \in \mathcal{A}$ yields that $m = \Phi(\bar{x})$ and thus $\bar{x} \in \mathcal{A}$ is a minimizer sought for. □

Remarks. 1) In application, we usually have $X = W^{1,p}(\Omega; \mathbf{R}^N)$ and $\mathcal{A}$ a *Dirichlet class*; that is,

$$\mathcal{A} = \mathcal{D}^p_\varphi(\Omega) = \{u \in W^{1,p}(\Omega; \mathbf{R}^N) \mid u|_{\partial\Omega} = \varphi\},$$

where $\varphi \in W^{1,p}(\Omega; \mathbf{R}^N)$ is given. Then X is *reflexive* for all $1 < p < \infty$. But if $p = 1, \infty$, X is not reflexive. In the case $p = \infty$, the topology for $\mathcal{A}$ in the theorem need be replaced by the *weak star topology* of $W^{1,\infty}(\Omega; \mathbf{R}^N)$.

2) In the calculus of variations, the *lower semicontinuity* of an energy functional is mostly essential for many problems, while the boundedness and coercivity are sometimes relatively easy to obtain. □

3.2. Lower semicontinuity in Sobolev spaces

As discussed before, in many variational problems lower semicontinuity is an essential condition for exitence of minimizers. In next two sections, we study the (sequential) lower semicontinuity of a multiple integral functional $I(u)$ in the Sobolev space $W^{1,p}(\Omega; \mathbf{R}^N)$. Assume

$$I(u) = \int_\Omega F(x, u, Du)\, dx.$$

We first prove a semicontinuity result, due mainly to Tonelli.

Theorem 3.3. *Let $F(x, s, \xi) \geq 0$ be* **smooth** *and* **convex** *in ξ. Assume F, F_ξ are both* continuous *in (x, s, ξ). Then the functional $I(u)$ defined above is* sequentially weakly (weakly star if $p = \infty$) lower semicontinuous *on $W^{1,p}(\Omega; \mathbf{R}^N)$ for all $1 \leq p \leq \infty$.*

Proof. We need only to prove $I(u)$ is *s.w.s.l.c.* on $W^{1,1}(\Omega; \mathbf{R}^N)$. To this end, assume $\{u_k\}$ is a sequence weakly convergent to u in $W^{1,1}(\Omega; \mathbf{R}^N)$. We need to show

$$I(u) \leq \liminf_{k\to\infty} I(u_k).$$

By the Sobolev embedding theorem it follows that (via a subsequence) $u_k \to u$ in $L^1(\Omega; \mathbf{R}^N)$. We can also assume $u_k(x) \to u(x)$ for almost every $x \in \Omega$. Now, for any given $\delta > 0$ we choose a compact set $K \subset \Omega$ such that

(i) $u_k \to u$ uniformly on K and $|\Omega \setminus K| < \delta$ (by *Egorov's theorem*);

(ii) u, Du are continuous on K (by *Lusin's theorem*).

Since $F(x, s, \xi)$ is *smooth* and *convex* in ξ, it follows that

$$F(x, s, \eta) \geq F(x, s, \xi) + F_{\xi^i_\alpha}(x, s, \xi)\, (\eta^i_\alpha - \xi^i_\alpha) \quad \forall \xi,\ \eta \in \mathbf{M}^{N\times n}.$$

Therefore, since $F \geq 0$,

$$I(u_k) \geq \int_K F(x, u_k, Du_k)\, dx$$

$$\geq \int_K \left[F(x, u_k, Du) + F_{\xi^i_\alpha}(x, u_k, Du)\, (D_\alpha u^i_k - D_\alpha u^i)\right]$$

$$= \int_K F(x, u_k, Du) + \int_K F_{\xi_\alpha^i}(x, u, Du)\,(D_\alpha u_k^i - D_\alpha u^i)$$

$$+ \int_K [F_{\xi_\alpha^i}(x, u_k, Du) - F_{\xi_\alpha^i}(x, u_k^i, Du)]\,(D_\alpha u_k^i - D_\alpha u^i).$$

Since $F(x, s, \xi)$ is uniformly continuous on bounded sets and $u_k(x) \to u(x)$ uniformly on K we have

$$\lim_{k\to\infty} \int_K F(x, u_k, Du)\,dx = \int_K F(x, u, Du)\,dx,$$

$$\lim_{k\to\infty} \|F_{\xi_\alpha^i}(x, u_k, Du) - F_{\xi_\alpha^i}(x, u_k^i, Du)\|_{L^\infty(K)} = 0.$$

Now since $F_{\xi_\alpha^i}(x, u, Du)$ is bounded on K and $D_\alpha u_k^i$ converges to $D_\alpha u^i$ weakly in $L^1(\Omega)$ as $k \to \infty$, we thus have

$$\lim_{k\to\infty} \int_K F_{\xi_\alpha^i}(x, u, Du)\,(D_\alpha u_k^i - D_\alpha u^i)\,dx = 0.$$

From these estimates, we use Lemma 3.4 below to deduce that

$$\liminf_{k\to\infty} I(u_k) \geq \int_K F(x, u, Du).$$

If $F(x, u, Du) \in L^1(\Omega)$, i.e., $I(u) < \infty$, then for any given $\epsilon > 0$, we use *Lebesgue's absolute continuity theorem* to determine $\delta > 0$ so that

$$\int_E F(x, u, Du) \geq \int_\Omega F(x, u, Du) - \epsilon, \quad \forall E \subset \Omega,\ |\Omega \setminus E| < \delta.$$

On the other hand, if $I(u) = \infty$ then for any given large number $M > 0$ we choose $\delta > 0$ so that

$$\int_E F(x, u, Du)\,dx > M, \quad \forall E \subset \Omega,\ |\Omega \setminus E| < \delta.$$

In any of these two cases, by letting either $\epsilon \to 0$ or $M \to \infty$, we obtain

$$\liminf_{k\to\infty} I(u_k) \geq I(u).$$

The theorem is proved. □

Lemma 3.4. $\liminf_{k\to\infty} (a_k + b_k) \geq \liminf_{k\to\infty} a_k + \liminf_{k\to\infty} b_k$.

Using the theorem, we obtain the following existence result for convex functionals.

Theorem 3.5. *In addition to the hypotheses of the previous theorem, assume there exists* $1 < p < \infty$ *such that*

$$F(x, s, \xi) \geq c\,|\xi|^p - C, \quad c > 0,\ C \text{ are some constants.}$$

If for some $\varphi \in W^{1,p}(\Omega; \mathbf{R}^N)$ *one has* $I(\varphi) < \infty$ *then the minimization problem* $\inf_{u\in\mathcal{A}} I(u)$ *has a minimizer in* $\mathcal{A}$*, where* $\mathcal{A} = \mathcal{D}^p_\varphi(\Omega)$ *is the* Dirichlet class *of* φ *defined before.*

Remark. Both theorems in this section hold for more general functions $F(x, s, \xi)$. For example, we can replace the continuity condition by the **Carathéodory** condition; a function $F(x, s, \xi)$ is called a *Carathéodory* function if and only if F is measurable in x for all (s, ξ) and continuous in (s, ξ) for almost every x. □

3.3. Quasiconvexity and lower semicontinuity

Note that $W^{1,\infty}(\Omega;\mathbf{R}^N)$ can be identified with the space of all *Lipschitz maps* from Ω to $\mathbf{R}^N$. A sequence $\{u_k\}$ converges to u in the weak star topology of $W^{1,\infty}(\Omega;\mathbf{R}^N)$ if and only if $\{u_k\}$ converges to u in the sense of **Lipschitz convergence**; that is,

1) $u_k \to u$ uniformly in $C(\bar\Omega;\mathbf{R}^N)$;

2) the Lipschitz norms of u_k and u are bounded.

Then we have the following necessary condition, mainly due to Morrey, for the lower semicontinuity under the *Lipschitz convergence* of the multiple integral

$$I(u) = \int_\Omega F(x,u(x),Du(x))\,dx.$$

Theorem 3.6. *Assume $F(x,s,\xi)$ is continuous on $\bar\Omega\times\mathbf{R}^N\times\mathbf{M}^{N\times n}$. If the functional $I(u)$ defined by F as above is* s.l.s.c. *with respect to the* Lipschitz convergence *on $W^{1,\infty}(\Omega;\mathbf{R}^N)$, then the following condition holds for all $x_0\in\Omega$, $s_0\in\mathbf{R}^N$, $\xi_0\in\mathbf{M}^{N\times n}$ and all $\phi\in C_0^\infty(\Omega;\mathbf{R}^N)$:*

$$F(x_0,s_0,\xi_0) \le \fint_\Omega F(x_0,s_0,\xi_0+D\phi(x))\,dx. \tag{3.1}$$

In this case, function $F(x,s,\xi)$ is called **quasiconvex** *in ξ.*

Proof. Let Q be a fixed open cube containing $\bar\Omega$ with center $\bar x$ and side-length $2L$. We prove this theorem by several lemmas.

Lemma 3.7. *Suppose*

$$F(x_0,s_0,\xi_0) \le \fint_Q F(x_0,s_0,\xi_0+D\phi(x))\,dx \tag{3.2}$$

holds for all $\phi\in C_0^\infty(Q;\mathbf{R}^N)$. Then (3.1) *holds.*

Proof. For any $\phi\in C_0^\infty(\Omega;\mathbf{R}^N)$ we extend ϕ by zero onto Q; then $\phi\in C_0^\infty(Q;\mathbf{R}^N)$. Inserting it into (3.2) yields (3.1). □

In the following, let $x_0\in\Omega$, $s_0\in\mathbf{R}^N$, $\xi_0\in\mathbf{M}^{N\times n}$ be given. Define $\tilde u(x) = s_0+\xi_0\cdot(x-x_0)$. Let also $\phi\in C_0^\infty(Q;\mathbf{R}^N)$ be given.

Assume $Q'\subset\subset\Omega$ is an arbitrarily given cube containing x_0 with side-length $2l$. For any positive integer ν we divide each side of Q' into 2^ν intervals of equal length, each being equal to $2^{-\nu+1}l$. This divides Q' into $2^{n\nu}$ small cubes $\{Q_j^\nu\}$ with $j=1,2,...,2^{n\nu}$. Denote the center of each cube Q_j^ν by $\bar x_j^\nu$ and define a function $u_\nu\colon\Omega\to\mathbf{R}^N$ as follows.

$$u_\nu(x) = \begin{cases} \tilde u(x) & \text{if } x\in\Omega\setminus\cup_{j=1}^{2^{n\nu}}Q_j^\nu; \\ \tilde u(x) + \frac{2^{-\nu}l}{L}\,\phi\big(\bar x + \frac{2^\nu L}{l}\,(x-\bar x_j^\nu)\big) & \text{if } x\in Q_j^\nu,\ 1\le j\le 2^{n\nu}. \end{cases}$$

Lemma 3.8. *u_ν is a Lipschitz function on Ω and $\{u_\nu\}$ converges to $\tilde u$ as $\nu\to\infty$ in the sense of* Lipschitz convergence *defined above.*

Proof. Note that, by the definition of $\tilde u$

$$Du_\nu(x) = \begin{cases} \xi_0 & \text{if } x\in\Omega\setminus\cup_{j=1}^{2^{n\nu}}Q_j^\nu; \\ \xi_0 + D\phi\big(\bar x + \frac{2^\nu L}{l}\,(x-\bar x_j^\nu)\big) & \text{if } x\in Q_j^\nu,\ 1\le j\le 2^{n\nu}. \end{cases}$$

Therefore $\{Du_\nu\}$ is uniformly bounded. The lemma follows directly from the definition of *Lipschitz convergence.* □

To continue the proof, we notice that

$$I(\tilde u) = \int_\Omega F(x, \tilde u(x), \xi_0)\, dx$$

and that

$$\begin{aligned} I(u_\nu) &= \int_\Omega F(x, u_\nu(x), Du_\nu)\, dx \\ &= \int_{\Omega\setminus Q'} F(x, \tilde u, \xi_0)\, dx + \int_{Q'} F(x, u_\nu, Du_\nu)\, dx. \end{aligned}$$

Therefore, by the lower semicontinuity of I, we thus have

$$\int_{Q'} F(x, \tilde u, \xi_0)\, dx \le \liminf_{\nu\to\infty} \int_{Q'} F(x, u_\nu, Du_\nu)\, dx. \tag{3.3}$$

From the uniform continuity of $F(x, s, \xi)$ on bounded sets and the fact that $u_\nu \to \tilde u$ uniformly on Ω we have

$$\liminf_{\nu\to\infty} \int_{Q'} F(x, u_\nu, Du_\nu)\, dx = \liminf_{\nu\to\infty} \int_{Q'} F(x, \tilde u, Du_\nu)\, dx. \tag{3.4}$$

We now compute

$$\begin{aligned} \int_{Q'} F(x, \tilde u, Du_\nu)\, dx &= \sum_{j=1}^{2^{n\nu}} \int_{Q_j^\nu} F\Big(x, \tilde u, \xi_0 + D\phi\big(\bar x + \frac{2^\nu L}{l}\,(x - \bar x_j^\nu)\big)\Big)\, dx \\ &= \sum_{j=1}^{2^{n\nu}} \int_{Q_j^\nu} F\Big(x_j^\nu, \tilde u(x_j^\nu), \xi_0 + D\phi\big(\bar x + \frac{2^\nu L}{l}\,(x - \bar x_j^\nu)\big)\Big)\, dx \\ &= \sum_{j=1}^{2^{n\nu}} \Big(\frac{l}{2^\nu L}\Big)^n \int_Q F\big(x_j^\nu, \tilde u(x_j^\nu), \xi_0 + D\phi(y)\big)\, dy \\ &= \sum_{j=1}^{2^{n\nu}} \tilde F(x_j^\nu)\, |Q_j^\nu|, \end{aligned} \tag{3.5}$$

where $x_j^\nu \in Q_j^\nu$ are some points by the mean value theorem of integration, and

$$\tilde F(x) = ⨍_Q F(x, \tilde u(x), \xi_0 + D\phi(y))\, dy.$$

This function is continuous on Q' and the sum in (3.5) is simply the *Riemann sum* of the integral of $\tilde F$ over Q'. Therefore, we arrive at

$$\lim_{\nu\to\infty} \int_{Q'} F(x, \tilde u, Du_\nu)\, dx = \int_{Q'} \tilde F(x)\, dx,$$

which by (3.4) implies

$$\int_{Q'} F(x, \tilde u(x), \xi_0)\, dx \le \int_{Q'} \tilde F(x)\, dx.$$

This inequality holds for any cube $Q' \subset\subset \Omega$ containing x_0; therefore,

$$F(x_0, \tilde u(x_0), \xi_0) \le \tilde F(x_0).$$

This is nothing but

$$F(x_0, s_0, \xi_0) \le ⨍_Q F(x_0, s_0, \xi_0 + D\phi(y))\, dy.$$

Finally, the proof of Theorem 3.6 is complete. □

Later we shall show that the quasiconvexity is also sufficient for the lower semicontinuity of the multiple integrals of our study. But, before proving the sufficiency theorems, we would like to point out some important properties of this *quasiconvexity* condition related to the lower semicontinuity; properties related to other convexity conditions will be discussed in next section.

Since quasiconvexity is only a condition for the dependence of $F(x,s,\xi)$ on ξ, in the following we assume $F = F(\xi)$ depends only on ξ. Therefore, F is *quasiconvex* if and only if

$$F(\xi) \le \fint_\Omega F(\xi + D\phi(x))\,dx$$

holds for all $\phi \in C_0^\infty(\Omega; \mathbf{R}^N)$. The following result, due to Meyers, will be useful to relax the zero boundary condition on ϕ.

Theorem 3.9. *Let $F\colon \mathbf{M}^{N\times n} \to \mathbf{R}$ be continuous and quasiconvex. For every bounded set $Q \subset \mathbf{R}^n$ and every sequence $\{z_k\}$ in $W^{1,\infty}(Q; \mathbf{R}^N)$ converging to zero in the sense of Lipschitz convergence, we have*

$$F(\xi) \le \liminf_{k\to\infty} \fint_Q F(\xi + Dz_k(x))\,dx$$

for every $\xi \in \mathbf{M}^{N\times n}$.

Proof. Let $Q_\nu = \{x \in Q \mid \operatorname{dist}(x, \partial Q) > 1/\nu\}$. Then $Q_\nu \subset\subset Q$ and $|Q\setminus Q_\nu| \to 0$ as $\nu \to \infty$. Choose a cut-off function $\zeta_\nu \in C_0^\infty(Q)$ such that

$$0 \le \zeta_\nu \le 1, \quad \zeta_\nu\big|_{Q_\nu} = 1, \quad M_\nu = \|D\zeta_\nu\|_{L^\infty} < \infty.$$

Since $z_k \to 0$ uniformly on Q we can choose a subsequence $\{k_\nu\}$ such that

$$\|z_{k_\nu}\|_{L^\infty} \le (M_\nu + 1)^{-1} \quad \forall\, \nu = 1, 2, ...$$

and we may also assume

$$\lim_{\nu\to\infty} \int_Q F(\xi + Dz_{k_\nu}(x))\,dx = \liminf_{k\to\infty} \int_Q F(\xi + Dz_k(x))\,dx.$$

Define $\phi_\nu = \zeta_\nu\, z_{k_\nu}$. Then $\phi_\nu \in W_0^{1,\infty}(Q; \mathbf{R}^N)$ and we can use them as test functions in the definition of quasiconvexity to obtain

$$\begin{aligned} |Q|\, F(\xi) &\le \int_Q F(\xi + D\phi_\nu(x))\,dx \\ &= \int_{Q_\nu} F(\xi + Dz_{k_\nu}) + \int_{Q\setminus Q_\nu} F(\xi + \zeta_\nu\, Dz_{k_\nu} + z_{k_\nu} \otimes D\zeta_\nu) \\ &= \int_Q F(\xi + Dz_{k_\nu}(x))\,dx + \epsilon_\nu, \end{aligned}$$

where

$$\epsilon_\nu = \int_{Q\setminus Q_\nu} [F(\xi + \zeta_\nu\, Dz_{k_\nu} + z_{k_\nu} \otimes D\zeta_\nu) - F(\xi + Dz_{k_\nu}(x))]\,dx.$$

Since $F(\xi)$ is bounded on bounded sets and $|Q \setminus Q_\nu| \to 0$ as $\nu \to \infty$, we easily have $\epsilon_\nu \to 0$ as $\nu \to \infty$. Therefore,

$$|Q|\, F(\xi) \le \liminf_{k\to\infty} \int_Q F(\xi + Dz_k(x))\,dx.$$

This completes the proof. □

We now prove the sufficiency of quasiconvexity for the lower semicontinuity of the functional

$$I(u) = \int_\Omega F(x, u(x), Du(x))\, dx$$

under the Lipschitz convergence on Ω.

Theorem 3.10. *Assume $F(x, s, \xi)$ is continuous on $\bar{\Omega} \times \mathbf{R}^N \times \mathbf{M}^{N\times n}$ and is* quasiconvex *in ξ. Then the functional I defined above is s.l.s.c. with respect to Lipschitz convergence on Ω.*

Proof. Let $\{z_k\}$ be any sequence converging to 0 in the sense of Lipschitz convergence on Ω, and let $u \in W^{1,\infty}(\Omega; \mathbf{R}^N)$ be any given function. We need to show

$$\int_\Omega F(x, u, Du) \le \liminf_{k\to\infty} \int_\Omega F(x, u + z_k, Du + Dz_k). \tag{3.6}$$

For any given $\epsilon > 0$, we choose finitely many disjoint cubes Q_j contained in Ω such that

$$I(u) \le \int_{\cup Q_j} F(x, u, Du)\, dx + \epsilon$$

and

$$I(u + z_k) \ge \int_{\cup Q_j} F(x, u + z_k, Du + Dz_k)\, dx - \epsilon,$$

for all $k = 1, 2, \cdots$. In what follows, we prove for each cube $Q = Q_j$

$$I_Q(u) \equiv \int_Q F(x, u, Du)\, dx \le \liminf_{k\to\infty} I_Q(u + z_k).$$

This, by Lemma 3.4, will certainly imply the conclusion of the theorem. To this end, for each positive integer ν, we divide Q into small cubes $\{Q_j^\nu\}$ with center $\bar{x}_j^\nu$ as in the proof of Theorem 3.6:

$$Q = \bigcup_{j=1}^{2^{n\nu}} Q_j^\nu \cup E, \quad |E| = 0.$$

Define

$$(u)_j^\nu = \fint_{Q_j^\nu} u(x)\, dx, \quad (Du)_j^\nu = \fint_{Q_j^\nu} Du(x)\, dx,$$

and

$$U^\nu(x) = \sum_{j=1}^{2^{n\nu}} (u)_j^\nu \cdot \chi_{Q_j^\nu}(x), \quad M^\nu = \sum_{j=1}^{2^{n\nu}} (Du)_j^\nu \cdot \chi_{Q_j^\nu}(x).$$

Note that

$$\|U^\nu\|_{L^\infty} + \|M^\nu\|_{L^\infty} \le \|u\|_{W^{1,\infty}}$$

and that the sequences $\{U^\nu\}$ and $\{M^\nu\}$ converge almost everywhere to u and Du on Q as $\nu \to \infty$, respectively. We now estimate $I_Q(u + z_k)$.

$$I_Q(u + z_k) = \int_Q F(x, u + z_k, Du + Dz_k) = a_k + b_k^\nu + c_k^\nu + d^\nu + I_Q(u),$$

where

$$\begin{aligned}
a_k &= \int_Q [F(x,u+z_k,Du+Dz_k)-F(x,u,Du+Dz_k)]\,dx,\\
b_k^\nu &= \sum_{j=1}^{2^{n\nu}} \int_{Q_j^\nu} [F(x,u,Du+Dz_k)-F(\bar{x}_j^\nu,(u)_j^\nu,(Du)_j^\nu+Dz_k)]\,dx,\\
c_k^\nu &= \sum_{j=1}^{2^{n\nu}} \int_{Q_j^\nu} [F(\bar{x}_j^\nu,(u)_j^\nu,(Du)_j^\nu+Dz_k)-F(\bar{x}_j^\nu,(u)_j^\nu,(Du)_j^\nu)]\,dx,\\
d^\nu &= \sum_{j=1}^{2^{n\nu}} \int_{Q_j^\nu} [F(\bar{x}_j^\nu,(u)_j^\nu,(Du)_j^\nu)-F(x,u,Du)]\,dx.
\end{aligned}$$

By the uniform continuity of $F(x,s,\xi)$ on bounded sets and the pointwise convergence of $\{U^\nu\}$ and $\{M^\nu\}$ we have

$$\lim_{k\to\infty} a_k = 0, \quad \lim_{\nu\to\infty} d^\nu = 0$$

and $\lim_{\nu\to\infty} b_k^\nu = 0$ uniformly with respect to k. We apply Theorem 3.9 to each Q_j^ν to obtain, by Lemma 3.4,

$$\liminf_{k\to\infty} c_k^\nu \geq 0$$

for all $\nu = 1,2,\cdots$. Therefore, again by Lemma 3.4,

$$\liminf_{k\to\infty} I_Q(u+z_k) \geq I_Q(u),$$

as desired. The proof is complete. □

Remarks. 1) Both Theorems 3.6 and 3.10 are valid also for *Carathéodory* functions $F(x,s,\xi)$.

2) Quasiconvexity is also the "right" condition for (sequential) lower semicontinuity of integral functionals in the weak topology of $W^{1,p}(\Omega;\mathbf{R}^N)$. The most general theorem in this direction is the following theorem due to Acerbi and Fusco. □

Theorem 3.11. *Let $F(x,s,\xi)$ be a* Carathéodory *function. Assume for some $1 \leq p < \infty$*

$$0 \leq F(x,s,\xi) \leq a(x) + C\,(|s|^p + |\xi|^p),$$

where $C > 0$ is a constant and $a(x) \geq 0$ is a locally integrable *function in Ω. Then functional $I(u) = \int_\Omega F(x,u,Du)\,dx$ is w.s.l.s.c. on $W^{1,p}(\Omega;\mathbf{R}^N)$* **if and only if** *$F(x,s,\xi)$ is quasiconvex in ξ.*

Theorem 3.12 (Existence of minimizers)**.** *Let $F(x,s,\xi)$ be* Carathéodory *and* quasiconvex *in ξ and satisfy*

$$\max\{0,\, c\,|\xi|^p - C\} \leq F(x,s,\xi) \leq a(x) + C\,(|s|^p + |\xi|^p)$$

for some $1 < p < \infty$, where $c > 0$, C are constants and $a(x) \geq 0$ is a locally integrable function in Ω. Then the minimization problem

$$\min_{u\in\mathcal{D}_\varphi^p(\Omega)} \int_\Omega F(x,u(x),Du(x))\,dx$$

has a minimizer for any given $\varphi \in W^{1,p}(\Omega;\mathbf{R}^N)$, where $\mathcal{D}_\varphi^p(\Omega)$ is the Dirichlet class *of φ defined before.*

3.4. Properties of quasiconvex functions

Again since quasiconvexity is only a property of a function $F(x,s,\xi)$ on ξ, for simplicity, we consider functions F depending only on ξ. Such a function F is *quasiconvex* if and only if

$$F(\xi) \le \fint_\Omega F(\xi + D\phi(x))\, dx, \quad \forall \xi \in \mathbf{M}^{N\times n} \tag{3.7}$$

holds for all $\phi \in C_0^\infty(\Omega;\mathbf{R}^N)$. We prove this property is independent of the domain Ω.

Theorem 3.13. *Let $F\colon \mathbf{M}^{N\times n} \to \mathbf{R}$ be continuous and* (3.7) *hold for all $\phi \in C_0^\infty(\Omega;\mathbf{R}^N)$. Then for any bounded open set $G \subset \mathbf{R}^n$ with $|\partial G| = 0$ one has*

$$F(\xi) \le \fint_G F(\xi + D\psi(y))\, dy, \quad \forall \xi \in \mathbf{M}^{N\times n} \tag{3.8}$$

holds for all $\psi \in C_0^\infty(G;\mathbf{R}^N)$.

Proof. Note that since F is continuous (3.7) holds for all $\phi \in W_0^{1,\infty}(\Omega;\mathbf{R}^N)$. Let $G \subset \mathbf{R}^n$ be any bounded open set with $|\partial G| = 0$, and $\psi \in C_0^\infty(G;\mathbf{R}^N)$ be any given test function. Assume $\bar y \in G$. For any $x \in \Omega$ and $\epsilon > 0$ let

$$G(x,\epsilon) = \{z \in \mathbf{R}^n \mid z = x + \epsilon\,(y - \bar y) \text{ for some } y \in G\}.$$

Then there exists an $\epsilon_x > 0$ such that $x \in \overline{G(x,\epsilon)} \subset \Omega$ for all $x \in \Omega$ and $0 < \epsilon < \epsilon_x$. This means the family

$$\{\overline{G(x,\epsilon)} \mid x \in \Omega,\ 0 < \epsilon < \epsilon_x\}$$

covers Ω in the sense of *Vitali covering*. Therefore, there exists a countable *disjoint* subfamily $\{G(x_j,\epsilon_j)\}$ and a set E of measure zero such that

$$\Omega = \bigcup_{j=1}^\infty \overline{G(x_j,\epsilon_j)} \cup E. \tag{3.9}$$

We now define a function $\phi\colon \Omega \to \mathbf{R}^N$ as follows.

$$\phi(x) = \begin{cases} 0 & \text{if } x \in \bigcup_{j=1}^\infty \partial[G(x_j,\epsilon_j)] \cup E, \\ \epsilon_j\, \psi\big(\bar y + \frac{x - x_j}{\epsilon_j}\big) & \text{if } x \in G(x_j,\epsilon_j) \text{ for some } j. \end{cases}$$

One can verify that $\phi \in W_0^{1,\infty}(\Omega;\mathbf{R}^N)$ and

$$D\phi(x) = D\psi\big(\bar y + \frac{x - x_j}{\epsilon_j}\big) \quad \forall x \in G(x_j,\epsilon_j).$$

Therefore, from (3.7), it follows that

$$\begin{aligned} F(\xi)\,|\Omega| &\le \int_\Omega F(\xi + D\phi(x))\,dx \\ &= \sum_{j=1}^\infty \int_{G(x_j,\epsilon_j)} F\left(\xi + D\psi\big(\bar y + \frac{x - x_j}{\epsilon_j}\big)\right) dx \\ &= \sum_{j=1}^\infty \epsilon_j^n \int_G F(\xi + D\psi(y))\,dy \\ &= \frac{|\Omega|}{|G|} \int_G F(\xi + D\psi(y))\,dy, \end{aligned}$$

where the last equality follows since, by (3.9), $\sum_{j=1}^\infty \epsilon_j^n = |\Omega|/|G|$. We have thus proved (3.8). ☐

In the following, let Σ be the unit cube in $\mathbf{R}^n$; that is

$$\Sigma = \{x \in \mathbf{R}^n \mid 0 < x_\alpha < 1,\ \alpha = 1, 2, \cdots, n\}.$$

Note that $|\Sigma| = 1$. Let $f\colon \mathbf{R}^n \to \mathbf{R}^N$ be any given function. We say f is **Σ-periodic** if $f(\cdots, x_\alpha, \cdots)$ is 1-periodic in x_α for all $\alpha = 1, 2, \cdots, n$. Quasicovexity can be also characterized by the following condition.

Theorem 3.14. *Let $F\colon \mathbf{M}^{N\times n} \to \mathbf{R}$ be continuous. Then F is quasiconvex if and only if*

$$F(\xi) \le \int_\Sigma F(\xi + D\phi(x))\, dx \quad \forall \xi \in \mathbf{M}^{N\times n} \tag{3.10}$$

for all Σ-periodic Lipschitz functions $\phi \in W^{1,\infty}(\mathbf{R}^n; \mathbf{R}^N)$.

Proof. Since any function $\psi \in C_0^\infty(\Sigma; \mathbf{R}^N)$ can be extended as a Σ-periodic function on $\mathbf{R}^n$ we easily see that (3.10) implies (3.8) for $G = \Sigma$ thus the quasiconvexity of F. We have only to prove (3.10) holds for all continuous quasiconvex functions F. Let $\phi \in W^{1,\infty}(\mathbf{R}^n; \mathbf{R}^N)$ be a Σ-periodic function. Define

$$\phi_j(x) = \frac{1}{j}\,\phi(jx)$$

for all $j = 1, 2, \cdots$. It is easily seen that $\phi_j \to 0$ in the sense of *Lipschitz convergence* on $W^{1,\infty}(\Sigma; \mathbf{R}^N)$. Therefore the theorem of Meyers, Theorem 3.9, and the quasiconvexity of F implies

$$F(\xi) \le \liminf_{j\to\infty} \int_\Sigma F(\xi + D\phi_j(x))\, dx.$$

Note that

$$\int_\Sigma F(\xi + D\phi_j(x))\, dx = \int_\Sigma F(\xi + D\phi(jx))\, dx$$
$$= j^{-n} \int_{j\Sigma} F(\xi + D\phi(y))\, dy$$

and that, besides a set of measure zero,

$$j\Sigma = \bigcup_{\nu=1}^{j^n} (\bar{x}_\nu + \Sigma),$$

where $\bar{x}_\nu$ are the left-lower corner points of the subcubes obtained by dividing the sides of $j\Sigma$ into j-equal subintervals. Since $D\phi(x)$ is Σ-periodic, we thus have

$$\int_{j\Sigma} F(\xi + D\phi(y))dy = \sum_{\nu=1}^{j^n} \int_{\bar{x}_\nu+\Sigma} F(\xi + D\phi(y))dy = j^n \int_\Sigma F(\xi + D\phi(x))dx,$$

and therefore

$$F(\xi) \le \int_\Sigma F(\xi + D\phi(x))\, dx$$

as needed; the proof is complete. □

Theorem 3.15. *Every continuous quasiconvex function is rank-one convex.*

Proof. There are many proofs for this result. We present a proof based on the previous theorem. Let F be a continuous quasiconvex function. We need to show that for any $\xi \in \mathbf{M}^{N\times n}$, $q \in \mathbf{R}^N$, $p \in \mathbf{R}^n$ the function $f(t) = F(\xi + tq\otimes p)$ is a convex function of $t \in \mathbf{R}$. Since $f(t)$ is continuous the convexity is equivalent to

$$f\left(\frac{t+s}{2}\right) \le \frac{f(t)+f(s)}{2} \quad \forall t,\, s \in \mathbf{R},$$

which is equivalent to

$$2\,F(\tilde{\xi}) \le F(\tilde{\xi} + \tilde{q} \otimes p) + F(\tilde{\xi} - \tilde{q} \otimes p),$$

where

$$\tilde{\xi} = \xi + \frac{t+s}{2}\, q \otimes p, \quad \tilde{q} = \frac{t-s}{2}\, q.$$

Therefore the theorem follows from the following result. □

Lemma 3.16. *Let F be a continuous quasiconvex function. Then*

$$2\,F(\xi) \le F(\xi + q \otimes p) + F(\xi - q \otimes p) \quad \forall \xi \in \mathbf{M}^{N\times n} \tag{3.11}$$

holds for all $q \in \mathbf{R}^N$ and $p \in \mathbf{R}^n$.

Proof. Since F is continuous it suffices to prove this inequality for all $p \in \mathbf{R}^n$ with all p_α being *rational numbers.* Let $\rho\colon \mathbf{R} \to \mathbf{R}$ be the "sawtooth" function which is 1-periodic such that $\rho(t) = t$ for $0 \le t \le \frac{1}{2}$ and $\rho(t) = 1 - t$ for $\frac{1}{2} \le t \le 1$. We choose a large *integer* T such that all Tp_α are integers. Define $\phi(x) = \frac{1}{T}\,\rho(Tx \cdot p)\, q$. Then it is easily seen that $\phi \in W^{1,\infty}(\mathbf{R}^n; \mathbf{R}^N)$ is Σ-*periodic* and

$$D\phi(x) = \rho'(Tx \cdot p)\, q \otimes p.$$

By Theorem 3.14 we have

$$\begin{aligned} F(\xi) &\le \int_\Sigma F(\xi + D\phi(x))\, dx \\ &= |\Sigma \cap \{\rho'(Tx\cdot p) = 1\}|\, F(\xi + q \otimes p) \\ &\quad + |\Sigma \cap \{\rho'(Tx\cdot p) = -1\}|\, F(\xi - q \otimes p). \end{aligned}$$

Since it is easy to see that

$$H(x_1, x_2, \cdots, x_n) = (1 - x_1,\, 1 - x_2, \cdots, 1 - x_n)$$

defines a diffeomorphism from $\Sigma \cap \{\rho'(Tx\cdot p) = 1\}$ onto $\Sigma \cap \{\rho'(Tx\cdot p) = -1\}$, we have thus

$$|\Sigma \cap \{\rho'(Tx\cdot p) = 1\}| = |\Sigma \cap \{\rho'(Tx\cdot p) = -1\}| = 1/2.$$

The proof is thus complete. □

Remark. Another proof is as follows. Let F be quasiconvex. If F is of class C^2, then

$$f(t) = \int_\Omega F(\xi + tD\phi(x))\, dx$$

takes its minimum at $t = 0$. Therefore $f''(0) \ge 0$; that is,

$$\int_\Omega F_{\xi^i_\alpha \xi^j_\beta}(\xi)\, D_\alpha \phi^i(x) D_\beta \phi^j(x)\, dx \ge 0$$

for all $\phi \in C_0^\infty(\Omega; \mathbf{R}^N)$. This implies the *weak Legendre-Hadamard* condition

$$F_{\xi^i_\alpha \xi^j_\beta}(\xi)\, q^i q^j\, p_\alpha p_\beta \ge 0.$$

This shows that F is *rank-one convex*; that is, F satisfies (3.11). If F is only continuous, then, for any $\epsilon > 0$, let $F^\epsilon = F * \rho_\epsilon$ be the regularization of F. Then F^ϵ is of class C^∞ and can be shown to be quasiconvex, and hence F^ϵ satisfies (3.11) for all $\epsilon > 0$; letting $\epsilon \to 0$ yields that F satisfies (3.11) and thus is rank-one convex. □

Lemma 3.17 (Jensen's inequality)**.** *Let (E, μ) be a measure space with total mass $\mu(E) = 1$ and let $h \colon E \to \mathbf{R}^L$ be an integrable function on E. If $G \colon \mathbf{R}^L \to \mathbf{R}$ is a convex function, then*

$$G\left(\int_E h(x)\, d\mu\right) \le \int_E G(h(x))\, d\mu.$$

Proof. Since $G \colon \mathbf{R}^L \to \mathbf{R}$ is convex, for each $F \in \mathbf{R}^L$ there exists $l_F \in \mathbf{R}^L$ (note that $l_F = DG(F)$ for almost every F) such that

$$G(A) \ge G(F) + l_F \cdot (A - F) \quad \forall A \in \mathbf{R}^L.$$

Let $F = \int_E h(x)\, d\mu$. Then $G(h(x)) \ge G(F) + l_F \cdot (h(x) - F)$ for all $x \in E$, and integrating over $x \in E$ yields

$$\int_E G(h(x))\, d\mu \ge G(F) + l_F \cdot \int_E (h(x) - F)\, d\mu = G(F),$$

which proves Jensen's inequality. □

Lemma 3.18 (Divergence Theorem)**.** *Let Ω be a domain with smooth boundary $\partial\Omega$ and let $\nu(x) = (\nu_1, ..., \nu_n)$ be the unit outer normal and dS the surface integral element on $\partial\Omega$. Then for any vector field $\phi \in C^1(\bar{\Omega}; \mathbf{R}^N)$ one has*

$$\int_\Omega D\phi(x)\, dx = \int_{\partial\Omega} \phi(x) \otimes \nu(x)\, dS.$$

Proposition 3.19. *If $F(\xi)$ is convex in ξ then F is quasiconvex.*

Proof. This follows easily from Jensen's inequality and the divergence theorem given above. □

3.5. Polyconvex functions and null-Lagrangians

Unlike the convexity and rank-one convexity, quasiconvexity is a global property since the inequality (3.7) is required to hold for all test functions. It is thus generally impossible to verify whether a given function $F(\xi)$ is quasiconvex. We have already seen that every convex function is quasiconvex. However, there is a class of functions which are quasiconvex but not necessarily convex. This class, mainly due to Morrey, has been called the **polyconvex** functions by Ball.

In order to introduce the *polyconvex* functions of Ball, we need some notation. Let $\sigma = \min\{n, N\}$. For each integer $k \in [1, \sigma]$, and any two ordered sequences of integers

$$1 \le i_1 < i_2 < \cdots < i_k \le N, \quad 1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_k \le n,$$

let $J^{i_1 i_2 \cdots i_k}_{\alpha_1 \alpha_2 \cdots \alpha_k}(\xi)$ be the *determinant* of the $k \times k$ matrix whose (q, p) position element is $\xi^{i_q}_{\alpha_p}$ for each $1 \le p,\, q \le k$. Note that, by the usual notation,

$$J^{i_1 i_2 \cdots i_k}_{\alpha_1 \alpha_2 \cdots \alpha_k}(Du(x)) = \frac{\partial(u^{i_1}, u^{i_2}, \cdots, u^{i_k})}{\partial(x_{\alpha_1}, x_{\alpha_2}, \cdots, x_{\alpha_k})} = \det\left(\frac{\partial u^{i_q}}{\partial x_{\alpha_p}}\right).$$

Let $\mathcal{J}(\xi)$ be the collection of all $J^{i_1 i_2 \cdots i_k}_{\alpha_1 \alpha_2 \cdots \alpha_k}(\xi)$ for all $k \in [1, \sigma]$ and all ordered integral sequences $\{i_q\}$, $\{\alpha_p\}$. We embed $\mathcal{J}(\xi)$ to a large dimensional $\mathbf{R}^L$ with the same rule for all ξ, where

$$L = L(n, N) = \sum_{k=1}^{\sigma} \binom{N}{k} \binom{n}{k}.$$

In this way, we have defined a map $\mathcal{J} \colon \mathbf{M}^{N \times n} \to \mathbf{R}^L$.

Definition 3.2. A function $F(\xi)$ on $\mathbf{M}^{N\times n}$ is called a **polyconvex** function if there exists a *convex* function $G\colon \mathbf{R}^L \to \mathbf{R}$ such that $F(\xi) = G(\mathcal{J}(\xi))$ for all $\xi \in \mathbf{M}^{N\times n}$; that is, $F = G \circ \mathcal{J}$ on $\mathbf{M}^{N\times n}$.

Remark. For a polyconvex function we may have different convex functions in its representation. For example, let $n = N = 2$ and $F(\xi) = |\xi|^2 - \det\xi$. In this case, consider $\mathcal{J}(\xi) = (\xi, \det\xi) \in \mathbf{R}^5$. Then we have

$$F(\xi) = G_1(\mathcal{J}(\xi)), \quad F(\xi) = G_2(\mathcal{J}(\xi)),$$

where

$$G_1(\xi, t) = |\xi|^2 - t, \quad G_2(\xi, t) = (\xi_1^1 - \xi_2^2)^2 + (\xi_1^2 + \xi_2^1)^2 + t$$

are both convex functions of (ξ, t). □

We need following result to study the properties of polyconvex functions.

Theorem 3.20. *Let $\mathcal{J}(\xi)$ be defined as above, and let Σ be the unit cube in $\subset \mathbf{R}^n$. Then it follows that*

$$\int_\Sigma \mathcal{J}\big(\xi + D\phi(x)\big)\,dx = \mathcal{J}(\xi)$$

for all $\phi \in C_0^\infty(\Sigma; \mathbf{R}^N)$ and $\xi \in \mathbf{M}^{N\times n}$.

Proof. Since each $\mathcal{J}(\xi)$ is given by a $k\times k$-determinant, without loss of generality, we only prove this identity for $\mathcal{J}(\xi) = J_k(\xi) = J^{12\cdots k}_{12\cdots k}(\xi)$, where $1 \le k \le \sigma = \min\{n, N\}$. For simplicity, let $u(x) = \xi x + \phi(x)$. Let

$$x' = (x_1, \cdots, x_k), \quad x'' = (x_{k+1}, \cdots, x_n) \text{ if } k+1 \le n.$$

Let Σ', Σ'' be the unit cubes in x', x'' variables, respectively. Fix $x'' \in \Sigma''$, for $t \ge 0$, consider maps V_t, $U_t\colon \Sigma' \to \mathbf{R}^k$ such that

$$V_t^i(x') = tx_i + (\xi x)^i, \quad U_t^i(x') = tx_i + u^i(x', x'').$$

We can choose $t > 0$ sufficiently large so that V_t, U_t are both *diffeomorphisms* on Σ', and therefore

$$\int_{\Sigma'} \det(DU_t(x'))\,dx' = \int_{U_t(\Sigma')} dy' = \int_{V_t(\Sigma')} dy' = \int_{\Sigma'} \det(DV_t(x'))\,dx'.$$

Since both sides are polynomials of t of degree k, it follows that this equality holds for all t. When $t = 0$ this implies

$$\int_{\Sigma'} J_k(\xi + D\phi(x', x''))\,dx' = \int_{\Sigma'} J_k(\xi)\,dx'. \tag{3.12}$$

Integrating (3.12) over $x'' \in \Sigma''$ we deduce

$$\int_\Sigma J_k(\xi + D\phi(x))\,dx = J_k(\xi), \tag{3.13}$$

completing the proof. □

Theorem 3.21. *A polyconvex function is quasiconvex.*

Proof. Let $F\colon \mathbf{M}^{N\times n}\to\mathbf{R}$ be a polyconvex function. Then there exists a convex function $G\colon \mathbf{R}^L\to\mathbf{R}$ such that $F(\xi)=G(\mathcal{J}(\xi))$ for all ξ. Given $\xi\in\mathbf{M}^{N\times n}$ and $\phi\in C_0^\infty(\Sigma;\mathbf{R}^N)$, let $h(x)=\mathcal{J}(\xi+D\phi(x))$. Then Jensen's inequality implies

$$G\left(\int_\Sigma h(x)\,dx\right)\le\int_\Sigma G(h(x))\,dx.$$

By the theorem above, the lefthand side is $G(\mathcal{J}(\xi))=F(\xi)$ and therefore

$$F(\xi)\le\int_\Sigma G(h(x))\,dx=\int_\Sigma G(\mathcal{J}(\xi+D\phi(x)))\,dx=\int_\Sigma F(\xi+D\phi(x))\,dx,$$

proving that F is quasiconvex. □

Note that, using the method in the proof of Theorem 3.20, we also see that

$$\int_\Sigma \mathcal{J}(Du(x))\,dx=\int_\Sigma \mathcal{J}(Du(x)+D\phi(x))\,dx$$

for all $u\in C^1(\bar\Sigma;\mathbf{R}^N)$ and $\phi\in C_0^\infty(\Sigma;\mathbf{R}^N)$. This property introduces the following definition.

Definition 3.3. A function $F\colon\mathbf{M}^{N\times n}\to\mathbf{R}$ is called a **null-Lagrangian** on $\mathbf{M}^{N\times n}$ if

$$\int_\Sigma F\big(Du(x)+D\phi(x)\big)\,dx=\int_\Sigma F(Du(x))\,dx$$

holds for all $u\in C^1(\bar\Sigma;\mathbf{R}^N)$ and $\phi\in C_0^\infty(\Sigma;\mathbf{R}^N)$.

Theorem 3.22. *Let $F\colon\mathbf{M}^{N\times n}\to\mathbf{R}$ be continuous and let*

$$I_\Omega(u)=\int_\Omega F(Du(x))\,dx,$$

where Ω is any smooth bounded domain in $\mathbf{R}^n$. Then the following conditions are equivalent:

(1) *F is a* null-Lagrangian *on $\mathbf{M}^{N\times n}$;*
(2) *$F(\xi)=\int_\Sigma F(\xi+D\varphi(x))\,dx$ for all $\xi\in\mathbf{M}^{N\times n}$ and $\varphi\in C_0^\infty(\Sigma;\mathbf{R}^N)$;*
(3) *F is of C^1 and the* Euler-Lagrange *equation for functional I_Ω is satisfied by all functions in $C^1(\bar\Omega;\mathbf{R}^N)$;*
(4) *the functional I_Ω is* continuous *with respect to the Lipschitz convergence on $W^{1,\infty}(\Omega;\mathbf{R}^N)$.*

Proof. Note that (3) is the reason for the name of *null-Lagrangians.* We now prove the theorem. It is easy to see (1) $\Longrightarrow$ (2). Note that (2) is equivalent to that both F and $-F$ are quasiconvex; the latter is equivalent to that both I_Ω and $-I_\Omega$ are lower semicontinuous with respect to the Lipschitz convergence on $W^{1,\infty}(\Omega;\mathbf{R}^N)$. Therefore, (4) $\Longleftrightarrow$ (2). It remains to show (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1). Let us first prove (3) $\Longrightarrow$ (1). To this end, given $u\in C^1(\bar\Sigma;\mathbf{R}^N)$ and $\phi\in C_0^\infty(\Sigma;\mathbf{R}^N)$, let

$$f(t)=\int_\Sigma F(Du(x)+tD\phi(x))\,dx.$$

Then (3) implies

$$f'(t)=\int_\Sigma F_{\xi^i_\alpha}(D(u+t\phi)(x))\,D_\alpha\phi^i(x)\,dx=0,$$

and thus $f(1)=f(0)$; therefore, F is null-Lagrangian and hence (3) $\Longrightarrow$ (1). The proof of (2) $\Longrightarrow$ (3) will follow from several lemmas proved below. □

Lemma 3.23. *If F is of C^∞, then* (2) $\Longrightarrow$ (3).

Proof. Note that (2) implies

$$\int_\Omega F(\xi + D\phi(x))\,dx = F(\xi)\,|\Omega| \quad \forall\, \phi \in C_0^\infty(\Omega; \mathbf{R}^N).$$

Since F is of C^∞, this implies that, for any $\phi,\, \psi \in C_0^\infty(\Omega; \mathbf{R}^N)$, the function

$$f(t) = \int_\Omega F(t\,D\phi(x) + D\psi(x))\,dx$$

is constant and of C^1. Therefore, $f'(0) = 0$, which gives

$$\int_\Omega F_{\xi^i_\alpha}(D\psi(x))\,D_\alpha\phi^i(x)\,dx = 0.$$

Now given $u \in C^1(\bar\Omega; \mathbf{R}^N)$, we can select a sequence $\{\psi_j\}$ in $C_0^\infty(\Omega; \mathbf{R}^N)$ such that $\psi_j \to u$ in $C^1(\operatorname{supp}\phi; \mathbf{R}^N)$. Using the identity above with $\psi = \psi_j$ and letting $j \to \infty$ yield

$$\int_\Omega F_{\xi^i_\alpha}(Du(x))\,D_\alpha\phi^i(x)\,dx = 0.$$

This shows the Euler-Lagrange equation for I_Ω holds for any $u \in C^1(\bar\Omega; \mathbf{R}^N)$, and thus (3) follows. □

From this lemma, the proof of (2) $\Longrightarrow$ (3) will be complete if we show that (2) $\Longrightarrow$ F is of C^∞. In fact, we can prove the following.

Proposition 3.24. *If F satisfies (2) then $F(\xi)$ is a polynomial in ξ.*

This result will be proved later. We say F is **rank-one affine** if $F(\xi + tq \otimes p)$ is affine in t for all $\xi \in \mathbf{M}^{N\times n}$, $q \in \mathbf{R}^N$, $p \in \mathbf{R}^n$.

Lemma 3.25. *If F satisfies (2) then F is rank-one affine.*

Proof. Since (2) $\Longleftrightarrow$ that both F and $-F$ are quasiconvex, by Theorem 3.15, (2) implies the fact that both F and $-F$ are rank-one convex, which is equivalent to that F is rank-one affine. □

We need some notation. Let $\mu^i_\alpha = e^i \otimes e_\alpha$, where $\{e^i\}$ and $\{e_\alpha\}$ are the standard bases of $\mathbf{R}^N$ and $\mathbf{R}^n$, respectively. For each $1 \le k \le \sigma = \min\{n, N\}$ and $1 \le i_1, \cdots, i_k \le N$, $1 \le \alpha_1, \cdots, \alpha_k \le n$, we define, inductively,

$$F^{i_1}_{\alpha_1}(\xi) = F(\xi + \mu^{i_1}_{\alpha_1}) - F(\xi),$$

$$F^{i_1\cdots i_k}_{\alpha_1\cdots\alpha_k}(\xi) = F^{i_1\cdots i_{k-1}}_{\alpha_1\cdots\alpha_{k-1}}(\xi + \mu^{i_k}_{\alpha_k}) - F^{i_1\cdots i_{k-1}}_{\alpha_1\cdots\alpha_{k-1}}(\xi).$$

Note that if F is a polynomial it follows

$$F^{i_1\cdots i_k}_{\alpha_1\cdots\alpha_k}(\xi) = \partial^k F(\xi)/\partial\xi^{i_1}_{\alpha_1}\cdots\partial\xi^{i_k}_{\alpha_k}.$$

Indeed, we have the same permutation invariance property.

Lemma 3.26. *Let $\{1', \cdots, k'\}$ be any permutation of $\{1, \cdots, k\}$. Then*

$$F^{i_1\cdots i_k}_{\alpha_1\cdots\alpha_k}(\xi) = F^{i_{1'}\cdots i_{k'}}_{\alpha_{1'}\cdots\alpha_{k'}}(\xi).$$

Proof. We use induction on k. Assume this is true for all $k \le s-1$. Suppose $\{1', 2', \cdots, s'\}$ is a permutation of $\{1, 2, \cdots, s\}$. We need to show

$$F^{i_1 \cdots i_s}_{\alpha_1 \cdots \alpha_s}(\xi) = F^{i_{1'} \cdots i_{s'}}_{\alpha_{1'} \cdots \alpha_{s'}}(\xi). \tag{3.14}$$

By definition

$$F^{i_1 \cdots i_s}_{\alpha_1 \cdots \alpha_s}(\xi) = F^{i_1 \cdots i_{s-1}}_{\alpha_1 \cdots \alpha_{s-1}}(\xi + \mu^{i_s}_{\alpha_s}) - F^{i_1 \cdots i_{s-1}}_{\alpha_1 \cdots \alpha_{s-1}}(\xi).$$

By induction assumption, (3.14) holds if $s' = s$. We thus assume $s' < s$. In this case, by induction assumption,

$$F^{i_1 \cdots i_s}_{\alpha_1 \cdots \alpha_s}(\xi) = F^{i_1 \cdots \hat{i}_{s'} \cdots i_{s'}}_{\alpha_1 \cdots \hat{\alpha}_{s'} \cdots \alpha_{s'}}(\xi + \mu^{i_s}_{\alpha_s}) - F^{i_1 \cdots \hat{i}_{s'} \cdots i_{s'}}_{\alpha_1 \cdots \hat{\alpha}_{s'} \cdots \alpha_{s'}}(\xi)$$

(where the $\hat{m}$ means omitting m)

$$\begin{aligned} &= F^{i_1 \cdots \hat{i}_{s'} \cdots}_{\alpha_1 \cdots \hat{\alpha}_{s'} \cdots}(\xi + \mu^{i_s}_{\alpha_s} + \mu^{i_{s'}}_{\alpha_{s'}}) - F^{i_1 \cdots \hat{i}_{s'} \cdots}_{\alpha_1 \cdots \hat{\alpha}_{s'} \cdots}(\xi + \mu^{i_s}_{\alpha_s}) \\ &\quad - F^{i_1 \cdots \hat{i}_{s'} \cdots}_{\alpha_1 \cdots \hat{\alpha}_{s'} \cdots}(\xi + \mu^{i_{s'}}_{\alpha_{s'}}) + F^{i_1 \cdots \hat{i}_{s'} \cdots}_{\alpha_1 \cdots \hat{\alpha}_{s'} \cdots}(\xi) \\ &= F^{i_1 \cdots \hat{i}_{s'} \cdots i_s}_{\alpha_1 \cdots \hat{\alpha}_{s'} \cdots \alpha_s}(\xi + \mu^{i_{s'}}_{\alpha_{s'}}) - F^{i_1 \cdots \hat{i}_{s'} \cdots i_s}_{\alpha_1 \cdots \hat{\alpha}_{s'} \cdots \alpha_s}(\xi), \end{aligned}$$

which, by induction assumption, equals

$$F^{i_{1'} \cdots i_{(s-1)'}}_{\alpha_{1'} \cdots \alpha_{(s-1)'}}(\xi + \mu^{i_{s'}}_{\alpha_{s'}}) - F^{i_{1'} \cdots i_{(s-1)'}}_{\alpha_{1'} \cdots \alpha_{(s-1)'}}(\xi) = F^{i_{1'} \cdots i_{s'}}_{\alpha_{1'} \cdots \alpha_{s'}}(\xi).$$

This proves the induction procedure and hence the lemma. □

Lemma 3.27. *If F is rank-one affine, then all $F^{i_1 \cdots i_k}_{\alpha_1 \cdots \alpha_k}$ are also rank-one affine. Moreover,*

$$F^{i_1 \cdots i_k}_{\alpha_1 \cdots \alpha_k}(\xi + t\, \mu^j_\beta) = F^{i_1 \cdots i_k}_{\alpha_1 \cdots \alpha_k}(\xi) + t\, F^{i_1 \cdots i_k j}_{\alpha_1 \cdots \alpha_k \beta}(\xi). \tag{3.15}$$

Proof. Use induction again on k. □

Proof of Proposition 3.24. If F satisfies (2) then F is rank-one affine. Write

$$\xi = \sum_{i=1}^N \sum_{\alpha=1}^n \xi^i_\alpha\, \mu^i_\alpha.$$

Then a successive use of the previous lemma shows that $F(\xi)$ is a polynomial of degree at most nN in ξ with coefficients determined by $F^{i_1 \cdots i_k}_{\alpha_1 \cdots \alpha_k}(0)$. □

Remark. In fact, one can prove that a rank-one affine function is also a null-Lagrangian. The proof involves some very complicated computations; we do not intend to present it here. Also, the following result actually characterizes all null-Lagrangians; the proof of this theorem is beyond this lecture. □

Theorem 3.28. *Let $F\colon \mathbf{M}^{N\times n} \to \mathbf{R}$ be continuous. Then F is a null-Lagrangian if and only if there exists an* affine *function $L\colon \mathbf{R}^L \to \mathbf{R}$ such that $F = L \circ \mathcal{J}$.*

We prove a compensated compactness property of the null-Lagrangians. For nonlinear operators with similar properties we refer to Coifman et al.

Theorem 3.29. *Let $J_k(Du)$ be any $k \times k$ subdeterminant. Let $\{u_j\}$ be any sequence weakly convergent to $\bar{u}$ in $W^{1,k}(\Omega; \mathbf{R}^N)$ as $j \to \infty$. Then $J_k(Du_j) \to J_k(D\bar{u})$ in the sense of distribution in Ω; that is,*

$$\lim_{j\to\infty} \int_\Omega J_k(Du_j(x))\, \phi(x)\, dx = \int_\Omega J_k(D\bar{u}(x))\, \phi(x)\, dx$$

for all $\phi \in C^\infty_0(\Omega)$.

Proof. We prove this by induction on k. Obviously, the theorem is true when $k = 1$. Assume it holds for J_s with $s \le k-1$. We need to show it also holds for $s = k$. Without loss of generality, we may assume

$$J_k(Du(x)) = \frac{\partial(u^1, u^2, \cdots, u^k)}{\partial(x_1, x_2, \cdots, x_k)}.$$

For any smooth function u, we observe that $J_k(Du)$ is actually a divergence:

$$J_k(Du(x)) = \sum_{\nu=1}^{k} \frac{\partial}{\partial x_\nu}\left(u^1 \frac{(-1)^{\nu+1}\partial(u^2, \cdots, u^k)}{\partial(x_1, \cdots, \hat{x_\nu}, \cdots, x_k)}\right), \tag{3.16}$$

where $\hat{x_\nu}$, again, means deleting x_ν. Let

$$J^{(\nu)}_{k-1}(Du(x)) = \frac{(-1)^{\nu+1}\partial(u^2, \cdots, u^k)}{\partial(x_1, \cdots, \hat{x_\nu}, \cdots, x_k)}.$$

Then (3.16) implies

$$\int_\Omega J_k(Du(x))\,\phi(x)\,dx = \sum_{\nu=1}^{k}\int_\Omega u^1(x)\,J^{(\nu)}_{k-1}(D(x))\,D_\nu\phi(x)\,dx. \tag{3.17}$$

By density argument, this identity still holds if $u \in W^{1,k}(\Omega; \mathbf{R}^N)$. Suppose $u_j \rightharpoonup \bar{u}$ in $W^{1,k}(\Omega; \mathbf{R}^N)$. By the Sobolev embedding theorem, $u_j \to \bar{u}$ in $L^k(\Omega; \mathbf{R}^N)$. Moreover, by the induction assumption, $J^{(\nu)}_{k-1}(Du_j) \to J^{(\nu)}_{k-1}(D\bar{u})$ in the sense of distribution. Note that since sequence $\{J^{(\nu)}_{k-1}(Du_j)\}$ is also bounded in $L^{\frac{k}{k-1}}(\Omega)$ it also weakly converges in $L^{\frac{k}{k-1}}(\Omega)$; by density the weak limit must be $J^{(\nu)}_{k-1}(D\bar{u})$. We can then apply (3.17) to conclude

$$\lim_{j\to\infty}\int_\Omega J_k(Du_j(x))\,\phi(x)\,dx = \int_\Omega J_k(D\bar{u}(x))\,\phi(x)\,dx,$$

as desired. The proof is complete. □

Remark. Although, when $u_j \rightharpoonup \bar{u}$ in $W^{1,k}(\Omega; \mathbf{R}^N)$, it follows $J_k(Du_j) \to J_k(D\bar{u})$ in the sense of distribution and $\{J_k(Du_j)\}$ is bounded in $L^1(\Omega)$, it is **not true** that $J_k(Du_j) \rightharpoonup J_k(D\bar{u})$ weakly in $L^1(\Omega)$. The following example is due to Ball and Murat. □

Example 3.4 (Ball and Murat '84)**.** Let B be the unit open ball in $\mathbf{R}^n$. Consider, for $j = 1, 2, \cdots$, the *radial* mappings

$$u_j(x) = \frac{U_j(|x|)}{|x|}x, \quad U_j(r) = \begin{cases} jr & \text{if } 0 \le r \le 1/j, \\ 2 - jr & \text{if } 1/j \le r \le 2/j, \\ 0 & \text{if } 2/j \le r < 1. \end{cases}$$

Computation shows that $u_j \rightharpoonup 0$ in $W^{1,n}(B; \mathbf{R}^n)$ as $j \to \infty$. But

$$\det Du_j(x) = (U_j(r)/r)^{n-1}U_j'(r)$$

for a.e. $x \in B$, where $r = |x|$, and hence

$$\int_{|x|<2/j} |\det Du_j(x)|\,dx = C$$

is a constant independent of j. This shows that $\{\det Du_j\}$ is not equi-integrable in B, and therefore it does not converges weakly in $L^1(B)$, or in $L^1_{loc}(B)$. The last observation is in sharp contrast to a well-known result of Müller, which states that if $\det Du_j(x) \ge 0$ a.e. in Ω and $u_j \to \bar{u}$ weakly in $W^{1,n}(\Omega; \mathbf{R}^n)$ then $\det Du_j \rightharpoonup \det D\bar{u}$ weakly in $L^1_{loc}(\Omega)$.

3.6. Questions and examples

We have introduced several convexity conditions: convex, rank-one convex, quasiconvex, polyconvex. For convexity and rank-one convexity we have the strict versions before, which are equivalent to the Legendre condition and Legendre-Hadamard condition (with $\nu > 0$), respectively. For quasiconvexity, Evans also introduced a strict version.

Definition 3.5. A function $F\colon \mathbf{M}^{N\times n} \to \mathbf{R}$ is called **uniformly strict quasiconvex** in $W^{1,p}$ provided that

$$\int_\Omega [F(\xi + D\phi(x)) - F(\xi)]\,dx \geq \gamma \int_\Omega |D\phi(x)|^p\,dx,$$

where $\gamma > 0$ is a constant, holds for all smooth domains $\Omega \subset \mathbf{R}^n$ and all $\phi \in W_0^{1,p}(\Omega; \mathbf{R}^N)$. This condition is useful for regularity theory of minimizers, which we will do later.

In our study so far, we have assumed $N \geq 2$. But if $N = 1$ or $n = 1$ we can easily see that all these convexities are the same as the usual convex condition. However, if $n, N \geq 2$, we have known that for continuous functions with finite values

$$\text{convex} \Longrightarrow \text{polyconvex} \Longrightarrow \text{quasiconvex} \Longrightarrow \text{rank-one convex}.$$

We give some examples which show that these are in general truly different conditions.

Example 3.6 (Dacorogna and Marcellini '88)**.** Let $n = N = 2$ and

$$F_\delta(\xi) = |\xi|^4 - \delta\,|\xi|^2 \det \xi.$$

Then

$$F_\delta \text{ is } \begin{cases} \text{convex} & \Longleftrightarrow \quad |\delta| \leq \frac{4}{3}\sqrt{2} \approx 1.89, \\ \text{polyconvex} & \Longleftrightarrow \quad |\delta| \leq 2, \\ \text{quasiconvex} & \Longleftrightarrow \quad |\delta| \leq 2 + \epsilon, \\ \text{rank-one convex} & \Longleftrightarrow \quad |\delta| \leq \frac{4}{3}\sqrt{3} \approx 2.31. \end{cases}$$

It is known that $\epsilon > 0$ for the quasiconvexity; whether or not $2 + \epsilon = \frac{4}{\sqrt{3}}$ is still open. Note that this example gives an explicit quasiconvex function which is not polyconvex. But it leaves open whether a rank-one convex function is quasiconvex.

Proposition 3.30. *If F is a polynomial of degree two (quadratic polynomial) then F is quasiconvex $\Longleftrightarrow$ F is rank-one convex.*

Proof. Assume F is a rank-one convex quadratic polynomial. We show F is quasiconvex. Since subtraction of an affine function from a function does not change the quasiconvexity or rank-one convexity, we thus assume F is a homogeneous quadratic polynomial given by

$$F(\xi) = A_{ij}^{\alpha\beta}\xi_\alpha^i \xi_\beta^j \quad \text{(summation notation is used here and below)}$$

with $A_{ij}^{\alpha\beta}$ are constants. Then

$$F(\xi + \eta) = F(\xi) + A_{ij}^{\alpha\beta}\xi_\alpha^i \eta_\beta^j + A_{ij}^{\alpha\beta}\xi_\beta^j \eta_\alpha^i + A_{ij}^{\alpha\beta}\eta_\alpha^i \eta_\beta^j. \tag{3.18}$$

Note that the rank-one convexity is equivalent to

$$A_{ij}^{\alpha\beta} q^i q^j\, p_\alpha p_\beta \geq 0.$$

Hence the linear system defined by constants $\tilde{A}_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta} + \epsilon\delta_{ij}\delta^{\alpha\beta}$ satisfies the Legendre-Hadamard condition, where $\epsilon > 0$ and δ_{ij}, $\delta^{\alpha\beta}$ are the usual delta notation. By a theorem

we prove before, the bilinear form defined by $\tilde{A}_{ij}^{\alpha\beta}$ is coercive in $W_0^{1,2}(\mathbf{R}^n;\mathbf{R}^N)$; this means that
$$\int_{\mathbf{R}^n} \tilde{A}_{ij}^{\alpha\beta} D_\alpha\phi^i(x) D_\beta\phi^j(x)\,dx \ge \epsilon \int_{\mathbf{R}^n} |D\phi(x)|^2\,dx$$
for all $\phi \in C_0^\infty(\mathbf{R}^n;\mathbf{R}^N)$; hence, by cancelling the ϵ-terms both sides, we have
$$\int_{\mathbf{R}^n} A_{ij}^{\alpha\beta} D_\alpha\phi^i(x) D_\beta\phi^j(x)\,dx \ge 0.$$
Therefore by (3.18)
$$\int_\Sigma F(\xi + D\phi(x))\,dx = F(\xi) + \int_{\mathbf{R}^n} A_{ij}^{\alpha\beta} D_\alpha\phi^i(x) D_\beta\phi^j(x)\,dx \ge F(\xi),$$
as required. The proof is complete. □

Proposition 3.31. *A rank-one convex third degree polynomial must be a null-Lagrangian and thus quasiconvex.*

Proof. Let F be a rank-one convex *third degree* polynomial. Then the polynomial $f(t) = F(\xi + tq \otimes p)$ is convex and of degree ≤ 3 in t, and hence the degree of $f(t)$ cannot be 3. Note that the coefficient of t^2 term in f is half of

$$F_{\xi^i_\alpha \xi^j_\beta}(\xi)\, p_\alpha p_\beta\, q^i q^j \ge 0, \tag{3.19}$$

which holds for all ξ, p, q. Since $F_{\xi^i_\alpha \xi^j_\beta}(\xi)$ is linear in ξ, condition (3.19) implies $F_{\xi^i_\alpha \xi^j_\beta}(\xi)\, p_\alpha p_\beta\, q^i q^j \equiv 0$. Therefore $f(t) = F(\xi + tq \otimes p)$ is affine in t and hence F is *rank-one affine.* Consequently, the result follows from the fact that a rank-one affine function must be a null-Lagrangian. □

The following now classical example of Šverák settles a long-standing open problem raised by Morrey in early fifties.

Theorem 3.32 (Šverák '92)**.** *If $n \ge 2$, $N \ge 3$ then there exists a rank-one convex function $F\colon \mathbf{M}^{N\times n} \to \mathbf{R}$ which is not quasiconvex.*

Proof. We only prove the theorem for $n = 2$, $N = 3$. Consider the periodic function $u\colon \mathbf{R}^2 \to \mathbf{R}^3$ by
$$u(x) = \frac{1}{2\pi}(\sin 2\pi x_1,\, \sin 2\pi x_2,\, \sin 2\pi(x_1 + x_2)).$$
Then
$$Du(x) = \begin{pmatrix} \cos 2\pi x_1 & 0 \\ 0 & \cos 2\pi x_2 \\ \cos 2\pi(x_1+x_2) & \cos 2\pi(x_1+x_2) \end{pmatrix}.$$
Let L be the linear span of the values of $Du(x)$ in $\mathbf{M} = \mathbf{M}^{3\times 2}$; that is,
$$L = \left\{ [r,s,t] \equiv \begin{pmatrix} r & 0 \\ 0 & s \\ t & t \end{pmatrix} \;\Big|\; r,\, s,\, t \in \mathbf{R} \right\}.$$
Note that a matrix $\xi = [r,s,t] \in L$ is of rank ≤ 1 if and only if at most one of $\{r,s,t\}$ is nonzero. Define $g\colon L \to \mathbf{R}$ by $g([r,s,t]) = -rst$. Using formula $2\cos\alpha \cdot \cos\beta = \cos(\alpha+\beta) + \cos(\alpha - \beta)$, we easily obtain
$$g(Du(x)) = -\frac{1}{4} - \frac{1}{4}\big(\sin 4\pi(x_1+x_2) + \cos 4\pi x_1 + \cos 4\pi x_2\big)$$

and hence by a direct computation that

$$\int_\Sigma g(Du(x))\,dx = \int_0^1 \int_0^1 g(Du(x))\,dx_1 dx_2 = -\frac{1}{4}. \tag{3.20}$$

We now extend g to the whole $\mathbf{M}$. Let $P\colon \mathbf{M} \to L$ be the orthogonal projection onto L and consider the *fourth degree polynomials*

$$F_{\epsilon,k}(\xi) = g(P\xi) + \epsilon\,(|\xi|^2 + |\xi|^4) + k\,|\xi - P\xi|^2. \tag{3.21}$$

Lemma 3.33. *For each $\epsilon > 0$ there exists a $k = k_\epsilon > 0$ such that F_{ϵ,k_ϵ} is rank-one convex.*

Proof. We use contradiction method. Suppose otherwise; then there exists an $\epsilon_0 > 0$ such that $F_{\epsilon_0,k}$ is not rank-one for any $k > 0$. Hence there exist $\xi_k \in \mathbf{M}$, $p^k \in \mathbf{R}^2$, $q_k \in \mathbf{R}^3$ with $|p^k| = |q_k| = 1$ such that

$$\frac{\partial^2 F_{\epsilon_0,k}(\xi_k)}{\partial \xi^i_\alpha \partial \xi^j_\beta}\, q^i_k q^j_k\, p^k_\alpha p^k_\beta \equiv D^2 F_{\epsilon_0,k}(\xi_k)\,[q_k \otimes p^k,\, q_k \otimes p^k] \le 0. \tag{3.22}$$

Computing $f''(0)$ for $f(t) = F_{\epsilon,k}(\xi + t\eta)$ yields

$$D^2 F_{\epsilon,k}(\xi)\,[\eta,\eta] = f''(0)$$

$$= D^2 g(P\xi)\,[P\eta, P\eta] + 2\epsilon\,|\eta|^2 + \epsilon\,(4|\xi|^2|\eta|^2 + 8(\xi\cdot\eta)^2) + 2k\,|\eta - P\eta|^2.$$

The term $D^2 g(P\xi)$ is linear in ξ; the third term is quadratic and positive definite in ξ if $\eta \ne 0$ (this is the reason the $|\xi|^4$-term is needed for $F_{\epsilon,k}$). From these and (3.22) we deduce $\{\xi_k\}$ is bounded as $k \to \infty$. Assume, via subsequence,

$$\xi_k \to \bar\xi,\;\; q_k \to \bar q,\;\; p^k \to \bar p.$$

Since $D^2 F_{\epsilon,j}(\xi)\,[\eta,\,\eta] \le D^2 F_{\epsilon,k}(\xi)\,[\eta,\,\eta]$ for all $k \ge j$, we deduce

$$D^2 g(P\bar\xi)\,[P(\bar q\otimes\bar p),\, P(\bar q\otimes\bar p)] + 2\epsilon_0 + 2j\,|P(\bar q\otimes\bar p) - \bar q\otimes\bar p\,| \le 0 \tag{3.23}$$

for all $j = 1, 2, \cdots$. Thus $P(\bar q\otimes\bar p) = \bar q\otimes\bar p$ and hence $\bar q\otimes\bar p \in L$. This implies $\bar q\otimes\bar p = [a,b,c]$, where at most one of a,b,c is nonzero. Therefore, function

$$t \mapsto g(P(\bar\xi + t\bar q\otimes\bar p)) = g(P\bar\xi + t\bar q\otimes\bar p)$$

is affine in t, and hence the first term in (3.23) vanishes. This yields the desired contradiction $\epsilon_0 \le 0$. The lemma is proved. □

We now complete the proof of Šverák's theorem. Let u be the periodic function above. We choose $\epsilon > 0$ small enough such that

$$\epsilon \int_\Sigma (|Du(x)|^2 + |Du(x)|^4)\,dx < \frac{1}{4}.$$

Let $F_\epsilon(\xi) = F_{\epsilon,k_\epsilon}(\xi)$ be a rank-one function by the previous lemma. Since $Du(x) \in L$, it follows by (3.20) that

$$\int_\Sigma F_\epsilon(Du(x))\,dx = \int_\Sigma g(Du) + \epsilon \int_\Sigma (|Du|^2 + |Du|^4) < 0 = F_\epsilon(0).$$

This shows that F_ϵ is not quasiconvex by a theorem we proved before. The theorem is now proved. □

Remarks. 1) Note that all functions $F_{\epsilon,k}$ defined by (3.21) are polynomials of degree 4. Therefore, we have shown that there are *quartic* polynomials on $\mathbf{M}^{N\times n}$ which are rank-one convex but not quasiconvex if $n \geq 2$ and $N \geq 3$. (See Propositions 3.30 and 3.31.)

2) For any function $g\colon \mathbf{M}^{N\times n} \to \mathbf{R}$, if we define $g^T\colon \mathbf{M}^{n\times N} \to \mathbf{R}$ by $g^T(\xi) = g(\xi^T)$, then it is easily seen that g^T is rank-one convex (polyconvex) *if and only if* g is rank-one convex (polyconvex). However, using Šverák's function $F_{\epsilon,k}$ above, S. Müller '98 recently proved that for each $\epsilon > 0$ there exists a k_ϵ such that $F^T_{\epsilon,k}$ is quasiconvex for all $k \geq k_\epsilon$; of course we already knew $F_{\epsilon,k}$ is not quasiconvex for any $k > 0$ if $\epsilon > 0$ is small enough.

3) It is still completely open whether or not there exists a rank-one convex function on $\mathbf{M}^{2\times n}$ which is not quasiconvex for $n \geq 2$. See also the example of Dacorogna and Marcellini. □

3.7. Relaxation principles

Given a function $F\colon \mathbf{M}^{N\times n} \to \mathbf{R}$, we are interested in the largest quasiconvex function less than or equal to F. This largest quasiconvex function is called the quasiconvex **envelope** or **relaxation** of F. This motivates the following definition.

Definition 3.7. The **quasiconvexification** of F is defined by

$$F^{qc}(\xi) = \inf_{\phi\in C_0^\infty(\Sigma;\mathbf{R}^N)} \int_\Sigma F(\xi + D\phi(x))\,dx, \quad \xi \in \mathbf{M}^{N\times n}.$$

Remarks. 1) From this definition, F^{qc} is only upper semicontinuous if F is continuous. Howerver the next theorem shows that in this case F^{qc} is in fact continuous (locally Lipschitz).

2) Recently, Ball et al solved a long-standing open problem that shows F^{qc} is of C^1 if F is of C^1 and satisfies a polynomial growth. □

Theorem 3.34. *Let $F \geq 0$ be continuous. Then F^{qc} is continuous and is the largest quasiconvex function below F, the quasiconvex envelope of F.*

Proof. For any quasiconvex function $G \leq F$,

$$G(\xi) \leq \int_\Sigma G(\xi + D\phi(x))\,dx \leq \int_\Sigma F(\xi + D\phi(x))\,dx.$$

Hence $G(\xi) \leq F^{qc}(\xi)$ by the definition of F^{qc}. It remains to prove F^{qc} is itself quasiconvex. We first observe that

$$F^{qc}(\xi) = \inf_{\phi\in W_0^{1,\infty}(\Omega;\mathbf{R}^N)} \fint_\Omega F(\xi + D\phi(x))\,dx \tag{3.24}$$

for any open set $\Omega \subset \mathbf{R}^n$ with $|\partial\Omega| = 0$. This can be proved by using the Vitali covering argument as before. We now prove a lemma.

Lemma 3.35. *For all piecewise affine Lipschitz continuous functions $\phi \in W_0^{1,\infty}(\Sigma;\mathbf{R}^N)$, it follows that*

$$F^{qc}(\xi) \leq \int_\Sigma F^{qc}(\xi + D\phi(x))\,dx.$$

Proof. Let $\Sigma = \cup_i \Omega_i \cup E$ such that $|E| = 0$ and each Ω_i is an open set with $|\partial\Omega_i| = 0$ and such that $\phi \in W_0^{1,\infty}(\Sigma;\mathbf{R}^N)$ takes constant gradients $D\phi = M_i$ on each Ω_i. Let $\epsilon > 0$ be given. By the above remark on the definition of F^{qc}, there exists $\psi_i \in W_0^{1,\infty}(\Omega_i; \mathbf{R}^N)$ such that

$$F^{qc}(\xi + M_i) \geq \fint_{\Omega_i} F(\xi + M_i + D\psi_i(x))\, dx - \epsilon.$$

With each ψ_i being extended to Σ, we set $\psi = \phi + \sum_i \psi_i$. Then $\psi \in W_0^{1,\infty}(\Sigma; \mathbf{R}^N)$ and we have

$$\int_\Sigma F^{qc}(\xi + D\phi(x))\, dx = \sum_i |\Omega_i| F^{qc}(\xi + M_i)$$

$$\geq \int_\Sigma F(\xi + D\psi(x))\, dx - \epsilon \geq F^{qc}(\xi) - \epsilon,$$

and the lemma follows. □

Now the lemma above is enough to show F^{qc} is rank-one convex (using "sawtooth" like piecewise affine functions) and therefore is locally Lipschitz continuous (since it is convex in each coordinate direction). Hence F^{qc} is quasiconvex by the same lemma above and density arguments. The proof of the theorem is complete. □

Suppose $F(\xi)$ is a continuous function on $\mathbf{M}^{N\times n}$ and satisfies

$$0 \leq F(\xi) \leq C\,(|\xi|^p + 1) \quad \forall \xi \in \mathbf{M}^{N\times n}$$

for some $1 \leq p < \infty$. We are interested in the *largest* w.s.l.s.c. functional $\tilde{I}(u)$ on $W^{1,p}(\Omega; \mathbf{R}^N)$ which is less than or equal to the functional

$$I(u) = \int_\Omega F(Du(x))\, dx.$$

This functional $\tilde{I}(u)$ is called the **envelope** or **relaxation** of I in the weak topology of $W^{1,p}(\Omega; \mathbf{R}^N)$. It turns out under the condition given above, $\tilde{I}(u)$ is given by the integral functional of F^{qc}.

Theorem 3.36. *Let $F(\xi)$ be continuous and satisfy $0 \leq F(\xi) \leq C\,(|\xi|^p + 1)$ for $1 \leq p < \infty$. Then the envelope $\tilde{I}(u)$ of I in the weak topology of $W^{1,p}(\Omega; \mathbf{R}^N)$ is given by*

$$\tilde{I}(u) = \int_\Omega F^{qc}(Du(x))\, dx.$$

Proof. Let $\hat{I}(u)$ be the integral functional defined by F^{qc}. Since F^{qc} is quasiconvex and satisfies the same growth condition as F, the functional $\hat{I}(u)$ is thus (sequential) w.l.s.c. on $W^{1,p}(\Omega; \mathbf{R}^N)$ by the theorem of Acerbi and Fusco. Therefore, $\hat{I} \leq \tilde{I}$. To prove the other direction, we first assume there exists a continuous function $g\colon \mathbf{M}^{N\times n} \to \mathbf{R}$ such that

$$\tilde{I}(u) = \int_\Omega g(Du(x))\, dx \quad \forall\, u \in W^{1,p}(\Omega; \mathbf{R}^N). \tag{3.25}$$

Then $g(\xi)$ must be quasiconvex and $g \leq F$, and thus $g \leq F^{qc}$; this proves $\tilde{I} \leq \hat{I}$. However, the proof of (3.25) is beyond the scope of this lecture and is omitted; see e.g., Buttazzo, Acerbi and Fusco, or Dacorogna. □

Finally we have the following theorem; the proof of the theorem is also omitted (see also Acerbi and Fusco, or Dacorogna).

Theorem 3.37 (Relaxation Principle)**.** *Assume*

$$c\,|\xi|^p \le F(\xi) \le C\,(|\xi|^p + 1)$$

holds for some constants $c > 0$, $C > 0$, $p > 1$. *Then*

$$\inf_{u \in \mathcal{D}^p_\varphi(\Omega)} \int_\Omega F(Du)\,dx = \min_{u \in \mathcal{D}^p_\varphi(\Omega)} \int_\Omega F^{qc}(Du)\,dx$$

for any $\varphi \in W^{1,p}(\Omega; \mathbf{R}^N)$, *where* $\mathcal{D}^p_\varphi(\Omega)$ *is the Dirichlet class of* φ.

Reamrk. This theorem seems too nice for the nonconvex variational problems since it changes a nonconvex problem which may not have a minimizer to a quasiconvex problem that has definitely at least one minimizer. But, there are costs for this: we may lose some information about the minimizing sequences; a minimizer so obtained for the relaxed problem may not characterize what seems more interesting in applications the finer and finer patterns of minimizing sequences. In the phase transition problems for certain materials, it accounts for the loss of information about the *microstructures* by a *macroscopic* effective processing (relaxation). For more information, we refer to Ball and James '92, Müller '98 and the references therein.

□

Regularity Theory for Linear Systems

4.1. Sobolev-Poincaré type inequalities

Let $\Omega \subset \mathbf{R}^n$ be a Lipschitz domain. Then, by Sobolev's embedding theorem, the immersion

$$W^{1,p}(\Omega; \mathbf{R}^N) \hookrightarrow L^p(\Omega; \mathbf{R}^N) \tag{4.1}$$

is a *compact* operator for all $1 \le p < \infty$. Based on this compact embedding, we can prove several useful *Poincaré-type* inequalities.

Proposition 4.1. *Let Ω be a bounded domain such that the immersion* (4.1) *is compact. Then for any $1 \le p < \infty$ there exists a constant $C_p = C_p(\Omega)$ such that*

$$\int_\Omega |u(x) - u_\Omega|^p \, dx \le C_p \int_\Omega |Du(x)|^p \, dx \tag{4.2}$$

holds for all $u \in W^{1,p}(\Omega; \mathbf{R}^N)$, where u_Ω is the average of u on Ω; that is

$$u_\Omega = \fint_\Omega u(x)\, dx = \frac{1}{|\Omega|} \int_\Omega u(x)\, dx.$$

Proof. Suppose the result is not true. Then we can find a sequence $\{u^k\}$ in $W^{1,p}(\Omega; \mathbf{R}^N)$ such that for $k = 1, 2, ...$

$$u^k_\Omega = 0, \quad \|u^k\|_{L^p(\Omega;\mathbf{R}^N)} = 1, \quad \|Du^k\|_{L^p(\Omega;\mathbf{R}^N)} \le 1/k.$$

By the compact embedding, we can find a subsequence of $\{u^k\}$, still denoted the same, such that $u^k \to \bar{u}$ in $L^p(\Omega; \mathbf{R}^N)$ as $k \to \infty$. Therefore, $\|\bar{u}\|_p = 1$, $\bar{u}_\Omega = 0$ and by the lower semicontinuity of the norm, $\|D\bar{u}\|_p = 0$. These are obvious contradictions. The result is proved. □

If we define $C_p(\Omega)$ to be the best constant in the Poincaré inequality (4.2), then by an easy scaling argument we have

$$C_p(R\Omega) = R^p \, C_p(\Omega), \quad \forall \; R > 0.$$

Hence we have the following useful Poincaré-type inequalities.

Proposition 4.2. *Let $1 \le p < \infty$. Then there exist constants $C_1 = C_1(n,p)$ and $C_2 = C_2(n,p)$ such that for all balls $B_R \subset \mathbf{R}^n$ and $u \in W^{1,p}(B_R;\mathbf{R}^N)$*

$$\int_{B_R} |u - u_{B_R}|^p \, dx \le C_1 \, R^p \int_{B_R} |Du|^p \, dx, \tag{4.3}$$

$$\int_{B_R\setminus B_{R/2}} |u - u_{B_R\setminus B_{R/2}}|^p \, dx \le C_2 \, R^p \int_{B_R\setminus B_{R/2}} |Du|^p \, dx. \tag{4.4}$$

We shall also use certain Poincaré type inequalities between L^p norm of u and L^q norm of Du, which are usually called the Sobolev-Poincaré type inequalities. We state two of such important inequalities without proof; see Gilbarg-Trudinger, Chapter 7.

Proposition 4.3. *For any $1 \le p < n$ there exists a constant $C_4 = C_4(n,p)$ such that for any domain Ω and $u \in W_0^{1,p}(\Omega;\mathbf{R}^N)$*

$$\int_\Omega |u|^{\frac{np}{n-p}} \, dx \le C_4 \left(\int_\Omega |Du|^p \, dx \right)^{\frac{n}{n-p}}. \tag{4.5}$$

We sometimes denote $p^* = \frac{np}{n-p}$, called the Sobolev conjugate of p.

Proposition 4.4. *For any $1 \le p < n$ there exists a constant $C_5 = C_5(n,p)$ such that for all balls $B_R \subset \mathbf{R}^n$ and $u \in W^{1,p}(B_R;\mathbf{R}^N)$*

$$\int_{B_R} |u - u_{B_R}|^{\frac{np}{n-p}} \, dx \le C_5 \left(\int_{B_R} |Du|^p \, dx \right)^{\frac{n}{n-p}}. \tag{4.6}$$

4.2. Caccioppoli-type estimates

Let $A(x,\xi) = A(x)\,\xi$ be a linear matrix function of ξ given by

$$(A(x,\xi))^i_\alpha = (A(x)\xi)^i_\alpha = A^{\alpha\beta}_{ij}(x)\xi^j_\beta,$$

where $A^{\alpha\beta}_{ij} \in L^\infty(\Omega)$. Consider the linear partial differential system

$$-D_\alpha\big(A^{\alpha\beta}_{ij}(x)\, D_\beta u^j\big) = g^i - D_\alpha f^i_\alpha, \quad i = 1, 2, \cdots, N, \tag{4.7}$$

where we assume g^i, $f^i_\alpha \in L^2_{loc}(\Omega)$. We also write this system as

$$-\operatorname{Div}(A(x)Du) = g - \operatorname{Div} f,$$

where $g = (g^i)$, $f = (f^i_\alpha)$. Recall that a function $u \in W^{1,2}_{loc}(\Omega;\mathbf{R}^N)$ is a *weak solution* of (4.7) if

$$\int_\Omega A(x)Du \cdot D\phi(x)\, dx = \int_\Omega (g(x)\cdot\phi(x) + f(x)\cdot D\phi(x))\, dx \tag{4.8}$$

holds for all $\phi \in C_0^\infty(\Omega;\mathbf{R}^N)$. Since $A^{\alpha\beta}_{ij} \in L^\infty(\Omega)$, the test function ϕ in (4.8) can be chosen in $W_0^{1,2}(\Omega';\mathbf{R}^N)$ for any subdomain $\Omega' \subset\subset \Omega$.

The regularity for system (4.7) relies on certain *ellipticity* conditions. We shall assume one of the following conditions for the coefficients $A^{\alpha\beta}_{ij}(x)$ holds with some constant $\nu > 0$.

$(H1)$ $\quad A^{\alpha\beta}_{ij}(x)\,\xi^i_\alpha\xi^j_\beta \ge \nu\,|\xi|^2.$

$(H2)$ $\quad A^{\alpha\beta}_{ij}$ are constants, $\quad A^{\alpha\beta}_{ij}\, p_\alpha p_\beta\, q^i q^j \ge \nu\,|p|^2|q|^2.$

$(H3)$ $\quad A^{\alpha\beta}_{ij} \in C(\bar\Omega)$, $\; A^{\alpha\beta}_{ij}(x)\, p_\alpha p_\beta\, q^i q^j \ge \nu\,|p|^2|q|^2.$

Assume $u \in W^{1,2}_{loc}(\Omega; \mathbf{R}^N)$ is a weak solution of (4.7). Almost all the estimates pertaining to regularity of u are derived using test functions of the form $\phi = \eta\,(u-\lambda)$, where η is a *cut-off* function which belongs to $W^{1,\infty}_0(\Omega')$ for certain $\Omega' \subset\subset \Omega$. Let $B_\rho \subset\subset B_R \subset\subset \Omega$ be concentric balls with center $a \in \Omega$. Let

$$\theta(t) = \begin{cases} 1 & \text{if } 0 \le t \le \rho, \\ \frac{R-t}{R-\rho} & \text{if } \rho \le t \le R, \\ 0 & \text{if } t > R. \end{cases}$$

Let $\zeta = \zeta_{\rho,R}(x) = \theta(|x-a|)$. Then $\zeta \in W^{1,\infty}_0(\Omega)$ with $\operatorname{supp}\zeta \subseteq \bar{B}_R$ and

$$0 \le \zeta \le 1, \quad \zeta|_{B_\rho} \equiv 1, \quad |D\zeta| \le \frac{\chi_{\rho,R}}{R-\rho}, \tag{4.9}$$

where $\chi_{\rho,R} = \chi_{B_R \setminus B_\rho}(x)$ is the characteristic function of $B_R \setminus B_\rho$. Define

$$\psi = \zeta\,(u-\lambda), \quad \phi = \zeta^2(u-\lambda) = \zeta\,\psi.$$

Then $\psi,\, \phi \in W^{1,2}_0(B_R; \mathbf{R}^N)$ and

$$D\phi = \zeta D\psi + \psi \otimes D\zeta, \quad D\psi = \zeta Du + (u-\lambda) \otimes D\zeta.$$

Using ϕ as a test function in (4.8) yields

$$\begin{aligned} \int_{B_R} (g \cdot \phi + f \cdot D\phi) &= \int_{B_R} A(x)Du \cdot D\phi \\ &= \int_{B_R} A(x)Du \cdot \zeta D\psi + \int_{B_R} A(x)Du \cdot \psi \otimes D\zeta. \end{aligned}$$

Note that

$$\begin{aligned} A(x)D\psi \cdot D\psi &= A(x)Du \cdot \zeta D\psi + A(x)(u-\lambda) \otimes D\zeta \cdot D\psi, \\ A(x)Du \cdot \psi \otimes D\zeta &= A(x)\zeta Du \cdot (u-\lambda) \otimes D\zeta \\ &= A(x)D\psi \cdot (u-\lambda) \otimes D\zeta \\ &- A(x)(u-\lambda) \otimes D\zeta \cdot (u-\lambda) \otimes D\zeta. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} \int_{B_R} A(x)D\psi \cdot D\psi &= \int_{B_R} A(x)(u-\lambda) \otimes D\zeta \cdot D\psi \\ &+ \int_{B_R} (g \cdot \phi + f \cdot D\phi) \\ &- \int_{B_R} A(x)D\psi \cdot (u-\lambda) \otimes D\zeta \\ &+ \int_{B_R} A(x)(u-\lambda) \otimes D\zeta \cdot (u-\lambda) \otimes D\zeta. \end{aligned}$$

(Note that the first and third terms on the righthand side would cancel out if $A(x)\xi \cdot \eta$ is symmetric in ξ, η.)

For the moment, we assume the following Gårding inequality holds:

$$\int_{B_R} A(x)D\psi \cdot D\psi \ge \nu_0 \int_{B_R} |D\psi|^2 - \nu_1 \int_{B_R} |\psi|^2, \tag{4.10}$$

where $\nu_0 > 0$, $\nu_1 \geq 0$ are constants. Then it follows that

$$\begin{aligned}\nu_0 \int_{B_R} |D\psi|^2 &\leq \int_{B_R} A(x)D\psi \cdot D\psi + \nu_1 \int_{B_R} |\psi|^2 \\ &\leq \Big|\int_{B_R} g\cdot\phi\Big| + \int_{B_R} |f|\cdot|D\psi| + \int_{B_R} |f|\cdot \frac{\chi_{\rho,R}\,|u-\lambda|}{R-\rho} \\ &+ C\int_{B_R} |D\psi|\cdot\frac{\chi_{\rho,R}\,|u-\lambda|}{R-\rho} + C\int_{B_R}\frac{\chi_{\rho,R}\,|u-\lambda|^2}{(R-\rho)^2} \\ &+ \nu_1 \int_{B_R} |u-\lambda|^2.\end{aligned}$$

Using the Young inequality

$$|a|\cdot|b| \leq \epsilon|a|^2 + C_\epsilon|b|^2$$

in this estimate, we deduce

$$\int_{B_R} |D\psi|^2 \leq C\left[\int_{B_R\setminus B_\rho}\frac{|u-\lambda|^2}{(R-\rho)^2} + \Big|\int_{B_R} g\cdot\phi\Big| + \int_{B_R}|f|^2 + \nu_1\int_{B_R}|u-\lambda|^2\right]. \tag{4.11}$$

We now notice that under the hypothesis (H1) or (H2) the Gårding inequality (4.10) holds with $\nu_0 = \nu$, $\nu_1 = 0$ and under (H3) the inequality (4.10) also holds. Since $\psi|_{B_\rho} = u - \lambda$, this last estimate (4.11) proves the following theorems.

Theorem 4.5. *Let $u \in W^{1,2}_{loc}(\Omega;\mathbf{R}^N)$ be a weak solution of* (4.7). *Assume either condition $(H1)$ or $(H2)$ holds. Then*

$$\int_{B_\rho} |Du|^2 \leq C\left[\int_{B_R\setminus B_\rho}\frac{|u-\lambda|^2}{(R-\rho)^2} + \Big|\int_{B_R} g\cdot\zeta^2(u-\lambda)\Big| + \int_{B_R}|f|^2\right] \tag{4.12}$$

for all concentric balls $B_\rho \subset\subset B_R \subset\subset \Omega$ and constants $\lambda\in\mathbf{R}^N$, where $\zeta = \zeta_{\rho,R}$ and $C > 0$ is a constant depending on the L^∞-norm of $A^{\alpha\beta}_{ij}$.

Theorem 4.6. *Assume condition $(H3)$ holds. Then*

$$\int_{B_\rho} |Du|^2 \leq C\left[\int_{B_R\setminus B_\rho}\frac{|u-\lambda|^2}{(R-\rho)^2} + \Big|\int_{B_R} g\cdot\zeta^2(u-\lambda)\Big| + \int_{B_R}(|f|^2 + |u-\lambda|^2)\right] \tag{4.13}$$

for all concentric balls $B_\rho \subset\subset B_R \subset\subset \Omega$ and constants $\lambda\in\mathbf{R}^N$.

Corollary 4.7. *Let $u \in W^{1,2}_{loc}(\Omega;\mathbf{R}^N)$ be a weak solution of* (4.7). *Assume either condition $(H1)$ or $(H2)$ holds. Then*

$$\int_{B_{R/2}} |Du|^2 \leq C\left[\int_{B_R\setminus B_{R/2}}\frac{|u-\lambda|^2}{R^2} + \int_{B_R}\Big(|u-\lambda|\cdot|g| + |f|^2\Big)\right] \tag{4.14}$$

for all balls $B_R \subset\subset \Omega$ and constants $\lambda\in\mathbf{R}^N$.

Remarks. 1) The estimates (4.12), (4.13) and (4.14) are usually referred to as the *Caccioppoli-type* inequalities or Caccioppoli estimates.

2) In both (4.12), (4.13), we leave the term $\int_{B_R} g\cdot\zeta^2(u-\lambda)$ in the estimates. We shall see later that this term needs a special consideration when we deal with higher regularity for weak solutions, especially when g is of the form of quotient difference. □

As an application of these Caccioppoli estimates, we prove the following results.

Proposition 4.8. *Suppose $u \in W^{1,2}_{loc}(\mathbf{R}^n; \mathbf{R}^N)$ is a weak solution of*

$$-D_\alpha(A^{\alpha\beta}_{ij}(x) D_\beta u^j) = 0, \tag{4.15}$$

where coefficients $A^{\alpha\beta}_{ij}(x)$ satisfy $(H1)$ or $(H2)$. If $|Du| \in L^2(\mathbf{R}^n)$, then u is a constant.

Proof. By (4.14), it follows that

$$\int_{B_{R/2}} |Du|^2 \le \frac{C}{R^2} \int_{B_R \setminus B_{R/2}} |u - \lambda|^2.$$

We choose

$$\lambda = \frac{1}{|B_R \setminus B_{R/2}|} \int_{B_R \setminus B_{R/2}} u(x)\, dx.$$

Then the Poincaré inequality (see Proposition 4.2) shows that

$$\int_{B_R \setminus B_{R/2}} |u - \lambda|^2 \le c(n)\, R^2 \int_{B_R \setminus B_{R/2}} |Du|^2.$$

Therefore

$$\int_{B_{R/2}} |Du|^2 \le C \int_{B_R \setminus B_{R/2}} |Du|^2.$$

Adding $C \int_{B_{R/2}} |Du|^2$ to both sides of this inequality (this sometimes called the *hole-filling* technique of Widman), we obtain

$$\int_{B_{R/2}} |Du|^2 \le \frac{C}{C+1} \int_{B_R} |Du|^2.$$

Letting $R \to \infty$ we have

$$\int_{\mathbf{R}^n} |Du|^2\, dx \le \frac{C}{C+1} \int_{\mathbf{R}^n} |Du|^2\, dx.$$

Since $\frac{C}{C+1} < 1$ we have $\int_{\mathbf{R}^n} |Du|^2 = 0$ and thus $Du \equiv 0$; hence $u \equiv$ constant. □

Proposition 4.9. *Assume either condition $(H1)$ or $(H2)$ holds. Then any bounded weak solution $u \in W^{1,2}_{loc}(\mathbf{R}^2; \mathbf{R}^N)$ to the equation (4.15) for $n = 2$ must be constant.*

Proof. Let $|u| \le M$; then by the Caccioppoli inequality (4.14) with $\lambda = 0$ we have

$$\int_{B_{R/2}} |Du|^2\, dx \le CM < \infty, \quad \forall\, R > 0.$$

This implies $|Du| \in L^2(\mathbf{R}^2)$; hence by the result above, u is a constant. □

4.3. Method of difference quotients

Let $\{e_1, \cdots, e_n\}$ be the standard basis of $\mathbf{R}^n$. Define

$$\tau_{h,s} u(x) = \frac{u(x + he_s) - u(x)}{h}, \quad h \ne 0.$$

If u is defined on $\Omega \subset \mathbf{R}^n$, then $\tau_{h,s} u$ is defined on

$$\Omega_{h,s} = \{x \in \Omega \,|\, x + he_s \in \Omega\}.$$

Note that

$$\Omega_h = \{x \in \Omega \,|\, \operatorname{dist}(x; \partial\Omega) > |h|\} \subset \Omega_{h,s}.$$

We have the following properties of $\tau_{h,s} u$.

1) If $u \in W^{1,p}(\Omega; \mathbf{R}^N)$ then $\tau_{h,s}u \in W^{1,p}(\Omega_{h,s}; \mathbf{R}^N)$ and

$$D(\tau_{h,s}u) = \tau_{h,s}Du.$$

2) If either u or v has compact support $\Omega' \subset\subset \Omega$ then

$$\int_\Omega u \cdot \tau_{h,s}v \, dx = -\int_\Omega v \cdot \tau_{-h,s}u \, dx \quad \forall \, |h| < \mathrm{dist}(\Omega'; \partial\Omega).$$

3) $\tau_{h,s}(\phi \, u)(x) = \phi(x) \, \tau_{h,s}u(x) + u(x + he_s) \, \tau_{h,s}\phi(x)$.

Proposition 4.10. *(a) Let $u \in W^{1,p}(\Omega)$. Then $\tau_{h,s}u \in L^p(\Omega')$ for any $\Omega' \subset\subset \Omega$ satisfying $|h| < \mathrm{dist}(\Omega'; \partial\Omega)$. Moreover, we have*

$$\|\tau_{h,s}u\|_{L^p(\Omega')} \leq \|D_s u\|_{L^p(\Omega)}.$$

(b) Let $u \in L^p(\Omega)$, $1 < p < \infty$, and $\Omega' \subset\subset \Omega$. If there exists a constant $K > 0$ such that

$$\liminf_{h \to 0} \|\tau_{h,s}u\|_{L^p(\Omega')} \leq K,$$

then the weak derivative $D_s u$ exists and satisfies $\|D_s u\|_{L^p(\Omega')} \leq K$.

Proof. (a) Let us suppose initially that $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$. Then, for $h > 0$,

$$\tau_{h,s}u(x) = \frac{1}{h} \int_0^h D_s u(x_1, \cdots, x_{s-1}, x_s + t, x_{s+1}, \cdots, x_n) \, dt$$

so that by Hölder's inequality

$$|\tau_{h,s}u(x)|^p \leq \frac{1}{h} \int_0^h |D_s u(x_1, \cdots, x_{s-1}, x_s + t, x_{s+1}, \cdots, x_n)|^p \, dt,$$

and hence

$$\int_{\Omega'} |\tau_{h,s}u(x)|^p \, dx \leq \frac{1}{h} \int_0^h \int_{B_h(\Omega')} |D_s u|^p \, dx \, dt \leq \int_\Omega |D_s u|^p \, dx,$$

where $B_h(\Omega') = \{x \in \Omega \,|\, \mathrm{dist}(x; \Omega') < h\}$. The extension to arbitrary functions in $W^{1,p}(\Omega)$ follows by a straight-forward approximation argument.

(b) Since $1 < p < \infty$, there exists a sequence $\{h_m\}$ tending to zero and a function $v \in L^p(\Omega')$ with $\|v\|_p \leq K$ such that $\tau_{h_m,s}u \rightharpoonup v$ in $L^p(\Omega')$. This means for all $\phi \in C_0^\infty(\Omega')$

$$\lim_{m \to \infty} \int_{\Omega'} \phi \, \tau_{h_m,s}u \, dx = \int_{\Omega'} \phi \, v \, dx.$$

Now for $|h_m| < \mathrm{dist}(\mathrm{supp} \, \phi; \partial\Omega')$, we have

$$\int_{\Omega'} \phi \, \tau_{h_m,s}u \, dx = -\int_{\Omega'} u \, \tau_{-h_m,s}\phi \, dx \to -\int_{\Omega'} u \, D_s\phi \, dx.$$

Hence

$$\int_{\Omega'} \phi \, v \, dx = -\int_{\Omega'} u \, D_s\phi \, dx,$$

which shows $v = D_s u \in L^p(\Omega')$ and $\|D_s u\|_{L^p(\Omega')} \leq K$. □

4.4. Hilbert-space regularity

We now assume $u \in W^{1,2}_{loc}(\Omega; \mathbf{R}^N)$ is a weak solution of linear system

$$-\operatorname{Div}(A(x)\, Du) = g - \operatorname{Div} f.$$

This means

$$\int_\Omega A(x)\, Du(x) \cdot D\phi(x)\, dx = \int_\Omega g(x) \cdot \phi(x)\, dx + \int_\Omega f(x) \cdot D\phi(x)\, dx$$

holds for all $\phi \in W^{1,2}_0(\Omega'; \mathbf{R}^N)$. If $|h| < \operatorname{dist}(\Omega'; \partial\Omega)$ this implies

$$\int_\Omega A(x+he_s)\, Du(x+he_s) \cdot D\phi(x)\, dx$$
$$= \int_\Omega g(x+he_s) \cdot \phi(x)\, dx + \int_\Omega f(x+he_s) \cdot D\phi(x)\, dx.$$

Substract two equations and divide by h to get

$$\begin{aligned} \int_\Omega A(x+he_s)\, D\tau_{h,s}u \cdot D\phi &= \int_\Omega \tau_{h,s} g(x) \cdot \phi(x)\, dx \\ &+ \int_\Omega \tau_{h,s} f(x) \cdot D\phi(x)\, dx \\ &- \int_\Omega \tau_{h,s} A(x)\, Du(x) \cdot D\phi(x)\, dx. \end{aligned}$$

This shows that $v = \tau_{h,s}u$ is a weak solution of system

$$-\operatorname{Div}(A(x+he_s)\, Dv) = \tau_{h,s} g - \operatorname{Div}(\tau_{h,s} f) + \operatorname{Div}(\tau_{h,s} A\, Du) \tag{4.16}$$

on Ω'. We now assume that Gårding's inequality (4.10) holds. Then we can invoke the estimate (4.11) with $\lambda = 0$, $\rho = R/2$ to obtain

$$\int_{B_R} |D\psi|^2 \le C\Big[\int_{B_R} \frac{1}{R^2} |\tau_{h,s}u|^2 + \Big| \int_{B_R} \tau_{h,s} g \cdot \phi \Big| \tag{4.17}$$
$$+ \int_{B_R} \big(|\tau_{h,s} f|^2 + \nu_1 |\tau_{h,s}u|^2 + |\tau_{h,s}A|^2 |Du|^2 \big) \Big],$$

where $\psi = \zeta\, \tau_{h,s}u$, $\phi = \zeta^2\, \tau_{h,s}u$ and $\zeta = \zeta_{R/2,R}$ is defined as before. Note that

$$\begin{aligned} -\int_{B_R} \tau_{h,s} g \cdot \phi &= \int_\Omega \tau_{h,s} g \cdot \phi = \int_\Omega g \cdot \tau_{-h,s}\phi \\ &= \int_\Omega g \cdot \zeta(x - he_s)\, \tau_{-h,s}\psi + \int_\Omega g \cdot \psi\, \tau_{-h,s}\zeta \\ &\equiv I + II. \end{aligned}$$

We estimate I, II as follows.

$$\begin{aligned} |I| &\le \int_{\Omega'} |g| \cdot |\tau_{-h,s}\psi| \\ &\le \epsilon \int_{\Omega'} |\tau_{-h,s}\psi|^2 + C_\epsilon \int_{\Omega'} |g|^2 \\ &\le \epsilon \int_\Omega |D_s\psi|^2 + C_\epsilon \int_{\Omega'} |g|^2 \\ &\le \epsilon \int_{B_R} |D\psi|^2 + C_\epsilon \int_{\Omega'} |g|^2, \end{aligned}$$

where $\Omega' \subset\subset \Omega$ is a domain containing $\bar{B}_R$.

$$\begin{aligned} |II| &\leq \int_{B_R} |g|\,|\tau_{h,s}u|\,|\tau_{-h,s}\zeta| \\ &\leq \frac{C}{R^2}\int_{\Omega'} |g|^2 + C\int_{B_R} |\tau_{h,s}u|^2. \end{aligned}$$

Combining these estimates with (4.17) yields

$$\int_{B_R} |D\psi|^2 \leq C(R)\int_{\Omega'} \Big(|\tau_{h,s}u|^2 + |g|^2 + |\tau_{h,s}f|^2 + |\tau_{h,s}A|^2|Du|^2\Big).$$

Since $\psi = \tau_{h,s}u$ and $D\psi = \tau_{h,s}Du$ on $B_{R/2}$ we have

$$\int_{B_{R/2}} |\tau_{h,s}Du|^2 \leq C(R)\int_{\Omega'} \Big(|\tau_{h,s}u|^2 + |g|^2 + |\tau_{h,s}f|^2 + |\tau_{h,s}A|^2|Du|^2\Big). \tag{4.18}$$

Finally, if we assume $f \in W^{1,2}_{loc}(\Omega; \mathbf{M}^{N\times n})$ and $A(x)$ is Lipschitz continuous with Lipschitz constant K then

$$\int_{\Omega'} |\tau_{h,s}f|^2 \leq \int_{\Omega''} |Df|^2, \quad |\tau_{h,s}A(x)| \leq K,$$

where $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, and hence by (4.18) we have

$$\int_{B_{R/2}} |\tau_{h,s}Du|^2(x)\,dx \leq M < \infty \quad \forall\ |h| << 1,$$

and thus D_sDu exists and belongs to $L^2(B_{R/2}; \mathbf{M}^{N\times n})$ for all $s = 1, 2, \cdots, n$. This implies $u \in W^{2,2}_{loc}(\Omega; \mathbf{R}^N)$. Therefore, we have proved the following theorem.

Theorem 4.11. *Suppose $A \in C(\bar{\Omega})$ is Lipschitz continuous and the Gårding inequality (4.10) holds. If $g \in L^2_{loc}(\Omega; \mathbf{R}^N)$, $f \in W^{1,2}_{loc}(\Omega; \mathbf{M}^{N\times n})$ and $u \in W^{1,2}_{loc}(\Omega; \mathbf{R}^N)$ is a weak solution of the system*

$$-\operatorname{Div}(A(x)\,Du) = g - \operatorname{Div} f$$

then $u \in W^{2,2}_{loc}(\Omega; \mathbf{R}^N)$.

The following higher regularity result can be proved by the standard bootstrap method.

Theorem 4.12. *Suppose $u \in W^{1,2}_{loc}(\Omega; \mathbf{R}^N)$ is a weak solution of the system*

$$-\operatorname{Div}(A(x)\,Du) = g - \operatorname{Div} f$$

with $A \in C^{k,1}(\bar{\Omega})$ (that is, D^kA is Lipschitz continuous) satisfying the Gårding inequality (4.10) and $g \in W^{k,2}_{loc}(\Omega; \mathbf{R}^N)$, $f \in W^{k+1,2}_{loc}(\Omega; \mathbf{M}^{N\times n})$. Then $u \in W^{k+2,2}_{loc}(\Omega; \mathbf{R}^N)$.

Proof. Let $\psi \in C^\infty_0(\Omega; \mathbf{R}^N)$; then we use $\phi = D_s\psi$ as a test function for the system to obtain

$$\int_\Omega D_s(A(x)\,Du)\cdot D\psi\,dx = \int_\Omega D_sg\cdot\psi + \int_\Omega D_sf\cdot D\psi.$$

Since $D_s(A(x)\,Du) = (D_sA)\,Du + A(x)\,DD_su$ we thus have

$$\int_\Omega A(x)\,D(D_su)\cdot D\psi = \int_\Omega D_sg\cdot\psi + \int_\Omega \big(D_sf - (D_sA)\,Du\big)\cdot D\psi.$$

This shows $v = D_su \in W^{1,2}_{loc}(\Omega; \mathbf{R}^N)$ is a weak solution of

$$-\operatorname{Div}(A(x)\,Dv) = D_sg - \operatorname{Div}(D_sf - (D_sA)\,Du),$$

and hence $v \in W^{2,2}_{loc}(\Omega; \mathbf{R}^N)$; that is, $u \in W^{3,2}_{loc}(\Omega; \mathbf{R}^N)$. The result for general k then follows from induction. □

Remark. Note that if A, g, f are all of C^∞ then any weak solution $u \in W^{1,2}_{loc}(\Omega; \mathbf{R}^N)$ must be in $C^\infty(\Omega; \mathbf{R}^N)$. We also have the following result. □

Theorem 4.13. *Assume (H2) holds. Let $u \in W^{1,2}_{loc}(\Omega; \mathbf{R}^N)$ be a weak solution of*

$$-D_\alpha(A^{\alpha\beta}_{ij}\, D_\beta u^j) = 0, \quad i = 1, 2, ..., N. \tag{4.19}$$

Then $u \in W^{k,2}_{loc}(\Omega; \mathbf{R}^N)$ for all $k = 1, 2, ...$ and

$$\|u\|_{W^{k,2}(B_{R/2};\mathbf{R}^N)} \le C(k, R)\, \|u\|_{L^2(B_R;\mathbf{R}^N)}$$

for any ball $B_R \subset\subset \Omega$.

Proof. By the Caccioppoli-type inequality, we have for any weak solution u of system (4.19)

$$\int_{B_\rho} |Du|^2\, dx \le \frac{C}{(R-\rho)^2} \int_{B_R} |u|^2\, dx.$$

The regularity result shows that $u \in W^{k,2}_{loc}(\Omega; \mathbf{R}^N)$ for all k and then it follows that any derivative $D^k u$ is also a weak solution of (4.19). Therefore, the conclusion will follow from a successive use of the above Caccioppoli inequality with a finite number of $R/2 = \rho_1 < \rho_2 < \cdots < \rho_K = R$. □

4.5. Morrey and Campanato spaces

Let $\Omega \subset \mathbf{R}^n$ be a bounded open domain. For $x \in \mathbf{R}^n$, $\rho > 0$ let

$$\Omega(x, \rho) = \{y \in \Omega \mid |y - x| < \rho\}.$$

Definition 4.1. For $1 \le p < \infty$, $\lambda \ge 0$ we define the **Morrey space** $L^{p,\lambda}(\Omega; \mathbf{R}^N)$ by

$$L^{p,\lambda}(\Omega; \mathbf{R}^N) = \left\{ u \in L^p(\Omega; \mathbf{R}^N) \;\Big|\; \sup_{\substack{a\in\Omega \\ 0<\rho<\operatorname{diam}\Omega}} \rho^{-\lambda} \int_{\Omega(a,\rho)} |u(x)|^p\, dx < \infty \right\}.$$

We define a norm by

$$\|u\|_{L^{p,\lambda}(\Omega;\mathbf{R}^N)} = \sup_{\substack{a\in\Omega \\ 0<\rho<\operatorname{diam}\Omega}} \left(\rho^{-\lambda} \int_{\Omega(a,\rho)} |u(x)|^p\, dx \right)^{1/p}.$$

Proposition 4.14. *$L^{p,\lambda}(\Omega; \mathbf{R}^N)$ is a Banach space.*

Lemma 4.15 (Lebesgue Differentiation Theorem)**.** *If $v \in L^1_{loc}(\Omega)$ then*

$$\lim_{\rho\to 0} \fint_{B_\rho(x)} |v(x) - v(y)|\, dy = 0$$

for almost every $x \in \Omega$.

Proposition 4.16. *(a) If $\lambda > n$ then $L^{p,\lambda}(\Omega; \mathbf{R}^N) = \{0\}$.*

(b) $L^{p,0}(\Omega; \mathbf{R}^N) \cong L^p(\Omega; \mathbf{R}^N)$; $L^{p,n}(\Omega; \mathbf{R}^N) \cong L^\infty(\Omega; \mathbf{R}^N)$.

(c) If $1 \le p \le q < \infty$, $\frac{n-\lambda}{p} \ge \frac{n-\mu}{q}$, then $L^{q,\mu}(\Omega; \mathbf{R}^N) \subset L^{p,\lambda}(\Omega; \mathbf{R}^N)$.

Proof. (a) By Lebesgue's differentiation theorem,

$$|u(a)| = \lim_{\rho\to 0} \fint_{\Omega(a,\rho)} |u(x)|\,dx, \quad \forall\ a.e.\ a \in \Omega. \tag{4.20}$$

Now, by Hölder's inequality,

$$\begin{aligned}\fint_{\Omega(a,\rho)} |u(x)|\,dx &\le \left(\fint_{\Omega(a,\rho)} |u(x)|^p\,dx\right)^{1/p} \\ &\le C\,\|u\|_{L^{p,\lambda}(\Omega;\mathbf{R}^N)}\rho^{-n/p}\,\rho^{\lambda/p} = C\,\rho^{\frac{\lambda-n}{p}}\,\|u\|_{L^{p,n}(\Omega;\mathbf{R}^N)}.\end{aligned} \tag{4.21}$$

If $\lambda > n$, letting $\rho \to 0$ we have $u(a) = 0$ for almost every $a \in \Omega$; thus $u \equiv 0$.

(b) That $L^{p,0}(\Omega;\mathbf{R}^N) \cong L^p(\Omega;\mathbf{R}^N)$ easily follows from the definition. We prove $L^{p,n}(\Omega;\mathbf{R}^N) \cong L^\infty(\Omega;\mathbf{R}^N)$. If $u \in L^\infty(\Omega;\mathbf{R}^N)$, then

$$\rho^{-n}\int_{\Omega(a,\rho)} |u(x)|^p\,dx \le C\,\|u\|_\infty^p$$

so that $\|u\|_{L^{p,n}} \le C\,\|u\|_\infty$. Suppose now $u \in L^{p,n}(\Omega;\mathbf{R}^N)$. Then by (4.20), (4.21)

$$|u(a)| = \lim_{\rho\to 0}\fint_{\Omega(a,\rho)} |u| \le C\,\|u\|_{L^{p,n}(\Omega;\mathbf{R}^N)}.$$

Hence $\|u\|_{L^\infty(\Omega;\mathbf{R}^N)} \le C\,\|u\|_{L^{p,n}(\Omega;\mathbf{R}^N)}$. Therefore $L^{p,n}(\Omega;\mathbf{R}^N) \cong L^\infty(\Omega;\mathbf{R}^N)$.

(c) We first note that $u \in L^{p,\lambda}(\Omega;\mathbf{R}^N)$ if and only if

$$\int_{\Omega(a,\rho)} |u(x)|^p\,dx \le C\,\rho^\lambda$$

for all $a \in \Omega$ and $0 < \rho < \rho_0 = \min\{1, \operatorname{diam}\Omega\}$. Suppose $u \in L^{q,\mu}(\Omega;\mathbf{R}^N)$. Then, by Hölder's inequality, for all $a \in \Omega$, $0 < \rho < \rho_0 < 1$,

$$\begin{aligned}\int_{\Omega(a,\rho)} |u|^p\,dx &\le |\Omega(a,\rho)|^{1-\frac{p}{q}}\left(\int_{\Omega(a,\rho)} |u|^q\,dx\right)^{\frac{p}{q}} \\ &\le C\,\rho^{n-\frac{np}{q}}\left(\|u\|^q_{L^{q,\mu}(\Omega;\mathbf{R}^N)}\,\rho^\mu\right)^{\frac{p}{q}} \\ &\le C\,\rho^{\frac{\mu p}{q}+n-\frac{np}{q}}\,\|u\|^p_{L^{q,\mu}(\Omega;\mathbf{R}^N)} \\ &\le C\,\rho^\lambda\,\|u\|^p_{L^{q,\mu}(\Omega;\mathbf{R}^N)},\end{aligned}$$

where we have used the assumption $\frac{\mu p}{q} + n - \frac{np}{q} \ge \lambda$ and the fact $0 < \rho < 1$. Therefore, $u \in L^{p,\lambda}(\Omega;\mathbf{R}^N)$. □

Definition 4.2. For $1 \le p < \infty$, $\lambda \ge 0$ we define the **Campanato space** $\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)$ by

$$\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N) = \left\{ u \in L^p(\Omega;\mathbf{R}^N) \,\Big|\, \sup_{\substack{a\in\Omega \\ 0<\rho<\operatorname{diam}\Omega}} \rho^{-\lambda}\int_{\Omega(a,\rho)} |u - u_{a,\rho}|^p\,dx < \infty \right\},$$

where $u_{a,\rho}$ is the average of u on $\Omega(a,\rho)$. Define the seminorm and norm by

$$\begin{aligned}[u]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)} &= \sup_{\substack{a\in\Omega \\ 0<\rho<\operatorname{diam}\Omega}} \left(\rho^{-\lambda}\int_{\Omega(a,\rho)} |u - u_{a,\rho}|^p\,dx\right)^{1/p}, \\ \|u\|_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)} &= \|u\|_{L^p(\Omega;\mathbf{R}^N)} + [u]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)}.\end{aligned}$$

For $0 < \alpha \le 1$, we define the **Hölder space** $C^{0,\alpha}(\bar\Omega; \mathbf{R}^N)$ by

$$C^{0,\alpha}(\bar\Omega; \mathbf{R}^N) = \left\{ v \in L^\infty(\Omega; \mathbf{R}^N) \,\middle|\, |v(x) - v(y)| \le C\,|x-y|^\alpha,\ \forall\, x,\, y \in \Omega \right\}$$

and define the seminorm and norm by

$$\begin{aligned} [v]_{C^{0,\alpha}(\Omega;\mathbf{R}^N)} &= \sup_{\substack{x,\,y\in\Omega \\ x\neq y}} \frac{|v(x)-v(y)|}{|x-y|^\alpha}, \\ \|v\|_{C^{0,\alpha}(\Omega;\mathbf{R}^N)} &= \|v\|_{L^\infty(\Omega;\mathbf{R}^N)} + [v]_{C^{0,\alpha}(\Omega;\mathbf{R}^N)}. \end{aligned}$$

Proposition 4.17. *Both $\mathcal{L}^{p,\lambda}(\Omega; \mathbf{R}^N)$ and $C^{0,\alpha}(\bar\Omega; \mathbf{R}^N)$ are Banach spaces.*

Proposition 4.18. (*a*) *For any $p \ge 1$, $\lambda \ge 0$, $L^{p,\lambda}(\Omega; \mathbf{R}^N) \subset \mathcal{L}^{p,\lambda}(\Omega; \mathbf{R}^N)$.*

(*b*) *For any $0 < \alpha \le 1$, $C^{0,\alpha}(\bar\Omega; \mathbf{R}^N) \subset \mathcal{L}^{p,n+p\alpha}(\Omega; \mathbf{R}^N)$.*

Proof. (a) Note that

$$\left(\int_{\Omega(a,\rho)} |v(y) - v_{a,\rho}|^p\,dx \right)^{1/p} \le \|v\|_{L^p(\Omega(a,\rho)} + |v_{a,\rho}| \cdot |\Omega(a,\rho)|^{1/p}.$$

It turns out that we can exactly estimate the two terms on the right-hand side by

$$\|v\|_{L^p(\Omega(a,\rho))} \le \rho^{\lambda/p}\,\|v\|_{L^{p,\lambda}(\Omega;\mathbf{R}^N)},$$

$$|v_{a,\rho}| \cdot |\Omega(a,\rho)|^{1/p} \le \rho^{\lambda/p}\,\|v\|_{L^{p,\lambda}(\Omega;\mathbf{R}^N)}$$

so that it follows that

$$[v]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)} \le 2\,\|v\|_{L^{p,\lambda}(\Omega;\mathbf{R}^N)}.$$

Hence $L^{p,\lambda}(\Omega; \mathbf{R}^N) \subset \mathcal{L}^{p,\lambda}(\Omega; \mathbf{R}^N)$.

(b) Assume $v \in C^{0,\alpha}(\bar\Omega; \mathbf{R}^N)$. Then

$$\begin{aligned} |v(x) - v_{a,\rho}| &= |\fint_{\Omega(a,\rho)} (v(x) - v(y))\,dy| \\ &\le [v]_{C^{0,\alpha}(\bar\Omega;\mathbf{R}^N)} \cdot \fint_{\Omega(a,\rho)} |x-y|^\alpha\,dy \\ &\le [v]_{C^{0,\alpha}(\bar\Omega;\mathbf{R}^N)} \cdot (2\rho)^\alpha. \end{aligned}$$

Hence

$$\int_{\Omega(a,\rho)} |v(x) - v_{a,\rho}|^p\,dx \le C\,[v]^p_{C^{0,\alpha}(\bar\Omega;\mathbf{R}^N)} \cdot \rho^{n+p\alpha}$$

and hence

$$[v]_{\mathcal{L}^{p,n+p\alpha}(\Omega;\mathbf{R}^N)} \le C\,[v]_{C^{0,\alpha}(\bar\Omega;\mathbf{R}^N)}. \tag{4.22}$$

The proof is complete. □

In order to study the properties of Campanato functions, we need a condition on domain Ω introduced by Campanato.

Definition 4.3. We say that $\Omega \subset \mathbf{R}^n$ is **of type** A if there exists a constant $A > 0$ such that

$$|\Omega(a,\rho)| \ge A\,\rho^n, \quad \forall\ a \in \Omega,\ 0 < \rho < \operatorname{diam}\Omega. \tag{4.23}$$

This condition excludes that Ω may have sharp outward cusps; for instance, all Lipschitz domains are of type A.

Lemma 4.19. *Assume Ω is of type A and $u \in \mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)$. Then for any $0 < r < R < \infty$, $a \in \Omega$ it follows that*

$$|u_{a,R} - u_{a,r}| \le 2\,A^{-\frac{1}{p}}\,R^{\frac{\lambda}{p}}\,r^{-\frac{n}{p}} \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)}.$$

Proof.

$$\begin{aligned}
|u_{a,R} - u_{a,r}| \cdot |\Omega(a,r)|^{\frac{1}{p}} &= \|u_{a,R} - u_{a,r}\|_{L^p(\Omega(a,r))}\\
&\le \|u - u_{a,R}\|_{L^p(\Omega(a,r))} + \|u - u_{a,r}\|_{L^p(\Omega(a,r))}\\
&\le \|u - u_{a,R}\|_{L^p(\Omega(a,R))} + \|u - u_{a,r}\|_{L^p(\Omega(a,r))}\\
&\le [u]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)}\,R^{\frac{\lambda}{p}} + [u]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)}\,r^{\frac{\lambda}{p}}\\
&\le 2\,[u]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)}\,R^{\frac{\lambda}{p}}.
\end{aligned}$$

Hence the lemma follows from the assumption that $|\Omega(a,r)| \ge A\,r^n$. □

Proposition 4.20. *If Ω is of type A then $L^{p,\lambda}(\Omega;\mathbf{R}^N) \cong \mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)$ for $0 \le \lambda < n$.*

Proof. We only need to show $\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N) \subset L^{p,\lambda}(\Omega;\mathbf{R}^N)$. Let $u \in \mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)$. Given any $a \in \Omega$, $0 < \rho < \operatorname{diam}\Omega$, we have

$$\begin{aligned}
\|u\|_{L^p(\Omega(a,\rho))} &\le \|u - u_{a,\rho}\|_{L^p(\Omega(a,\rho))} + \|u_{a,\rho}\|_{L^p(\Omega(a,\rho))}\\
&\le [u]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)}\,\rho^{\frac{\lambda}{p}} + C\,|u_{a,\rho}|\,\rho^{\frac{n}{p}}.
\end{aligned}$$

We now choose an integer k large enough so that $\Omega(a, 2^k\rho) = \Omega$. Then, by Lemma 4.19, we have

$$\begin{aligned}
|u_{a,\rho}| &\le |u_{a,2^k\rho}| + \sum_{j=0}^{k-1} |u_{a,2^{j+1}\rho} - u_{a,2^j\rho}|\\
&\le |u_\Omega| + \sum_{j=0}^{k-1} 2\,A^{-\frac{1}{p}}\,(2^{j+1}\rho)^{\frac{\lambda}{p}}\,(2^j\rho)^{-\frac{n}{p}} \cdot [u]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)}\\
&\le |u_\Omega| + C\,\rho^{\frac{\lambda-n}{p}}\,[u]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)} \cdot \sum_{j=0}^{k} 2^{j(\lambda-n)/p}\\
&\le |u_\Omega| + C\,[u]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)}\,\rho^{\frac{\lambda-n}{p}},
\end{aligned}$$

where u_Ω is the average of u on Ω and therefore $|u_\Omega| \le C(\Omega)\,\|u\|_{L^p(\Omega;\mathbf{R}^N)}$. Combining these estimates, we deduce

$$\|u\|_{L^p(\Omega(a,\rho))} \le C\,[u]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)}\,\rho^{\frac{\lambda}{p}} + C\,\|u\|_{L^p(\Omega;\mathbf{R}^N)}\,\rho^{\frac{n}{p}}$$

and, by dividing both sides by $\rho^{\frac{\lambda}{p}}$ and noting $\lambda < n$,

$$\rho^{-\frac{\lambda}{p}}\,\|u\|_{L^p(\Omega(a,\rho))} \le C\,[u]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)} + C(\Omega)\,\|u\|_{L^p(\Omega;\mathbf{R}^N)}.$$

This proves

$$\|u\|_{L^{p,\lambda}(\Omega;\mathbf{R}^N)} \le C(\Omega)\,\|u\|_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)}.$$

□

Remark. Note that $\mathcal{L}^{p,n}(\Omega;\mathbf{R}^N) \not\cong L^{p,n}(\Omega;\mathbf{R}^N) \cong L^\infty(\Omega;\mathbf{R}^N)$. For example, let $p = \lambda = 1$, $n = N = 1$ and $\Omega = (0,1)$. Then $u = \ln x$ is in $\mathcal{L}^{1,1}(0,1)$ but not in $L^{1,1}(0,1) \cong L^\infty(0,1)$. In fact, $\mathcal{L}^{p,n}(\Omega;\mathbf{R}^N) \cong BMO(\Omega;\mathbf{R}^N)$, which is called the *John-Nirenberg space.* □

Theorem 4.21 (Campanato '63)**.** *Let Ω be of type A. Then for $n < \lambda \leq n+p$,*

$$\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N) \cong C^{0,\alpha}(\bar{\Omega};\mathbf{R}^N), \quad \alpha = \frac{\lambda - n}{p},$$

whereas for $\lambda > n+p$ we have $\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N) = \{constants\}$.

Proof. Assume $\lambda > n$ and $v \in \mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)$. For any $x \in \Omega$ and $R > 0$ we define

$$\tilde{v}(x) = \lim_{k\to\infty} v_{x,\frac{R}{2^k}}.$$

Lemma 4.22. *$\tilde{v}$ is well-defined and independent of $R > 0$.*

Proof. We first show the limit defining $\tilde{v}(x)$ exists. We need to show the sequence $\{v_{x,\frac{R}{2^k}}\}$ is Cauchy. For $h > k$ we have, by Lemma 4.19,

$$\begin{aligned} |v_{x,\frac{R}{2^h}} - v_{x,\frac{R}{2^k}}| &\leq \sum_{j=k}^{h-1} |v_{x,\frac{R}{2^j}} - v_{x,\frac{R}{2^{j+1}}}| \\ &\leq 2\,A^{-\frac{1}{p}}\,[v]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)}\,R^{\frac{\lambda-n}{p}} \cdot \sum_{j=k}^{h-1} 2^{\frac{j(n-\lambda)}{p}}, \end{aligned}$$

which, since $\lambda > n$, tends to zero if $k, h \to \infty$. Therefore $\{v_{x,\frac{R}{2^k}}\}$ is Cauchy and the limit $\tilde{v}(x)$ exists. Also in the inequality above, if $k = 0$ and $h \to \infty$ we also deduce

$$|v_{x,R} - \tilde{v}(x)| \leq C\,[v]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)} \cdot R^{\frac{\lambda-n}{p}}. \tag{4.24}$$

We now prove $\tilde{v}(x)$ is independent of $R > 0$. This follows easily since by Lemma 4.19

$$\lim_{k\to\infty} |v_{x,\frac{R}{2^k}} - v_{x,\frac{r}{2^k}}| = 0.$$

The lemma is proved. □

By Lebesgue's differentiation theorem, we also have $\tilde{v}(x) = v(x)$ for almost every $x \in \Omega$. Therefore $\tilde{v} = v$ in $\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)$.

Lemma 4.23. *$\tilde{v} \in C^{0,\alpha}(\bar{\Omega};\mathbf{R}^N)$, where $\alpha = \frac{\lambda-n}{p}$.*

Proof. Let $x, y \in \Omega$ and $x \neq y$. Let $R = |x-y|$. By (4.24) it follows that

$$\begin{aligned} |\tilde{v}(x) - \tilde{v}(y)| &\leq |\tilde{v}(x) - v_{x,2R}| + |\tilde{v}(y) - v_{y,2R}| + |v_{x,2R} - v_{y,2R}| \\ &\leq C\,[v]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)} \cdot R^\alpha + |v_{x,2R} - v_{y,2R}|. \end{aligned}$$

We need to estimate $|v_{x,2R} - v_{y,2R}|$. To this end, let $S = \Omega(x,2R) \cap \Omega(y,2R)$. Then $\Omega(x,R) \subset S$ and hence

$$|S| \geq |\Omega(x,R)| \geq A\,R^n.$$

On the other hand, we have

$$\begin{aligned} |S|^{\frac{1}{p}} \cdot |v_{x,2R} - v_{y,2R}| &= \|v_{x,2R} - v_{y,2R}\|_{L^p(S)} \\ &\leq \|v_{x,2R} - v\|_{L^p(S)} + \|v_{y,2R} - v\|_{L^p(S)} \\ &\leq \|v_{x,2R} - v\|_{L^p(\Omega(x,2R))} + \|v_{y,2R} - v\|_{L^p(\Omega(y,2R))} \\ &\leq 2\,[v]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)} \cdot (2R)^{\lambda/p}. \end{aligned}$$

Combining the above two estimates we have

$$|v_{x,2R} - v_{y,2R}| \leq C\,[v]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)} \cdot R^{\frac{\lambda-n}{p}} = C\,[v]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)} \cdot R^\alpha$$

and hence

$$|\tilde{v}(x) - \tilde{v}(y)| \le C\,[v]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)} \cdot |x-y|^\alpha.$$

This shows

$$[\tilde{v}]_{C^{0,\alpha}(\bar{\Omega};\mathbf{R}^N)} \le C\,[v]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)}.$$

To complete the proof of this lemma, we only have to observe that, by (4.24) with $R = \operatorname{diam}\Omega$,

$$\begin{aligned}
\|\tilde{v}\|_{L^\infty(\Omega;\mathbf{R}^N)} &\le |v_\Omega| + C\,[v]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)} \cdot R^\alpha \\
&\le C(\Omega)\,\|v\|_{L^p(\Omega;\mathbf{R}^N)} + C(\Omega)\,[v]_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)} \\
&= C(\Omega)\,\|v\|_{\mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)}.
\end{aligned}$$

This proves the lemma. □

We have thus proved that if $\lambda > n$ then every $v \in \mathcal{L}^{p,\lambda}(\Omega;\mathbf{R}^N)$ has a representation $\tilde{v}$ which belongs to $C^{0,\alpha}(\bar{\Omega};\mathbf{R}^N)$ with $\alpha = (\lambda - n)/p$. If $\lambda > n + p$ then $\alpha > 1$ and any $u \in C^{0,\alpha}(\bar{\Omega};\mathbf{R}^N)$ must be a constant (why?). The proof of Campanato's theorem is complete. □

In order to use the Campanato spaces for elliptic systems, we also need some local version of these spaces. To disperse some technicalities, we prove the following lemma.

Lemma 4.24. *Let $p = 1, 2$ and $u \in L^p_{loc}(\Omega;\mathbf{R}^N)$. Then the map $E \mapsto \int_E |u - u_E|^p$ is nondecreasing in subsets $E \subset\subset \Omega$.*

Proof. We prove the case $p = 2$ first. Let $E \subset F \subset\subset \Omega$. Then

$$\begin{aligned}
\int_E |u - u_E|^2 &= \int_E |u - u_F + u_F - u_E|^2 \\
&= \int_E |u - u_F|^2 + 2\,(u_F - u_E) \cdot \int_E (u - u_F) + |E| \cdot |u_F - u_E|^2 \\
&= \int_E |u - u_F|^2 - |E| \cdot |u_F - u_E|^2 \\
&\le \int_F |u - u_F|^2.
\end{aligned}$$

We now prove the case $p = 1$. Note that

$$\begin{aligned}
\int_E |u - u_E| &= \int_E |u - u_F + u_F - u_E| \\
&\le \int_E |u - u_F| + \int_E |u_F - u_E| \\
&= \int_F |u - u_F| - \int_{F \setminus E} |u - u_F| + |E| \cdot |u_F - u_E|.
\end{aligned}$$

Thus we need to prove

$$|E| \cdot |u_F - u_E| \le \int_{F \setminus E} |u - u_F|. \tag{4.25}$$

Note that, by Jensen's inequality,

$$\begin{aligned}
\fint_{F\setminus E}|u-u_F| &\geq \left|\fint_{F\setminus E}(u-u_F)\right| \\
&= \frac{1}{|F\setminus E|}\left||F\setminus E|\cdot u_F-\int_{F\setminus E}u\right| \\
&= \frac{1}{|F\setminus E|}\left||F|\cdot u_F-|E|\cdot u_F-\int_F u+\int_E u\right| \\
&= \frac{1}{|F\setminus E|}\big||F|\cdot u_F-|E|\cdot u_F-|F|\cdot u_F+|E|\cdot u_E\big| \\
&= \frac{|E|}{|F\setminus E|}\,|u_F-u_E|,
\end{aligned}$$

and hence (4.25) follows. □

Theorem 4.25. *Let $p=1,\,2$ and $u\in L^p_{loc}(\Omega;\mathbf{R}^N)$. Assume there exists a constant $C_u>0$ and $\alpha>0$ such that*

$$\int_{B_\rho}|u-u_{B_\rho}|^p\,dx\leq C_u\,\rho^\lambda$$

holds for all balls $B_\rho\subset\subset\Omega$. Then for any subdomain $\Omega'\subset\subset\Omega$ we have $u\in\mathcal{L}^{p,\lambda}(\Omega';\mathbf{R}^N)$ and moreover

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega';\mathbf{R}^N)}\leq C(\Omega')\,[C_u^{1/p}+\|u\|_{L^p(\Omega';\mathbf{R}^N)}].$$

Proof. Let $\Omega'\subset\subset\Omega$ be given. We will show $u\in\mathcal{L}^{p,\lambda}(\Omega';\mathbf{R}^N)$. Let $d=\operatorname{dist}(\Omega';\partial\Omega)$. Given any $a\in\Omega'$ and $0<\rho<\operatorname{diam}(\Omega')$. If $\rho<\operatorname{dist}(a;\partial\Omega)$ we have by the previous lemma,

$$\int_{\Omega'(a,\rho)}|u-u_{\Omega'(a,\rho)}|^p\,dx\leq\int_{B_\rho(a)}|u-u_{B_\rho(a)}|^p\,dx\leq C_u\,\rho^\lambda.$$

If $\rho\geq\operatorname{dist}(a;\partial\Omega)$, then $\rho\geq d>0$ and hence

$$\int_{\Omega'(a,\rho)}|u-u_{\Omega'(a,\rho)}|^p\,dx\leq 2^p\int_{\Omega'(a,\rho)}|u|^p\,dx\leq\frac{2^p\,\|u\|^p_{L^p(\Omega';\mathbf{R}^n)}}{d^\lambda}\,\rho^\lambda.$$

Therefore, for all $a\in\Omega'$, $0<\rho<\operatorname{diam}(\Omega')$ it follows that

$$\int_{\Omega'(a,\rho)}|u-u_{\Omega'(a,\rho)}|^p\leq\left[C_u+\frac{2^p\,\|u\|^p_{L^p(\Omega';\mathbf{R}^n)}}{d^\lambda}\right]\rho^\lambda,$$

and hence by definition $u\in\mathcal{L}^{p,\lambda}(\Omega';\mathbf{R}^N)$ and moreover

$$[u]_{\mathcal{L}^{p,\lambda}(\Omega';\mathbf{R}^N)}\leq C(\Omega')\,[C_u^{1/p}+\|u\|_{L^p(\Omega';\mathbf{R}^N)}].$$

The proof is complete. □

Theorem 4.26 (Morrey)**.** *Let $u\in W^{1,p}_{loc}(\Omega;\mathbf{R}^N)$. Suppose for some $\beta>0$ we have*

$$\int_{B_\rho}|Du|^p\,dx\leq C\,\rho^{n-p+\beta},\quad\forall\,B_\rho\subset\subset\Omega.$$

Then for any $\Omega'\subset\subset\Omega$ of type A, we have $u\in C^{0,\frac{\beta}{p}}(\bar\Omega';\mathbf{R}^N)$.

Proof. Using the Poincaré type inequality

$$\int_{B_R} |u - u_{B_R}|\, dx \le C_n\, R \int_{B_R} |Du|\, dx, \tag{4.26}$$

we have for all balls $B_\rho \subset\subset \Omega$,

$$\begin{aligned} \int_{B_\rho} |u - u_{B_\rho}|\, dx &\le C_n\, \rho \int_{B_\rho} |Du|\, dx \\ &\le C_n\, \rho\, \|Du\|_{L^p(B_\rho;\mathbf{M}^{N\times n})} \cdot |B_\rho|^{1-\frac{1}{p}} \\ &\le C\, \rho \cdot \rho^{\frac{n-p+\beta}{p}} \cdot \rho^{n(1-\frac{1}{p})} \\ &= C\, \rho^{n+\frac{\beta}{p}}. \end{aligned}$$

Therefore, by Theorem 4.25, $u \in \mathcal{L}^{1,n+\frac{\beta}{p}}(\Omega'; \mathbf{R}^N) \cong C^{0,\frac{\beta}{p}}(\bar{\Omega}'; \mathbf{R}^N)$. □

When $\beta = 0$ Morrey's theorem has to be replaced by the *John-Nirenberg* estimate; see G-T, P. 166, Theorem 7.21.

Theorem 4.27 (John-Nirenberg). *Let $u \in W^{1,1}(\Omega; \mathbf{R}^N)$ where Ω is convex. Suppsose there exists a constant K such that*

$$\int_{\Omega(a,R)} |Du|\, dx \le K\, R^{n-1} \quad \forall\, a \in \Omega,\ R < \operatorname{diam}\Omega. \tag{4.27}$$

Then there exist positive constants σ_0 and C depending only on n such that

$$\int_\Omega \exp\Big(\frac{\sigma}{K}\,|u - u_\Omega|\Big)\, dx \le C\,(\operatorname{diam}\Omega)^n, \tag{4.28}$$

where $\sigma = \sigma_0\, |\Omega|\, (\operatorname{diam}\Omega)^{-n}$.

Remark. The set of all functions $u \in W^{1,1}(\Omega; \mathbf{R}^N)$ satisfying (4.27) is the space $BMO(\Omega; \mathbf{R}^N)$ introduced by John and Nirenberg, and for Ω cubes or balls it follows that

$$BMO(\Omega; \mathbf{R}^N) \cong \mathcal{L}^{p,n}(\Omega; \mathbf{R}^N), \quad \forall\, p \ge 1.$$

For the proof of all these results and more on BMO-spaces, we refer to Gilbarg-Trudinger's book for a proof based on the Riesz potential, and Giaquinta's book on the Calderon-Zygmund cube decomposition. □

4.6. Estimates for systems with constant coefficients

We consider systems with constant coefficients. Let $A = A_{ij}^{\alpha\beta}$ be constants satisfying hypothesis (H2). We first have some Campanato estimates for homogeneous systems.

Theorem 4.28. *Let $u \in W^{1,2}_{loc}(\Omega; \mathbf{R}^N)$ be a weak solution of*

$$D_\alpha(A_{ij}^{\alpha\beta}\, D_\beta u^j) = 0, \quad i = 1, 2, ..., N. \tag{4.29}$$

Then there exists a constant c depending on $A_{ij}^{\alpha\beta}$ such that for any concentric balls $B_\rho \subset\subset B_R \subset\subset \Omega$,

$$\int_{B_\rho} |u|^2\, dx \le c \cdot (\frac{\rho}{R})^n \int_{B_R} |u|^2\, dx, \tag{4.30}$$

$$\int_{B_\rho} |u - u_{B_\rho}|^2\, dx \le c \cdot (\frac{\rho}{R})^{n+2} \int_{B_R} |u - u_{B_R}|^2\, dx. \tag{4.31}$$

Proof. We do scaling first. Let $B_R = B_R(a)$, where $a \in \Omega$. Define

$$v(y) = u(a + Ry),$$

where $y \in D \equiv \{y \in \mathbf{R}^n \,|\, a + Ry \in \Omega\}$, which includes $\bar{B}_1(0)$ in the y-space. Note that $v \in W^{1,2}_{loc}(D; \mathbf{R}^N)$ is a weak solution of

$$D_{y_\alpha}(A^{\alpha\beta}_{ij} D_{y_\beta} v^j) = 0.$$

Then the Caccioppoli estimates show that

$$\|v\|_{W^{k,2}(B_{1/2(0)};\mathbf{R}^N)} \leq C(k)\,\|v\|_{L^2(B_1(0);\mathbf{R}^N)}, \quad \forall\, k = 1, 2, ...$$

and hence for all $0 < t \leq 1/2$ it follows that

$$\begin{aligned} \int_{B_t(0)} |v|^2\,dy &\leq c(n)\,t^n \sup_{y \in B_{1/2}(0)} |v(y)|^2 \\ &\leq c(n)\,t^n\,\|v\|^2_{W^{k,2}(B_{1/2}(0);\mathbf{R}^N)} \\ &\leq c(n,k)\,t^n\,\|v\|^2_{L^2(B_1(0);\mathbf{R}^N)}, \end{aligned}$$

where we have chosen integer $k > n/2$ and used the Sobolev embedding $W^{k,2}(B_{1/2}(0);\mathbf{R}^N) \hookrightarrow C^{0,\alpha}(B_{1/2}(0);\mathbf{R}^N)$ for some $0 < \alpha < 1$. Now if $t \geq 1/2$ we easily have

$$\int_{B_t(0)} |v|^2\,dy \leq 2^n\,t^n \int_{B_1(0)} |v|^2\,dy.$$

Therefore we have proved

$$\int_{B_t(0)} |v|^2\,dy \leq C(n)\,t^n \int_{B_1(0)} |v|^2\,dy, \quad \forall\, 0 < t < 1.$$

Rescaling back to $u(x)$ and letting $\rho = tR$ we have

$$\int_{B_\rho(a)} |u|^2\,dx \leq C(n)\,(\frac{\rho}{R})^n \cdot \int_{B_R(a)} |u|^2\,dx, \quad \forall\,\rho < R < \mathrm{dist}(a;\partial\Omega);$$

this proves (4.30). Note that Du is also a weak solution of (4.29); therefore, by (4.30) it follows that

$$\int_{B_\rho(a)} |Du|^2\,dx \leq C(n)\,(\frac{\rho}{R})^n \cdot \int_{B_R(a)} |Du|^2\,dx, \quad \forall\,\rho < R < \mathrm{dist}(a;\partial\Omega).$$

Suppose $0 < \rho < R/2$. Then we use the Poincaré inequality, the previous estimate and the Caccioppoli inequality to obtain

$$\begin{aligned} \int_{B_\rho} |u - u_{B_\rho}|^2\,dx &\leq c(n)\,\rho^2 \cdot \int_{B_\rho} |Du|^2\,dx \\ &\leq C(n)\,\rho^2\,(\frac{\rho}{R})^n \cdot \int_{B_{R/2}} |Du|^2\,dx \\ &\leq C(n)\,(\frac{\rho}{R})^{n+2} \cdot \int_{B_R} |u - u_{B_R}|^2\,dx. \end{aligned}$$

Now if $\rho \geq R/2$ we easily have

$$\begin{aligned}\int_{B_\rho} |u - u_{B_\rho}|^2\,dx &= \int_{B_\rho} |u - u_{B_R}|^2\,dx - |B_\rho|\cdot|u_{B_\rho} - u_{B_R}|^2 \\ &\leq \int_{B_R} |u - u_{B_R}|^2\,dx \\ &\leq 2^{n+2}\,(\frac{\rho}{R})^{n+2}\cdot\int_{B_R} |u - u_{B_R}|^2\,dx.\end{aligned}$$

Therefore, for all $0 < \rho < R < \operatorname{dist}(a;\partial\Omega)$,

$$\int_{B_\rho} |u - u_{B_\rho}|^2\,dx \leq C(n)\,(\frac{\rho}{R})^{n+2}\cdot\int_{B_R} |u(x) - u_{B_R}|^2\,dx.$$

The proof of both (4.30) and (4.31) is now complete. □

In (4.30) and (4.31), if we let $R \to \operatorname{dist}(a;\partial\Omega)$, we see that both estimates also hold for all balls $B_\rho \subset\subset B_R \subset \Omega$. We state this fact as follows.

Corollary 4.29. *Both estimates* (4.30) *and* (4.31) *hold for all balls* $B_\rho \subset\subset B_R \subset \Omega$.

In the following, we consider the nonhomogeneous elliptic systems with constant coefficients:

$$D_\alpha(A^{\alpha\beta}_{ij}\,D_\beta u^j) = D_\alpha f^i_\alpha, \quad i = 1, 2, \cdots. \tag{4.32}$$

Theorem 4.30. *Let* $A^{\alpha\beta}_{ij}$ *satisfy hypothesis* $(H2)$ *and* $u \in W^{1,2}_{loc}(\Omega;\mathbf{R}^N)$ *be a weak solution of* (4.32)*. Suppose* $f \in \mathcal{L}^{2,\lambda}_{loc}(\Omega;\mathbf{M}^{N\times n})$ *and* $0 \leq \lambda < n+2$*. Then* $Du \in \mathcal{L}^{2,\lambda}_{loc}(\Omega;\mathbf{M}^{N\times n})$.

Corollary 4.31. *Under the same assumptions, if* $f \in C^{0,\mu}_{loc}(\Omega;\mathbf{M}^{N\times n})$ *and* $0 < \mu < 1$ *then* $Du \in C^{0,\mu}_{loc}(\Omega;\mathbf{M}^{N\times n})$.

Proof of Theorem 4.30. Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Let $a \in \Omega'$ and $B_R(a) = B_R \subset \Omega''$. We write $u = v + w = v + (u - v)$, where $v \in W^{1,2}(B_R;\mathbf{R}^N)$ is the solution of the Dirichlet problem

$$\begin{cases}\operatorname{Div}(A\,Dv) = 0 & \text{in } B_R,\\ v|_{\partial B_R} = u.\end{cases}$$

The existence of solution v follows by the Lax-Milgram theorem. We now by Corollary 4.29 have for all $\rho < R$

$$\int_{B_\rho} |Dv - (Dv)_{B_\rho}|^2\,dx \leq c\cdot(\frac{\rho}{R})^{n+2}\int_{B_R} |Dv - (Dv)_{B_R}|^2\,dx. \tag{4.33}$$

From this we have

$$\begin{aligned}&\int_{B_\rho} |Du - (Du)_{B_\rho}|^2\,dx \\ =\ &\int_{B_\rho} |Dv + Dw - (Dv)_{B_\rho} - (Dw)_{B_\rho}|^2\,dx \\ \leq\ & C\cdot(\frac{\rho}{R})^{n+2}\int_{B_R} |Dv - (Dv)_{B_R}|^2\,dx + \int_{B_\rho} |Dw - (Dw)_{B_\rho}|^2\,dx \\ \leq\ & C_1\cdot(\frac{\rho}{R})^{n+2}\int_{B_R} |Du - (Du)_{B_R}|^2\,dx + C_2\int_{B_R} |Du - Dv|^2\,dx.\end{aligned}$$

Since $u - v \in W_0^{1,2}(B_R; \mathbf{R}^N)$, we use the Legendre-Hadamard condition to have

$$\begin{aligned}\nu \int_{B_R} |Du - Dv|^2\,dx &\le \int_{B_R} A\,D(u-v)\cdot D(u-v)\,dx \\ &= \int_{B_R} (f - f_{B_R})\cdot D(u-v)\,dx \\ &\le \frac{\nu}{2}\int_{B_R} |Du - Dv|^2\,dx + C_\nu \int_{B_R} |f - f_{B_R}|^2\,dx\end{aligned}$$

and hence

$$\int_{B_R} |Du - Dv|^2\,dx \le C_\nu \int_{B_R} |f - f_{B_R}|^2\,dx \le C_\nu\,[f]^2_{\mathcal{L}^{2,\lambda}(\Omega'';\mathbf{M}^{N\times n})} \cdot R^\lambda.$$

Combining what we proved above, we have

$$\begin{aligned}\int_{B_\rho} |Du - (Du)_{B_\rho}|^2\,dx &\le C_1 \cdot (\frac{\rho}{R})^{n+2} \int_{B_R} |Du - (Du)_{B_R}|^2\,dx \\ &+ C_3\,[f]^2_{\mathcal{L}^{2,\lambda}(\Omega'';\mathbf{M}^{N\times n})} \cdot R^\lambda.\end{aligned}$$

Let

$$\Phi(\rho) = \int_{B_\rho} |Du - (Du)_{B_\rho}|^2\,dx.$$

Using the Campanato lemma below, it follows that

$$\Phi(\rho) \le C_4 \left[\left(\frac{\rho}{R}\right)^\lambda \Phi(R) + [f]^2_{\mathcal{L}^{2,\lambda}(\Omega'';\mathbf{M}^{N\times n})} \cdot \rho^\lambda\right].$$

Now we have

$$\begin{aligned}\int_{\Omega'(a,\rho)} |Du - (Du)_{\Omega'(a,\rho)}|^2 &\le \int_{\Omega'(a,\rho)} |Du - (Du)_{B_\rho(a)}|^2 \\ &\le \int_{B_\rho} |Du - (Du)_{B_\rho}|^2 = \Phi(\rho) \\ &\le C_5\,\rho^\lambda \left(\|Du\|^2_{L^2(\Omega'';\mathbf{M}^{N\times n})} + [f]^2_{\mathcal{L}^{2,\lambda}(\Omega'';\mathbf{M}^{N\times n})}\right).\end{aligned}$$

Therefore

$$[Du]_{\mathcal{L}^{2,\lambda}(\Omega';\mathbf{M}^{N\times n})} \le C\left(\|Du\|_{L^2(\Omega'';\mathbf{M}^{N\times n})} + [f]_{\mathcal{L}^{2,\lambda}(\Omega'';\mathbf{M}^{N\times n})}\right).$$

The proof is complete. □

Lemma 4.32 (Campanato Lemma)**.** *Let $\Phi(t)$ be a nonnegative nondecreasing function. Consider the inequality*

$$\Phi(\rho) \le A\left[\left(\frac{\rho}{R}\right)^\alpha + \epsilon\right]\Phi(R) + B\,R^\beta \quad \forall\,\rho \le R \le R_0, \tag{4.34}$$

where A, B, α, β, ϵ are positive constants with $\alpha > \beta$. Then there exists $\epsilon_0 = \epsilon_0(A, \alpha, \beta)$ such that if (4.34) *holds for some $0 \le \epsilon \le \epsilon_0$ then*

$$\Phi(\rho) \le C\left[\left(\frac{\rho}{R}\right)^\beta \Phi(R) + B\,\rho^\beta\right] \quad \forall\,\rho \le R \le R_0,$$

where C is a constant depending only on α, β, A.

Proof. For $0 < \tau < 1$ and $R \le R_0$, (4.34) is equivalent to

$$\Phi(\tau R) \le A\,\tau^{\alpha}\,(1 + \epsilon\tau^{-\alpha})\,\Phi(R) + B\,R^{\beta}.$$

Let $\gamma \in (\beta,\,\alpha)$ be fixed and choose $\tau \in (0,\,1)$ so that $2A\tau^{\alpha} \le \tau^{\gamma}$. Let $\epsilon_0 = \tau^{\alpha}$. Then, if (4.34) holds for some $0 \le \epsilon \le \epsilon_0$, we have for every $R \le R_0$

$$\Phi(\tau R) \le \tau^{\gamma}\,\Phi(R) + B\,R^{\beta}$$

and therefore for all $k = 1, 2, \cdots$

$$\begin{aligned}\Phi(\tau^{k+1}R) &\le \tau^{\gamma}\,\Phi(\tau^{k}R) + B\,\tau^{k\beta}\,R^{\beta}\\ &\le \tau^{(k+1)\gamma}\,\Phi(R) + B\,\tau^{k\beta}\,R^{\beta}\sum_{j=0}^{k}\tau^{j(\gamma-\beta)}\\ &\le C\,\tau^{(k+1)\beta}(\Phi(R) + B\,R^{\beta}).\end{aligned}$$

Since $\Phi(t)$ is nondecreasing and $\tau^{k+2}R < \rho \le \tau^{k+1}R$ for some k, we have

$$\Phi(\rho) \le C\left(\frac{\rho}{R}\right)^{\beta}(\Phi(R) + B\,R^{\beta}) = C\,\left[\left(\frac{\rho}{R}\right)^{\beta}\Phi(R) + B\,\rho^{\beta}\right],$$

as desired. The proof is complete. □

4.7. Schauder estimates for systems with variable coefficients

In this section, we study the local regularity of weak solutions of systems with variable coefficients. We first prove the regularity in the Morrey space $L^{2,\lambda}_{loc}(\Omega)$ for the gradient of the weak solutions.

Theorem 4.33. *Let $A^{\alpha\beta}_{ij}(x)$ satisfy the hypothesis $(H3)$ and $u \in W^{1,2}_{loc}(\Omega;\mathbf{R}^N)$ be a weak solution of system*

$$D_{\alpha}(A^{\alpha\beta}_{ij}(x)\,D_{\beta}u^{j}) = D_{\alpha}f^{i}_{\alpha}. \tag{4.35}$$

Suppose $f \in L^{2,\lambda}_{loc}(\Omega;\mathbf{M}^{N\times n})$ and $0 \le \lambda < n$. Then $Du \in L^{2,\lambda}_{loc}(\Omega;\mathbf{M}^{N\times n})$.

Proof. Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Let $a \in \Omega'$ and $B_R(a) = B_R \subset \Omega''$. Using the standard Korn's freezing coefficients device, u is a weak solution of system with constant coefficients

$$\operatorname{Div}(A(a)\,Du) = \operatorname{Div}F, \quad F = f + (A(a) - A(x))\,Du.$$

Let $v \in W^{1,2}(B_R;\mathbf{R}^N)$ be the solution of the Dirichlet problem

$$\begin{cases}\operatorname{Div}(A(a)\,Dv) = 0 & \text{in } B_R,\\ v|_{\partial B_R} = u.\end{cases}$$

Then, as before, using (4.30) instead of (4.31) we have

$$\begin{aligned}\int_{B_\rho}|Du|^2 &\le c\cdot(\frac{\rho}{R})^n\int_{B_R}|Du|^2 + C\int_{B_R}|D(u-v)|^2\\ &\le c\cdot(\frac{\rho}{R})^n\int_{B_R}|Du|^2 + C\int_{B_R}|F|^2\\ &\le c\cdot(\frac{\rho}{R})^n\int_{B_R}|Du|^2 + C\int_{B_R}|f|^2 + C\,\omega(R)\int_{B_R}|Du|^2\\ &\le c\,\left[(\frac{\rho}{R})^n + \omega(R)\right]\int_{B_R}|Du|^2 + C\,\|f\|^2_{L^{2,\lambda}(\Omega'';\mathbf{M}^{N\times n})}\,R^{\lambda},\end{aligned}$$

where $\omega(R)$ is the uniform modulus of continuity of $A(x)$:

$$\omega(R) = \sup_{|x-y|\le R} |A(x) - A(y)|.$$

We choose $R_0 > 0$ sufficiently small so that $\omega(R) < \epsilon_0$ for all $R < R_0$, where ϵ_0 is the constant appearing in the Campanato lemma above. Therefore,

$$\int_{B_\rho} |Du|^2\,dx \le C(\Omega', \Omega'') \left(\|Du\|^2_{L^2(\Omega'';\mathbf{M}^{N\times n})} + \|f\|^2_{L^{2,\lambda}(\Omega'';\mathbf{M}^{N\times n})} \right) \rho^\lambda.$$

This, by a local version similar to the Campanato space, we have

$$\|Du\|_{L^{2,\lambda}(\Omega';\mathbf{M}^{N\times n})} \le C(\Omega', \Omega'') \left(\|Du\|_{L^2(\Omega'';\mathbf{M}^{N\times n})} + \|f\|_{L^{2,\lambda}(\Omega'';\mathbf{M}^{N\times n})} \right),$$

which proves the theorem. □

We now study the regularity of the gradient of weak solutions in the Hölder spaces. This is done by proving the regularity of gradient in the Campanato space $\mathcal{L}^{2,n+2\mu}_{loc}(\Omega)$ for some $\mu \in (0, 1)$.

Theorem 4.34. *Let $A^{\alpha\beta}_{ij} \in C^{0,\mu}(\Omega)$ with some $0 < \mu < 1$ satisfy the hypothesis $(H3)$ and $u \in W^{1,2}_{loc}(\Omega; \mathbf{R}^N)$ be a weak solution of system*

$$D_\alpha(A^{\alpha\beta}_{ij}(x)\, D_\beta u^j) = D_\alpha f^i_\alpha. \tag{4.36}$$

Suppose $f \in C^{0,\mu}_{loc}(\Omega; \mathbf{M}^{N\times n})$. Then $Du \in C^{0,\mu}_{loc}(\Omega; \mathbf{M}^{N\times n})$.

Proof. Similarly as above, we have

$$\begin{aligned}
\int_{B_\rho} |Du - (Du)_{B_\rho}|^2 \;&\le\; c \cdot (\frac{\rho}{R})^{n+2} \int_{B_R} |Du - (Du)_{B_R}|^2 + C \int_{B_R} |F - f_{B_R}|^2 \\
&\le\; c \cdot (\frac{\rho}{R})^{n+2} \int_{B_R} |Du - (Du)_{B_R}|^2 + C \int_{B_R} |f - f_{B_R}|^2 \\
&+\; C\, [A]^2_{C^{0,\mu}}\, R^{2\mu} \int_{B_R} |Du|^2 \\
&\le\; c \cdot (\frac{\rho}{R})^{n+2} \int_{B_R} |Du - (Du)_{B_R}|^2 + C\, [f]^2_{C^{0,\mu}(\Omega'')}\, R^{n+2\mu} \\
&+\; C\, [A]^2_{C^{0,\mu}}\, R^{2\mu} \int_{B_R} |Du|^2.
\end{aligned}$$

Since by the previous theorem $Du \in L^{2,n-\epsilon}_{loc}(\Omega; \mathbf{M}^{N\times n})$ for all $\epsilon > 0$ we obtain

$$\int_{B_\rho} |Du - (Du)_{B_\rho}|^2 \le A\, (\frac{\rho}{R})^{n+2} \int_{B_R} |Du - (Du)_{B_R}|^2 + B\, R^{n+2\mu-\epsilon}.$$

Using Campanato's lemma, we have

$$\int_{B_\rho} |Du - (Du)_{B_\rho}|^2 \le C\, \rho^{n+2\mu-\epsilon}$$

and hence $Du \in \mathcal{L}^{2,n+2\mu-\epsilon}_{loc}(\Omega; \mathbf{M}^{N\times n})$ for all $\epsilon > 0$. This implies $Du \in C^{0,\beta}_{loc}(\Omega; \mathbf{M}^{N\times n})$ for $\beta = \mu - \frac{\epsilon}{2}$. In particular, Du is locally bounded. Therefore, again, using the above estimates, it follows that

$$\begin{aligned}\int_{B_\rho} |Du-(Du)_{B_\rho}|^2 &\le c\cdot(\frac{\rho}{R})^{n+2}\int_{B_R}|Du-(Du)_{B_R}|^2\\ &+ C\,[f]^2_{C^{0,\mu}(\Omega'')}\,R^{n+2\mu}+C\,[A]^2_{C^{0,\mu}}\,R^{2\mu}\int_{B_R}|Du|^2\\ &\le c\cdot(\frac{\rho}{R})^{n+2}\int_{B_R}|Du-(Du)_{B_R}|^2\\ &+ C\,[f]^2_{C^{0,\mu}(\Omega'')}\,R^{n+2\mu}+C\,[A]^2_{C^{0,\mu}}\,R^{n+2\mu}\end{aligned}$$

and using Campanato's lemma again we have $Du \in L^{2,n+2\mu}_{loc}(\Omega;\mathbf{M}^{N\times n})$ and hence $Du \in C^{0,\mu}_{loc}(\Omega;\mathbf{M}^{N\times n})$. □

Finally we remark that the following higher order regularity result can be easily deduced.

Theorem 4.35. *Let $k\ge 0$, $0<\mu<1$ and $A^{\alpha\beta}_{ij}\in C^{k,\mu}(\Omega)$ satisfy the hypothesis $(H3)$ and $u\in W^{1,2}_{loc}(\Omega;\mathbf{R}^N)$ be a weak solution of system*

$$D_\alpha(A^{\alpha\beta}_{ij}(x)\,D_\beta u^j)=D_\alpha f^i_\alpha. \tag{4.37}$$

Suppose $f\in C^{k,\mu}_{loc}(\Omega;\mathbf{M}^{N\times n})$. Then $u\in C^{k+1,\mu}_{loc}(\Omega;\mathbf{R}^N)$.

4.8. Systems in non-divergence form and boundary estimates

In this section, we show that the Campanato estimates can also be proved for systems that are not in divergence form and also the global estimates are valid if the boundary $\partial\Omega$ of the domain $\Omega\subset\mathbf{R}^n$ satisfies certain smoothness condition.

We first prove the interior estimates for systems in the following form:

$$A^{\alpha\beta}_{ij}(x)\,D_{\alpha\beta}u^j=f^i;\quad i=1,2,\cdots,N.$$

By a weak solution u to this system we mean a function $u\in W^{2,2}_{loc}(\Omega;\mathbf{R}^N)$ such that the system is satisfied almost everywhere in Ω.

Theorem 4.36. *Let $A^{\alpha\beta}_{ij}$, $f^i\in C^{0,\mu}_{loc}(\Omega)$ and $0<\mu<1$. If $u\in W^{2,2}_{loc}(\Omega;\mathbf{R}^N)$ is a weak solution to the system above, then $u\in C^{2,\mu}_{loc}(\Omega;\mathbf{R}^N)$.*

We now consider the regularity up to the boundary. In what follows, we assume the boundary $\partial\Omega$ of the domain Ω is of $C^{1,\mu}$; that is, for any $x_0\in\partial\Omega$, there exist an open set $U\subset\mathbf{R}^n$ containing x_0 and a $C^{1,\mu}$-diffeomorphism $y=G\colon U\to\mathbf{R}^n$ such that

$$G(x_0)=0,\quad G(U\cap\Omega)=B_1^+=\{y\in\mathbf{R}^n\,|\,|y|<1,\ y_n>0\};$$

$$G(U\cap\partial\Omega)=\Gamma_1=\{y\in\mathbf{R}^n\,|\,|y|<1,\ y_n=0\}.$$

This G is called (locally) flattening the boundary.

Theorem 4.37. *Let $\partial\Omega$ be of $C^{1,\mu}$ with $0<\mu<1$ and $A^{\alpha\beta}_{ij}$, $f^i_\alpha\in C^{0,\mu}(\bar\Omega)$ and $g^j\in C^{1,\mu}(\bar\Omega)$. Let $A(x)$ satisfy the condition $(H3)$. If $u\in W^{1,2}(\Omega;\mathbf{R}^N)$ is a weak solution to the problem*

$$\operatorname{Div}(A(x)\,Du)=\operatorname{Div}f,\quad u|_{\partial\Omega}=g,$$

then $u\in C^{1,\mu}(\bar\Omega;\mathbf{R}^N)$.

Theorem 4.38. *Let $\partial\Omega$ be of $C^{1,\mu}$ with $0 < \mu < 1$ and $A_{ij}^{\alpha\beta} \in C^{0,\mu}(\bar{\Omega})$ satisfy $(H3)$. Assume $f^i \in C^{0,\mu}(\bar{\Omega})$, $g^j \in C^{2,\mu}(\bar{\Omega})$. If $u \in W^{2,2}(\Omega; \mathbf{R}^N)$ is a weak solution to the problem*

$$A_{ij}^{\alpha\beta}(x)\, D_{\alpha\beta} u^j = f^i; \quad u^j|_{\partial\Omega} = g^j,$$

then $u \in C^{2,\mu}(\bar{\Omega}; \mathbf{R}^N)$.

Partial Regularity for Nonlinear Systems

5.1. Reduction to linear and quasilinear systems

Let us consider a system in divergence form

$$\text{Div}\, A(Du(x)) = 0, \tag{5.1}$$

where $A(\xi) = (A^i_\alpha(\xi))$ is of C^1 and satisfies a *controllable* growth condition; that is, for all $\xi, \eta \in \mathbf{M}^{N\times n}$, letting $A^{\alpha\beta}_{ij}(\xi) = \partial A^i_\alpha(\xi)/\partial \xi^j_\beta$,

$$\begin{cases} |A^i_\alpha(\xi)| \le c\,|\xi|, \\ |A^{\alpha\beta}_{ij}(\xi)| \le L, \\ A^{\alpha\beta}_{ij}(\xi)\,\eta^i_\alpha \eta^j_\beta \ge \nu\,|\eta|^2; \quad \nu > 0. \end{cases} \tag{5.2}$$

Theorem 5.1. *Let $u \in W^{1,2}(\Omega; \mathbf{R}^N)$ be a weak solution of* (5.1)*. Then $u \in W^{2,2}_{loc}(\Omega; \mathbf{R}^N)$ and Du satisfies a quasilinear system*

$$\int_\Omega A^{\alpha\beta}_{ij}(Du(x))\, D_\beta D_s u^j(x)\, D_\alpha \phi^i(x)\, dx = 0 \tag{5.3}$$

for all $\phi \in W^{1,2}_0(\Omega; \mathbf{R}^N)$ with $\operatorname{supp}\phi \subset\subset \Omega$.

Proof. Since

$$\int_\Omega A(Du(x)) \cdot D\phi(x)\, dx = 0, \quad \forall\, \phi \in W^{1,2}_0(\Omega; \mathbf{R}^N),$$

we use the well-known method of difference quotients to "linearize" the system. Let ϕ have a compact support in Ω and $|h| < \text{dist}(\operatorname{supp}\phi; \partial\Omega)$ we have

$$\frac{1}{h}\int_\Omega [A(Du(x + he_s)) - A(Du(x))] \cdot D\phi(x)\, dx = 0.$$

Now for almost every $x \in \Omega$

$$\begin{aligned} & A^i_\alpha(Du(x+he_s)) - A^i_\alpha(Du(x)) \\ = & \int_0^1 \frac{d}{dt} A^i_\alpha\big(tDu(x+he_s) + (1-t)Du(x)\big)\,dt \\ = & \int_0^1 A^{\alpha\beta}_{ij}\big(tDu(x+he_s)+(1-t)Du(x)\big)\,D_\beta\big(u^j(x+he_s) - u^j(x)\big)\,dt. \end{aligned}$$

Thus

$$\frac{1}{h}\left[A(Du(x+he_s)) - A(Du(x))\right] = \tilde{A}_{(h)}(x)\,\tau_{h,s}Du(x),$$

where $\tilde{A}_{(h)}(x) = (\tilde{A}^{\alpha\beta}_{ij(h)}(x))$ is defined by

$$\tilde{A}^{\alpha\beta}_{ij(h)}(x) = \int_0^1 A^{\alpha\beta}_{ij}\big(tDu(x+he_s)+(1-t)Du(x)\big)\,dt.$$

For each h we have $\tilde{A}_{(h)}(x)$ satisfies the hypothesis (H1) in the previous chapter; that is

$$|\tilde{A}_{(h)}(x)| \le L, \quad \tilde{A}^{\alpha\beta}_{ij(h)}(x)\,\xi^i_\alpha \xi^j_\beta \ge \nu|\xi|^2.$$

Note also that

$$\int_\Omega \tilde{A}_{(h)}(x)\,\tau_{h,s}Du(x)\cdot D\phi(x)\,dx = 0. \tag{5.4}$$

As before, choosing $\phi = \zeta^2\,\tau_{h,s}u$ as in the Caccioppoli estimates, we can prove that

$$\int_{B_{R/2}} |\tau_{h,s}Du|^2\,dx \le C(R,\nu,L,\|Du\|_{L^2(\Omega)}) < \infty,$$

which is independent of h, s. Therefore, we have $u \in W^{2,2}_{loc}(\Omega;\mathbf{R}^N)$. Passing to the limit for $h \to 0$ in (5.4) we have

$$\int_\Omega A^{\alpha\beta}_{ij}(Du(x))\,D_\beta D_s u^j(x)\,D_\alpha\phi^i(x)\,dx = 0$$

for all $\phi \in W^{1,2}_0(\Omega;\mathbf{R}^N)$ with $\operatorname{supp}\phi \subset\subset \Omega$. Since we may not have $u \in W^{2,2}(\Omega;\mathbf{R}^N)$ globally, this may not hold for all $\phi \in W^{1,2}_0(\Omega;\mathbf{R}^N)$. This proves the theorem. □

For each fixed $s = 1,2,...$ define $v\colon \Omega \to \mathbf{R}^N$ by $v^j = D_s u^j$, and let

$$\tilde{A}^{\alpha\beta}_{ij}(x) = A^{\alpha\beta}_{ij}(Du(x)).$$

Then (5.3) becomes a *linear system* for $v : \Omega \to \mathbf{R}^N$

$$D_\alpha(\tilde{A}^{\alpha\beta}_{ij}(x)\,D_\beta v^j) = 0, \quad \forall\, i = 1,2,...,N. \tag{5.5}$$

The coefficients of this linear system are only known to belong to $L^\infty(\Omega)$ and satisfy the hypothesis (H1) in the previous chapter; the weak solution $v \in W^{1,2}_{loc}(\Omega;\mathbf{R}^N)$ and we do not have the higher regularity result from the linear theory we proved before.

Theorem 5.2. *Let A^i_α be of C^∞ and satisfy the controllable growth condition* (5.2) *and let $u \in W^{1,2}(\Omega;\mathbf{R}^N)$ be a weak solution of*

$$\operatorname{Div} A(Du(x)) = 0.$$

Let Ω_0 be any open set in Ω. Then $u \in C^\infty(\Omega_0;\mathbf{R}^N)$ if $u \in C^{1,\mu}_{loc}(\Omega_0;\mathbf{R}^N)$ for some $0 < \mu <$ 1.

Proof. If $u \in C^{1,\mu}_{loc}(\Omega_0; \mathbf{R}^N)$ for some $0 < \mu < 1$, it follows that

$$\tilde{A}^{\alpha\beta}_{ij}(x) = A^{\alpha\beta}_{ij}(Du) \in C^{0,\mu}_{loc}(\Omega_0).$$

Therefore from the linear system (5.5) and the Schauder estimates we proved before we have $v \in C^{1,\mu}_{loc}(\Omega_0; \mathbf{R}^N)$; this is true for all $s = 1, 2, ..., n$, so $u \in C^{2,\mu}_{loc}(\Omega_0; \mathbf{R}^N)$ and thus $\tilde{A}^{\alpha\beta}_{ij} \in C^{1,\mu}_{loc}(\Omega_0)$ and hence $v \in C^{2,\mu}_{loc}(\Omega_0; \mathbf{R}^N)$. By this "bootstrap" argument, the theorem follows. □

Remarks. 1) This theorem shows that the C^∞-regularity of weak solutions of a nonlinear elliptic system reduces to the $C^{1,\mu}$-regularity of the weak solutions. In the theorem, if $\Omega_0 = \Omega$ we obtain the full C^∞-regularity theory. However, for general systems, this is not the case. If Ω_0 is the largest open set such that a weak solution u belongs to $C^{1,\mu}_{loc}(\Omega_0; \mathbf{R}^N)$ for some $\mu \in (0, 1)$, then the set $\Omega \setminus \Omega_0$ is called the **singular set** for the solution u. Further study on the singular sets will be given later.

2) Since the $C^{1,\mu}$-regularity is the key issue for the weak solution u of nonlinear system (5.1), it is desirable to study the $C^{0,\mu}$-regularity for the gradient Du. By Theorem 5.1, Du satisfies the quasilinear system (5.3). This system can be written for the gradient field $U = (U^j_s) = (D_s u^j)$ from Ω to $\mathbf{R}^{nN}$ as follows

$$\int_\Omega \delta^{\kappa s} A^{\alpha\beta}_{ij}(U)\, D_\beta U^j_s\, D_\alpha \phi^i_\kappa\, dx = 0 \tag{5.6}$$

for all $\phi = (\phi^i_\kappa) \in C^\infty_0(\Omega; \mathbf{R}^{nN})$. Note that the coefficients of system (5.6), $\delta^{\kappa s} A^{\alpha\beta}_{ij}(U)$, are continuous on $U \in \mathbf{R}^{nN}$ and satisfy the *Legendre ellipticity* condition:

$$\delta^{\kappa s} A^{\alpha\beta}_{ij}(U)\, P^i_{\alpha\kappa} P^j_{\beta s} \ge \nu\, |P|^2 \equiv \nu \sum_{\alpha,\kappa=1}^{n} \sum_{i=1}^{N} |P^i_{\alpha\kappa}|^2.$$

The system (5.6) is called the **system in variation** of system (5.1). □

5.2. Full regularity for equations with one unknown function

Let $N = 1$; that is, we only deal with the scalar functions satisfying a single equation

$$D_\alpha A_\alpha(Du(x)) = 0. \tag{5.7}$$

Here we assume $A(p) = (A_\alpha(p))$ is of C^1 and satisfies the controllable growth condition; that is, for all p, $\eta \in \mathbf{R}^n$, letting $A^{\alpha\beta}(p) = \partial A_\alpha(p)/\partial p_\beta$,

$$\begin{cases} |A_\alpha(p)| \le c\, |p|, \\ |A^{\alpha\beta}(p)| \le L, \\ A^{\alpha\beta}(p)\, \eta_\alpha \eta_\beta \ge \nu\, |\eta|^2; \quad \nu > 0. \end{cases} \tag{5.8}$$

We use the same or similar notation as in the previous section. Then (5.5) reduces to

$$D_\alpha(\tilde{A}^{\alpha\beta}(x)\, D_\beta v) = 0, \tag{5.9}$$

with, again, the coefficients $\tilde{A}^{\alpha\beta}(x)$ belonging to $L^\infty(\Omega)$. However, in this case, we have the following extremely important result due to De Giorgi.

Theorem 5.3 (De Giorgi '57)**.** *Let $v \in W^{1,2}_{loc}(\Omega)$ be a weak solution of*

$$D_\alpha(a^{\alpha\beta}(x)\, D_\beta v) = 0, \tag{5.10}$$

where $a^{\alpha\beta} \in L^\infty(\Omega)$ *satisfying for a constant* $\nu > 0$

$$a^{\alpha\beta}(x)\, p_\alpha p_\beta \geq \nu\, |p|^2, \quad \forall\, p \in \mathbf{R}^n.$$

Then $v \in C^{0,\mu}_{loc}(\Omega)$ *for some* $0 < \mu < 1$.

Theorem 5.4. *Suppose* $u \in W^{1,2}(\Omega)$ *is a weak solution of*

$$D_\alpha(A_\alpha(Du(x))) = 0.$$

If the functions $A_\alpha \in C^\infty$ *satisfy the controllable growth condition* (5.8) *then* $u \in C^\infty(\Omega)$.

Proof. From De Giorgi's theorem, we deduce from the linear equation (5.9) that $v = D_s u \in C^{0,\mu}_{loc}(\Omega)$ for all $s = 1, 2, ..., n$ and hence

$$\tilde{A}^{\alpha\beta}(x) = A^{\alpha\beta}(Du(x)) \in C^{0,\mu}_{loc}(\Omega).$$

Therefore the Schauder estimates from (5.9) imply $v = D_s u \in C^{1,\mu}_{loc}(\Omega)$ for all $s = 1, 2, ...$; this implies $\tilde{A}^{\alpha\beta} \in C^{1,\mu}_{loc}(\Omega)$ and hence $v \in C^{2,\mu}_{loc}(\Omega)$. The theorem follows by this bootstrap argument. □

Proof of De Giorgi's Theorem. The proof given below is due to J. Moser '60 and uses an iteration method.

We say a function $w \in W^{1,2}(\Omega)$ is a **subsolution** (or a **supersolution**) of equation (5.10) if

$$\int_\Omega a^{\alpha\beta}(x)\, D_\beta w\, D_\alpha \varphi\, dx \leq 0 \ \ (\text{or} \geq 0), \quad \forall\, \varphi \in W^{1,2}_0(\Omega),\ \varphi \geq 0.$$

We first prove some useful lemmas.

Lemma 5.5. *Let* $w \geq 0$ *be a* **locally bounded** *function in* $W^{1,2}(\Omega)$.

(*a*) *If* w *is a subsolution then for any* $q > 1$ *there exists a constant* $c_1 = c_1(q) > 0$ *such that*

$$\sup_{B_{R/2}} w \leq c_1 \left(\fint_{B_R} w^q\, dx \right)^{\frac{1}{q}} \quad \forall\ B_{2R} \subset \Omega.$$

(*b*) *If* w *is a supersolution then for any* $0 < q < \frac{n}{n-2}$ *there exists a constant* $c_2 = c_2(q) > 0$ *such that*

$$\inf_{B_{R/2}} w \geq c_2 \left(\fint_{B_R} w^q\, dx \right)^{\frac{1}{q}} \quad \forall\ B_{2R} \subset \Omega.$$

Proof. Since the idea of proving both (a), (b) is essentially the same, using special test functions, we prove them jointly. Let $k > 0$ and let $\bar{w} = w + k$. For any $p \neq 0$ and cut-off function $\zeta \in W^{1,\infty}_0(B_{2R})$ let $\varphi = \zeta^2\, \bar{w}^p$. Since $\bar{w} \geq k > 0$ is *locally bounded* by assumption, $\varphi \geq 0$ is a legitimate test function and

$$D\varphi = 2\zeta \bar{w}^p\, D\zeta + p\zeta^2 \bar{w}^{p-1}\, Dw$$

so that testing by φ yields

$$p \int_{B_{2R}} a^{\alpha\beta}\, \bar{w}^{p-1} \zeta^2 D_\beta w\, D_\alpha w\, dx + 2 \int_{B_{2R}} a^{\alpha\beta}\, \zeta \bar{w}^p D_\beta w\, D_\alpha \zeta\, dx \begin{cases} \leq 0 & \text{if } w \text{ is a subsolution,} \\ \geq 0 & \text{if } w \text{ is a supersolution.} \end{cases}$$

In the following we assume that $p > 0$ (or $p < 0$) if w is a subsolution (or a supersolution). Therefore we have

$$\int_{B_{2R}} |Dw|^2 \bar{w}^{p-1} \zeta^2 \, dx \le \frac{c}{|p|} \int_{B_{2R}} |Dw| \, \bar{w}^{\frac{p-1}{2}} \, \bar{w}^{\frac{p+1}{2}} \zeta |D\zeta| \, dx$$

and hence

$$\int_{B_{2R}} |Dw|^2 \bar{w}^{p-1} \zeta^2 \, dx \le \frac{c}{p^2} \int_{B_{2R}} \bar{w}^{p+1} |D\zeta|^2 \, dx.$$

Note that

$$\begin{aligned} |D(\ln \bar{w})|^2 &= \bar{w}^{-2} \, |D\bar{w}|^2, \\ |D(\bar{w}^{\frac{p+1}{2}})|^2 &= \frac{(p+1)^2}{4} \, \bar{w}^{p-1} |Dw|^2, \\ |D(\zeta \bar{w}^{\frac{p+1}{2}})|^2 &\le 2|D(\bar{w}^{\frac{p+1}{2}})|^2 \zeta^2 + 2\bar{w}^{p+1} |D\zeta|^2. \end{aligned}$$

We have

$$\int_{B_{2R}} |D(\ln \bar{w})|^2 \, \zeta^2 \, dx \le C \int_{B_{2R}} |D\zeta|^2 \, dx, \tag{5.11}$$

$$\int_{B_{2R}} |D(\zeta \bar{w}^{\frac{p+1}{2}})|^2 \, dx \le c \, [(p+1)^2 + 1] \int_{B_{2R}} \bar{w}^{p+1} \, |D\zeta|^2 \, dx; \quad p \neq -1. \tag{5.12}$$

By Sobolev-Poincaré's inequality, letting $2^* = \frac{2n}{n-2}$, we have

$$\left[\int_{B_{2R}} \left(\zeta \bar{w}^{\frac{p+1}{2}} \right)^{2^*} dx \right]^{\frac{2}{2^*}} \le c \, [(p+1)^2 + 1] \int_{B_{2R}} \bar{w}^{p+1} \, |D\zeta|^2 \, dx; \quad p \neq -1.$$

We now let $B_{r_1} \subset B_{r_2} \subset B_{3R/2}$ and choose the cut-off function ζ as before such that $\zeta \in W_0^{1,\infty}(B_{2R})$ and $\zeta \equiv 1$ on B_{r_1}, $\zeta \equiv 0$ in $\Omega \setminus B_{r_2}$ and

$$0 \le \zeta \le 1; \quad |D\zeta| \le \frac{1}{r_2 - r_1}.$$

Let $\delta = \frac{2^*}{2} = \frac{n}{n-2}$ and $\gamma = p + 1$. We obtain

$$\left(\int_{B_{r_1}} \bar{w}^{\delta \gamma} \, dx \right)^{\frac{1}{\delta}} \le \frac{c \, (|\gamma| + 1)^2}{(r_2 - r_1)^2} \int_{B_{r_2}} \bar{w}^{\gamma} \, dx; \quad \gamma \neq 0. \tag{5.13}$$

We now iterate (5.13) as follows. Let for $i = 1, 2, ...$ and $0 < \theta < 3$

$$\gamma_i = \delta^i \gamma, \quad R_i = \frac{R}{2} + \frac{\theta R}{2^{i+1}}$$

and we find that

$$\begin{aligned}\left(\int_{B_{R_{i+1}}} \bar{w}^{\gamma_{i+1}}\,dx\right)^{\frac{1}{\delta^{i+1}}} &= \left(\int_{B_{R_{i+1}}} \bar{w}^{\delta\gamma_i}\,dx\right)^{\frac{1}{\delta}\frac{1}{\delta^i}}\\ &\le \left[\frac{c\,(1+|\gamma_i|)^2}{4^{-(i+1)}\,\theta^2\,R^2}\right]^{\frac{1}{\delta^i}}\cdot\left(\int_{B_{R_i}} \bar{w}^{\gamma_i}\,dx\right)^{\frac{1}{\delta^i}}\\ &\le \prod_{j=0}^{i}\left[\frac{c\,(1+|\gamma_j|)^2}{4^{-(j+1)}\,\theta^2\,R^2}\right]^{\frac{1}{\delta^j}}\int_{B_{(1+\theta)R/2}} \bar{w}^{\gamma}\,dx\\ &\le C\,\theta^{-n}\,R^{-n+\frac{n}{\delta^{i+1}}}\int_{B_{(1+\theta)R/2}} \bar{w}^{\gamma}\,dx,\end{aligned}$$

using the elementary calculation

$$\prod_{j=0}^{i}\left[\frac{c\,(1+|\gamma_j|)^2}{4^{-(j+1)}\,\theta^2\,R^2}\right]^{\frac{1}{\delta^j}} \le C\,\theta^{-n}\,R^{-n+\frac{n}{\delta^{i+1}}}.$$

Therefore,

$$\left(\fint_{B_{R_{i+1}}} \bar{w}^{\gamma_{i+1}}\,dx\right)^{\frac{1}{\delta^{i+1}}} \le C(\theta)\fint_{B_{(1+\theta)R/2}} \bar{w}^{\gamma}\,dx.$$

For $\gamma>0$, letting $\theta=1$, we have

$$\left(\fint_{B_{R_{i+1}}} \bar{w}^{\gamma_{i+1}}\,dx\right)^{\frac{1}{\gamma_{i+1}}} \le c_1\left(\fint_{B_R} \bar{w}^{\gamma}\,dx\right)^{\frac{1}{\gamma}}, \tag{5.14}$$

while for $\gamma<0$, letting $\theta=2$, we have

$$\left(\fint_{B_{R_{i+1}}} \bar{w}^{\gamma_{i+1}}\,dx\right)^{\frac{1}{\gamma_{i+1}}} \ge c_2\left(\fint_{B_{3R/2}} \bar{w}^{\gamma}\,dx\right)^{\frac{1}{\gamma}}. \tag{5.15}$$

Case (a): w is a subsolution. In this case $p>0$ and $\gamma=p+1>1$. In (5.14), letting $i\to\infty$ we obtain

$$\sup_{B_{R/2}}\bar{w} = \lim_{i\to\infty}\|\bar{w}\|_{L^{\gamma_{i+1}}(B_{R_{i+1}})} \le c_1\left(\fint_{B_R} \bar{w}^{\gamma}\,dx\right)^{\frac{1}{\gamma}};\quad \gamma>1.$$

Letting $k\to 0^+$ yields the same estimate for w; the part (a) is proved.

Case (b): w is a supersolution. In this case, $p<0$ and $\gamma=p+1<1$. Let $0<q<\frac{n}{n-2}=\delta$ be any number given. Let $0<q_0<q$ be a number determined by the lemma below and assume $k\ge 0$ is the integer satisfying $q_0\delta^k\le q<q_0\delta^{k+1}$. First of all, by Hölder's inequality, we have

$$\left(\fint_{B_R} \bar{w}^{q}\,dx\right)^{\frac{1}{q}} \le \left(\fint_{B_R} \bar{w}^{q_{k+1}}\,dx\right)^{\frac{1}{q_{k+1}}};\quad q_{k+1}=q_0\delta^{k+1}.$$

Using (5.13) as in (5.14) we have

$$\left(\fint_{B_R} \bar{w}^{q}\,dx\right)^{\frac{1}{q}} \le \left(\fint_{B_R} \bar{w}^{q_{k+1}}\,dx\right)^{\frac{1}{q_{k+1}}} \le C_1\left(\fint_{B_{3R/2}} \bar{w}^{q_0}\,dx\right)^{\frac{1}{q_0}}.$$

On the other hand, in (5.15) letting $i \to \infty$ we have

$$\inf_{B_{R/2}} \bar{w} = \lim_{i\to\infty} \|\bar{w}\|_{L^{\gamma_{i+1}}(B_{R_{i+1}})} \geq c_2 \left(\fint_{B_{3R/2}} \bar{w}^{-q_0}\, dx \right)^{\frac{1}{-q_0}}.$$

The part (b) and hence the lemma are proved if we prove the following lemma. □

Lemma 5.6. *In case* (*b*) *above, for any* $0 < q < \delta$ *there exist constants* $q_0 \in (0, q)$ *and* $C_2 > 0$ *such that*

$$\left(\fint_{B_{3R/2}} \bar{w}^{-q_0}\, dx \right)^{\frac{1}{-q_0}} \geq C_2 \left(\fint_{B_{3R/2}} \bar{w}^{q_0}\, dx \right)^{\frac{1}{q_0}}.$$

Proof. Let B_r be any ball lying in B_{2R}. Let ζ be the cut-off function such that

$$\zeta|_{B_r} = 1; \quad \zeta|_{\Omega\setminus B_{2r}} = 0; \quad |D\zeta| \leq 2/r.$$

Use this ζ and the estimate (5.11) above and we have

$$\int_{B_r} |D(\ln \bar{w})|^2\, dx \leq C\, r^{n-2}$$

so that by Hölder's inequality

$$\int_{B_r} |D(\ln \bar{w})|\, dx \leq K\, r^{n-1}.$$

We also assume K large enough so that $q_0 = \frac{\sigma}{K} \in (0, q)$, where $\sigma = \sigma_n$ is the constant in the John-Nirenberg theorem for BMO functions (see Theorem 4.27, Chapter 4). From that theorem, $u = \ln \bar{w} \in BMO(B_{3R/2})$ and, letting

$$u_0 = \fint_{B_{3R/2}} u(x)\, dx,$$

we have

$$\int_{B_{3R/2}} e^{q_0\,|u-u_0|}\, dx \leq C\, R^n,$$

and thus

$$\int_{B_{3R/2}} e^{q_0\,(u-u_0)}\, dx \leq C\, R^n, \quad \int_{B_{3R/2}} e^{q_0\,(u_0-u)}\, dx \leq C\, R^n.$$

Multiplying these two inequalities yields

$$\int_{B_{3R/2}} e^{q_0\,u}\, dx \cdot \int_{B_{3R/2}} e^{-q_0\,u}\, dx \leq C\, R^{2n}.$$

Since $e^{q_0\,u} = \bar{w}^{q_0}$ and $e^{-q_0\,u} = \bar{w}^{-q_0}$ rearranging this inequality we have

$$\left(\fint_{B_{3R/2}} \bar{w}^{-q_0}\, dx \right)^{\frac{1}{-q_0}} \geq C_2 \left(\fint_{B_{3R/2}} \bar{w}^{q_0}\, dx \right)^{\frac{1}{q_0}}.$$

The lemma is proved. □

Lemma 5.7. *Let* $v \in W^{1,2}(\Omega)$ *be a weak solution of* (5.10). *Then* v *is locally bounded and there exists* $c_3 > 0$ *such that for any ball* $B_{2R} \subset \Omega$ *we have*

$$\sup_{B_R} |v| \leq c_3 \left(\fint_{B_{2R}} |v|^2\, dx \right)^{\frac{1}{2}}.$$

Moreover, if $v \geq 0$ then we have the following **Harnack inequality:**

$$\sup_{B_{R/2}} v \leq C \inf_{B_{R/2}} v.$$

Proof. The Harnack inequality follows easily from Lemma 5.5. We have only to show the first assertion. We use a similar technique as in the proof of Lemma 5.5. Let $\bar{w} = v^+ = \max\{v, 0\}$. For any $p \geq 1$ and $M > 0$ let $H \in C^1[0, \infty)$ be defined by $H(s) = s^p$ for $0 \leq s \leq M$ and $H(s)$ is linear for $s \geq M$. Let $\zeta \in W_0^{1,\infty}(B_{2R})$. We define test function $\varphi = \zeta^2 G(\bar{w})$, where

$$G(s) = \int_0^s |H'(t)|^2 \, dt, \quad s \geq 0.$$

Since G is Lipschitz and $G(s) \leq C_M s$ this function φ is indeed a test function in $W_0^{1,2}(B_{2R})$. Let us look at some properties of G, H:

$$G(s) \leq sG'(s), \quad G'(s) = |H'(s)|^2, \quad H(s) + sH'(s) \leq 2ps^p. \tag{5.16}$$

Note that $D\varphi = \zeta^2 G'(\bar{w}) D\bar{w} + 2\zeta \, G(\bar{w}) \, D\zeta$ and

$$G(\bar{w}) = 0, \quad D\bar{w} = 0 \quad \text{on } \{x \in \Omega \,|\, v(x) \leq 0\},$$

hence testing equation (5.10) with φ yields

$$\begin{aligned}
\int_{B_{2R}} |D\bar{w}|^2 \, G'(\bar{w}) \, \zeta^2 &\leq C \int_{B_{2R}} \zeta \, |D\bar{w}| \, |D\zeta| \, G(\bar{w}) \\
&\leq C \int_{B_{2R}} \zeta \, |D\bar{w}| \, |D\zeta| \, \bar{w} \, G'(\bar{w}) \\
&\leq \frac{1}{2} \int_{B_{2R}} |D\bar{w}|^2 \, G'(\bar{w}) \, \zeta^2 + C \int_{B_{2R}} |\bar{w}|^2 \, |D\zeta|^2 \, G'(\bar{w}).
\end{aligned}$$

This, combined with (5.16), yields

$$\int_{B_{2R}} |D[\zeta \, H(\bar{w})]|^2 \, dx \leq C \, p^2 \int_{B_{2R}} |D\zeta|^2 \, \bar{w}^{2p} \, dx.$$

Using the Sobolev-Poincaré inequality, we have

$$\left(\int_{B_{2R}} [\zeta H(\bar{w})]^{2^*} \, dx \right)^{\frac{2}{2^*}} \leq C \, p^2 \int_{B_{2R}} \bar{w}^{2p} \, |D\zeta|^2 \, dx.$$

The righthand side of this inequality is independent of number M, so letting $M \to \infty$ we have

$$\left(\int_{B_{2R}} \zeta^{2^*} \, \bar{w}^{q\delta} \, dx \right)^{\frac{1}{\delta}} \leq C \, q^2 \int_{B_{2R}} \bar{w}^q \, |D\zeta|^2 \, dx,$$

where $\delta = \frac{n}{n-2}$, $q = 2p \geq 2$. Using the special cut-off function ζ as above, we have

$$\left(\int_{B_{r_1}} \bar{w}^{q\delta} \, dx \right)^{\frac{1}{\delta}} \leq \frac{C \, q^2}{(r_2 - r_1)^2} \int_{B_{r_2}} \bar{w}^q \, dx$$

for all $0 < r_1 < r_2 \leq 2R$. Hence using

$$q_i = 2\delta^i, \quad R_i = R + \frac{R}{2^i},$$

as in the proof of part (a) of Lemma 5.5 we deduce

$$\sup_{B_R} v \le \sup_{B_R} \bar{w} \le c_3 \left(\fint_{B_{2R}} \bar{w}^2 \, dx \right)^{\frac{1}{2}} \le c_3 \left(\fint_{B_{2R}} v^2 \, dx \right)^{\frac{1}{2}}.$$

The same estimate holds also for $-v$, and hence we have

$$\sup_{B_R} |v| \le c_3 \left(\fint_{B_{2R}} v^2 \, dx \right)^{\frac{1}{2}},$$

which proves the lemma. □

Completion of the Proof. Finally let us complete the proof of the De Giorgi theorem. Let

$$M(R) = \sup_{B_R} v, \quad m(R) = \inf_{B_R} v, \quad \omega(R) = M(R) - m(R).$$

Let $B_{2R_0} \subset\subset \Omega$. For $R \le R_0$ we have $M(R) - v \ge 0$ is a weak solution on B_R thus by Harnack's inequality,

$$M(R) - m(R/2) \le c\,(M(R) - M(R/2)).$$

Since $u - m(R) \ge 0$ is also a weak solution,

$$M(R/2) - m(R) \le c\,(m(R/2) - m(R)).$$

Adding these two inequalities yields

$$\omega(R) + \omega(R/2) \le c\,(\omega(R) - \omega(R/2))$$

so that

$$\omega(R/2) \le \frac{c-1}{c+1}\,\omega(R), \quad \forall R \le R_0.$$

From this and Lemma 5.8 below, it follows that

$$\omega(R) \le C\,(R/R_0)^{\mu}; \quad \mu = -\frac{\ln \eta}{\ln 2}, \quad \eta = \frac{c-1}{c+1},$$

and hence $v \in C^{0,\mu}_{loc}(\Omega)$. Since we can choose constant $c > 0$ uniformly for all R_0, the constant $\mu > 0$ can be chosen independently of R_0; hence we have proved the De Giorgi theorem.

Lemma 5.8 (De Giorgi)**.** *Assume $\theta(R) \ge 0$ is bounded on $(0, R_0)$ and satisfies for some constants $\tau, \eta \in (0, 1)$*

$$0 \le \theta(\tau R) \le \eta\,\theta(R), \quad \forall\, R \in (0, R_0).$$

Then there exists a constant $C > 0$ such that

$$\theta(R) \le C \cdot (R/R_0)^{\beta}, \quad \beta = \frac{\ln \eta}{\ln \tau}.$$

Proof. Let $0 \le \theta(R) \le \theta_0$ for $0 < R < R_0$. Then

$$\sup_{R \in [\tau R_0,\, R_0)} \frac{\theta(R)}{R^{\beta}} \le \frac{\theta_0}{\tau^{\beta}\, R_0^{\beta}} \equiv M_0.$$

We use induction to show that

$$\theta(R) \le M_0 \cdot R^{\beta}, \quad R \in [\tau^i R_0,\, \tau^{i-1} R_0), \quad i = 1, 2, \cdots.$$

Indeed, if $i = 1$ this is trivial by the definition of M_0. Suppose this inequality is proved for $i = k - 1$ then for $i = k$ and $R \in [\tau^k R_0, \tau^{k-1} R_0)$ we have $R/\tau \in [\tau^{k-1} R_0, \tau^{k-2} R_0)$ and hence by induction assumption $\theta(R/\tau) \le M_0 \, (R/\tau)^\beta$. Therefore, it follows that

$$\theta(R) = \theta(\tau \cdot \frac{R}{\tau}) \le \eta \cdot \theta(\frac{R}{\tau}) \le \eta \, M_0 \, \frac{R^\beta}{\tau^\beta} = M_0 \cdot R^\beta$$

since $\tau^\beta = \eta$ by the definition of β. This proves the lemma. □

5.3. No full regularity for elliptic systems

The full C^∞-regularity (for smooth data) would follow for elliptic systems if De Giorgi's theorem were true for systems. But this is not the case. The following example due to De Giorgi '68 showed that his result for equations cannot be extended to systems.

Example 5.1 (De Giorgi '68)**.** Let B be the unit ball in $\mathbf{R}^n$ with $n \ge 3$. Consider the coefficients

$$A_{ij}^{\alpha\beta}(x) = \delta^{\alpha\beta} \, \delta_{ij} + P_\alpha^i(x) \, P_\beta^j(x),$$

where

$$P_\alpha^i(x) = (n-2) \, \delta_{\alpha i} + n \, \frac{x_i x_\alpha}{|x|^2}.$$

It is easily seen that $A_{ij}^{\alpha\beta} \in L^\infty(B)$ and there exist constants $0 < \nu \le M$ such that

$$\nu \, |\xi|^2 \le A_{ij}^{\alpha\beta}(x) \, \xi_\alpha^i \xi_\beta^j \le M \, |\xi|^2.$$

Moreover one verifies that the function

$$u_0(x) = \frac{x}{|x|^\gamma}, \quad \gamma = \frac{n}{2} \, \left[1 - \left((2n-2)^2 + 1 \right)^{-\frac{1}{2}} \right]$$

belongs to $W^{1,2}(B; \mathbf{R}^n)$ and is a weak solution of the system

$$D_\alpha (A_{ij}^{\alpha\beta}(x) \, D_\beta u^j(x)) = 0.$$

But, $u_0 \notin C_{loc}^{0,\mu}(B; \mathbf{R}^n)$ since it is not bounded at $x = 0$.

Note that the function u_0 defined above is also the *unique minimizer* of the energy

$$J(v) = \int_B F(x, Dv) \, dx \equiv \int_B \Big(|Dv|^2 + \big(P(x) \cdot Dv \big) \Big) \, dx$$

among the Dirichlet class $\mathcal{D}_\varphi^{1,2}(B; \mathbf{R}^n)$ of the boundary value $\varphi(x) = x$, where $P(x)$ is the matrix $(P_\alpha^i(x))$ defined above.

Remarks. 1) Modifying De Giorgi's example, Giusti-Miranda '68 showed that a quasilinear elliptic system of type

$$\mathrm{Div}(A(u) \, Du) = 0$$

with *analytic* coefficients $A(u)$ can have $u = \frac{x}{|x|}$ as weak solution and thus have singularities in dimension $n \ge 3$.

2) In 1975, J. Nečas presented a functional of type

$$\int_\Omega F(Du(x)) \, dx; \quad u \colon \Omega \subset \mathbf{R}^n \to \mathbf{R}^{n^2}$$

satisfying, for $n \ge 5$, the strict Legendre ellipticity condition whose minimizer is function $u^{ij}(x) = \frac{x_i x_j}{|x|}$, which is Lipschitz but not C^1.

3) Most recently, in 1998, Müller and Šverák (ICM '98, Berlin) showed that there exists a smooth functional of type

$$I(u) = \int_\Omega F(Du(x))\,dx; \quad u\colon \Omega \subset \mathbf{R}^2 \to \mathbf{R}^2$$

satisfying the strong Legendre-Hadamard condition such that the Euler-Lagrange equation of $I(u)$ on whole $\mathbf{R}^2$ admits

(i) nontrivial Lipschitz solutions with compact support;

(ii) Lipschitz solutions that are nowhere C^1. □

We can conclude that vector valued minimizers or more generally the weak solutions of nonlinear elliptic systems are in general *non-regular* and we can only hope to have a *partial regularity*; that is regularity outside a certain closed set (called the *singular set*). The new (unexpected) example of Müller-Šverák shows that the weak solutions for certain strongly (Legendre-Hadamard) elliptic systems may be nowhere regular; this is in sharp contrast with all the existing regularity theories for nonlinear systems, which establish the *almost everywhere* regularity for weak solutions under the strong Legendre ellipticity condition or for energy minimizers under certain strong Legendre-Hadamard conditions (e.g., uniformly strict quasiconvexity).

There are also many open problems regarding the size of the singular sets and the condition of *ellipticity* which can guarantee at least almost everywhere regularity of energy minimizers in the calculus of variations. In the end of this chapter, we shall prove a theorem of Evans regarding the quasiconvexity and partial regularity in the calculus of variations.

5.4. Almost everywhere regularity: an indirect approach

Consider the following *quasilinear* system, which is usually the *system in variation* of some nonlinear system,

$$\operatorname{Div}(A(x,u)\,Du) = 0, \tag{5.17}$$

where the coefficient $A(x,u) = (A_{ij}^{\alpha\beta}(x,u))$ satisfies

$$|A_{ij}^{\alpha\beta}(x,u)| \le L, \quad A_{ij}^{\alpha\beta}(x,u)\,\xi_\alpha^i\xi_\beta^j \ge \nu\,|\xi|^2,\ \forall\,\xi \in \mathbf{M}^{N\times n}. \tag{5.18}$$

This condition compares to the following condition we used before for functions $\tilde{A}(x) = (\tilde{A}_{ij}^{\alpha\beta}(x))$ which we called the hypothesis (H1).

$$(H1) \qquad \|\tilde{A}\|_{L^\infty(\Omega)} \le L, \quad \tilde{A}_{ij}^{\alpha\beta}(x)\,\xi_\alpha^i\xi_\beta^j \ge \nu\,|\xi|^2,\ \forall\,\xi \in \mathbf{M}^{N\times n}.$$

Theorem 5.9. *Assume $A(x,u)$ is uniformly continuous on $(x,u) \in \bar{\Omega} \times \mathbf{R}^N$. Let $u \in W^{1,2}(\Omega;\mathbf{R}^N)$ be a weak solution of* (5.17). *Then there exists an open set $\Omega_0 \subset \Omega$ such that $|\Omega \setminus \Omega_0| = 0$ and $u \in C^{0,\mu}_{loc}(\Omega_0;\mathbf{R}^N)$ for each $0 < \mu < 1$.*

Proof. The main idea of the proof is the following. If u varies very little near a point $x = a$ in the sense that the quantity

$$E_u(a,R) \equiv R^{-n}\int_{B_R(a)} |u(x) - u_{B_R(a)}|^2\,dx \tag{5.19}$$

is smaller than a power of R as $R \to 0$ then the blowing up of u will converge to a solution of linear system with constant coefficients; the limit function (tangent map) will be regular (Hölder continuous) at point a. The convergence of the blowing up will also be strong enough to conclude a is also a regular point for u. The idea is central to the regularity theory for nonlinear systems. We shall give the details of the proof of the theorem below, after several lemmas. □

Lemma 5.10. *Let $u \in W^{1,2}_{loc}(\Omega; \mathbf{R}^N)$ be a weak solution of* (5.17)*. Then there exists a constant $C_1 = C_1(n, N, \nu, L)$ such that for all $a \in \Omega$ and $0 < \rho < R < \operatorname{dist}(a; \partial\Omega)$*

$$\int_{B_\rho(a)} |Du|^2 \, dx \le \frac{C_1}{(R-\rho)^2} \int_{B_R(a)} |u|^2 \, dx. \tag{5.20}$$

Proof. This follows from the standard Caccioppoli estimate for linear systems since we can consider $A(x, u(x)) = A(x)$ as in $L^\infty(\Omega)$ satisfying the hypothesis (H1). □

Lemma 5.11. *Let $\tilde{A}^{\alpha\beta}_{ij}$ be constants satisfying ($H1$) above with $\Omega = B_1(0)$. Then there exists a constant $C_2 = C_2(n, N, \nu, L)$ such that for any weak solution $v \in W^{1,2}_{loc} \cap L^2(B_1(0); \mathbf{R}^N)$ of*

$$D_\alpha(\tilde{A}^{\alpha\beta}_{ij} \, D_\beta v^j(x)) = 0$$

and all $0 < \rho < 1$ we have

$$E_v(0, \rho) \le C_2 \, \rho^2 \, E_v(0, 1),$$

where $E_v(a, R)$ is defined by (5.19)*.*

Proof. This is simply the Campanato inequality (4.31) we proved before. □

Lemma 5.12 (Compactness Lemma)**.** *Let $\tilde{A}_{(h)}(x)$ be a sequence of functions satisfying ($H1$) above with the same constants L, $\nu > 0$ in $\Omega = B_1(0)$. Assume $\tilde{A}_{(h)}(x)$ converges to $\tilde{A}(x)$ for almost every $x \in B_1(0)$ as $h \to 0$. Suppose $v_{(h)} \in W^{1,2}_{loc} \cap L^2(B_1(0); \mathbf{R}^N)$ is a weak solution of*

$$D_\alpha(\tilde{A}^{\alpha\beta}_{ij(h)}(x) \, D_\beta v^j_{(h)}(x)) = 0$$

and $v_{(h)} \rightharpoonup v$ weakly in $L^2(B_1(0); \mathbf{R}^N)$. Then $v \in W^{1,2}_{loc} \cap L^2(B_1(0); \mathbf{R}^N)$ and for all $0 < \rho < 1$

$$v_{(h)} \to v \text{ strongly in } L^2(B_\rho(0); \mathbf{R}^N),$$
$$Dv_{(h)} \rightharpoonup Dv \text{ weakly in } L^2(B_\rho(0); \mathbf{R}^N).$$

Moreover, v is a weak solution of

$$D_\alpha(\tilde{A}^{\alpha\beta}_{ij}(x) \, D_\beta v^j(x)) = 0.$$

Proof. First of all, since $\tilde{A}_{(h)}(x)$ satisfies the Legendre condition (H1), the weak solution $v_{(h)}$ satisfies a Caccioppoli-type inequality similar to (5.10). Since $\{v_{(h)}\}$ is bounded in $L^2(B_1(0); \mathbf{R}^N)$ we thus have for $0 < \rho < 1$ that $\{Dv_{(h)}\}$ is bounded in $L^2(B_\rho(0); \mathbf{M}^{N\times n})$; therefore, the convergence part of the lemma follows. We now prove the weak limit v is a weak solution of

$$D_\alpha(\tilde{A}^{\alpha\beta}_{ij}(x) \, D_\beta v^j(x)) = 0. \tag{5.21}$$

Let $\phi \in C^\infty_0(B_1(0); \mathbf{R}^N)$ and $\operatorname{supp}\phi \subset B_\rho(0)$ for some $0 < \rho < 1$. From the systems satisfied by $v_{(h)}$ we have

$$\int_{B_\rho(0)} \tilde{A}^{\alpha\beta}_{ij(h)}(x) \, D_\beta v^j_{(h)}(x) \, D_\alpha \phi^i(x) \, dx = 0.$$

By assumption and Lebesgue's bounded convergence theorem, $\tilde{A}_{(h)} \to \tilde{A}$ strongly in $L^2(B_\rho(0))$ as $h \to 0$; therefore we can pass to the limit in the previous equation to deduce

$$\int_{B_\rho(0)} \tilde{A}_{ij}^{\alpha\beta}(x)\, D_\beta v^j(x)\, D_\alpha \phi^i(x)\, dx = 0,$$

and hence v is a weak solution of (5.21); the proof is complete. □

Remark. We notice that the assumption that the sequence of coefficients satisfy the uniform *Legendre condition* cannot be replaced with the Legendre-Hadamard condition since in that case the Caccioppoli inequality (5.20) may not hold. □

The following important lemma is similar to the Schoen-Uhlenbeck ϵ-regularity theorem for *harmonic maps.*

Theorem 5.13 (Main Lemma)**.** *For all $0 < \tau < 1$ there exist two positive constants $\epsilon_0 = \epsilon_0(\tau, n, N, \nu, L)$ and $R_0 = R_0(\tau, n, N, \nu, L)$ such that if $u \in W^{1,2}(\Omega; \mathbf{R}^N)$ is a weak solution to system* (5.17) *then for any $a \in \Omega$ and $R < \min\{R_0, \operatorname{dist}(a; \partial\Omega)\}$ the condition*

$$E_u(a, R) < \epsilon_0^2 \tag{5.22}$$

implies

$$E_u(a, \tau R) \le 2\, C_2\, \tau^2\, E_u(a, R), \tag{5.23}$$

where $C_2 = C_2(n, N, \nu, L)$ is the constant in Lemma 5.11.

Proof. We use the contradiction method. Suppose the result is not true. Then there exist $\tau_0 \in (0, 1)$, a sequence of points $a_h \in \Omega$, a sequence $\epsilon_h \to 0^+$, a sequence $R_h \to 0^+$ and a sequence u_h of weak solutions of (5.17) such that

$$E_{u_h}(a_h, R_h) = \epsilon_h^2, \quad E_{u_h}(a_h, \tau_0 R_h) > 2\, C_2\, \tau_0^2\, \epsilon_h^2.$$

Now the following is the *blowing up* technique. Let

$$v_h(y) = \epsilon_h^{-1}\left[u_h(a_h + R_h y) - (u_h)_{B_{R_h}(a_h)}\right].$$

Let

$$\tilde{A}_{(h)}(y) = A\big(a_h + R_h y, \epsilon_h v_h(y) + (u_h)_{B_{R_h}(a_h)}\big), \quad y \in B_1(0).$$

Then it is easily seen that $v_h \in W^{1,2}(B_1(0); \mathbf{R}^N)$ is a weak solution of

$$\operatorname{Div}(\tilde{A}_{(h)}(y)\, D v_h(y)) = 0,$$

and satisfies $(v_h)_{B_1(0)} = 0$; moreover,

$$E_{v_h}(0, 1) = \int_{B_1(0)} |v_h(y)|^2\, dy = 1,$$

$$E_{v_h}(0, \tau_0) > 2\, C_2\, \tau_0^2. \tag{5.24}$$

Now, passing to a subsequence, still labeled the same, as $h \to 0$, we have

$$\begin{cases} v_h \rightharpoonup v \quad \text{weakly in } L^2(B_1(0); \mathbf{R}^N), \\ \epsilon_h\, v_h(y) \to 0 \quad \text{a.e. in } B_1(0), \\ A_{ij}^{\alpha\beta}\big(a_h, (u_h)_{B_{R_h}(a_h)}\big) \to \tilde{A}_{ij}^{\alpha\beta}, \quad \text{some constant}, \end{cases} \tag{5.25}$$

and hence using the uniform continuity of $A_{ij}^{\alpha\beta}(x, u)$ we have

$$\tilde{A}_{(h)}(y) \to \tilde{A} = (\tilde{A}_{ij}^{\alpha\beta}) \quad \text{a.e. in } B_1(0).$$

From the compactness lemma above, we obtain v is a weak solution of

$$\mathrm{Div}(\tilde{A}\, Dv(y)) = 0$$

and hence by Lemma 5.11 we must have

$$E_v(0,\tau_0) \le C_2\,\tau_0^2\,E_v(0,1).$$

On the other hand, using the semicontinuity of the norm in $L^2(B_1(0);\mathbf{R}^N)$ we have $E_v(0,1) \le 1$, and by (5.24) and the strong convergence of v_h to v in $L^2(B_\rho(0);\mathbf{R}^N)$ for all $\rho < 1$ we have

$$E_v(0,\tau_0) > 2\,C_2\,\tau_0^2,$$

which is a contradiction with the previous estimate we have above. □

Proof of Theorem 5.9. Let $0 < \mu < 1$ and choose $\tau \in (0,1)$ in such a way that $2\,C_2\,\tau^{2-2\mu} \le 1$. Let

$$\Omega_0 = \left\{ a \in \Omega \,\middle|\, E_u(a,R) < \epsilon_0^2 \quad \exists\, R < \min\{R_0,\ \mathrm{dist}(a;\partial\Omega)\} \right\}, \tag{5.26}$$

where ϵ_0, R_0 are the constants determined in the main lemma above. From the main lemma, we have for any $a \in \Omega_0$

$$E_u(a,\tau R) \le 2\,C_2\,\tau^2\,E_u(a,R) \le \tau^{2\mu}\,E_u(a,R)$$

and hence $E_u(a,\tau R) \le E_u(a,R) < \epsilon_0^2$, and therefore we can use the main lemma again to have

$$E_u(a,\tau^2 R) \le \tau^{4\mu}\,E_u(a,R).$$

By induction we get for every k

$$E_u(a,\tau^k R) \le \tau^{2\mu k}\,E_u(a,R)$$

and hence for every $\rho < R$ (note that $\rho^n E_u(a,\rho)$ is nondecreasing in ρ)

$$E_u(a,\rho) \le \tau^{-n-2\mu}\,(\frac{\rho}{R})^{2\mu}\,E_u(a,R). \tag{5.27}$$

On the other hand, since $E_u(a,R)$ is continuous in a, if $E_u(a,R) < \epsilon_0^2$ then there exists a ball $B_r(a) \subset \Omega$ such that for all $x \in B_r(a)$ we have

$$E_u(x,R) < \epsilon_0^2.$$

Therefore, Ω_0 is an *open set* in Ω and moreover, similarly as (5.27),

$$E_u(x,\rho) \le \tau^{-n-2\mu}\,(\frac{\rho}{R})^{2\mu}\,E_u(x,R), \quad \forall\, x \in B_r(a).$$

Hence for every $x \in B_r(a)$, we have

$$\int_{B_\rho(x)} |u(y) - u_{B_\rho(x)}|^2\,dy \le C_u \cdot \rho^{n+2\mu},$$

where $C_u = C_u(\tau,R) < \infty$ is some constant. By the (local) Campanato theorem, this shows that $u \in C^{0,\mu}_{loc}(B_r(a);\mathbf{R}^N)$. Hence $u \in C^{0,\mu}_{loc}(\Omega_0;\mathbf{R}^N)$.

To complete the proof, we show $|\Omega \setminus \Omega_0| = 0$. By Lebesgue's differentiation theorem, $E_u(x,R) \to 0$ as $R \to 0^+$ for almost every $x \in \Omega$ and, for these x, we have $x \in \Omega_0$ and hence $|\Omega \setminus \Omega_0| = 0$; moreover, the *singular set* is given by

$$\Omega \setminus \Omega_0 = \{a \in \Omega \mid \liminf_{R\to 0^+} E_u(a,R) > 0\}.$$

The proof of Theoerm 5.9 is now complete.

Remark. We would like to remark also that Theorem 5.9 holds even under the weaker assumption that the coefficients $A_{ij}^{\alpha\beta}(x,u)$ be only continuous in $\Omega \times \mathbf{R}^N$. More precisely the following theorem is true; its proof is similar to the one for Theorem 5.9 given above. (See also the proof for energy minimizers later.) □

Theorem 5.14. *Assume $A(x,u)$ is continuous on $(x,u) \in \Omega \times \mathbf{R}^N$. Let $u \in W^{1,2}(\Omega;\mathbf{R}^N)$ be a weak solution of* (5.17). *Then for every $M_0 > 0$, there exist positive constants ϵ_0, R_0 such that if for some $a \in \Omega$ and $R < \min\{R_0, \operatorname{dist}(a;\partial\Omega)\}$ we have*

$$|u_{B_R(a)}| < M_0, \quad E_u(a,R) < \epsilon_0^2$$

then u is of $C^{0,\mu}$ in a neighborhood of a for all $\mu \in (0,1)$. Therefore, in this case, $u \in C^{0,\mu}_{loc}(\Omega_0;\mathbf{R}^N)$, where

$$\Omega_0 = \{a \in \Omega \mid \liminf_{R\to 0^+} E_u(a,R) = 0 \text{ and } \sup_R |u_{B_R(a)}| < +\infty\},$$

and thus again $|\Omega \setminus \Omega_0| = 0$.

Applying this theorem to the system in variation (5.6) of the nonlinear system (5.1), we have the following theorem.

Theorem 5.15. *Let $A^i_\alpha(\xi)$ be of C^1 in $\xi \in \mathbf{M}^{N\times n}$ and satisfy the controllable growth condition* (5.2) *and let $u \in W^{1,2}(\Omega;\mathbf{R}^N)$ be a weak solution of*

$$\operatorname{Div} A(Du(x)) = 0.$$

Then $u \in C^{1,\mu}_{loc}(\Omega_0;\mathbf{R}^N)$ for each $\mu \in (0,1)$, where

$$\Omega_0 = \{a \in \Omega \mid \liminf_{R\to 0^+} E_{Du}(a,R) = 0 \text{ and } \sup_R |(Du)_{B_R(a)}| < +\infty\}$$

is an open set in Ω and $|\Omega \setminus \Omega_0| = 0$. Furthermore, if A^i_α is of C^∞ then $u \in C^\infty(\Omega_0;\mathbf{R}^N)$.

5.5. Reverse Hölder inequality with increasing supports

Let us consider a weak solution $u \in W^{1,2}(\Omega;\mathbf{R}^N)$ of the linear system

$$D_\alpha(A_{ij}^{\alpha\beta}(x)\, D_\beta u^j) = 0,$$

where $A_{ij}^{\alpha\beta} \in L^\infty(\Omega)$ satisfies the Legendre condition

$$A_{ij}^{\alpha\beta}(x)\, \xi^i_\alpha \xi^j_\beta \geq \nu\, |\xi|^2.$$

Then we have the Caccioppoli inequality

$$\int_{B_{R/2}} |Du|^2\, dx \leq \frac{c}{R^2} \int_{B_R} |u - u_{B_R}|^2\, dx, \quad \forall\ B_R \subset \Omega$$

and using the Sobolev-Poincaré inequality

$$\int_{B_R} |u - u_{B_R}|^2\, dx \leq C_{n,p} \left(\int_{B_R} |Du|^q\, dx\right)^{\frac{2}{q}},$$

where $2 = q^* = \frac{nq}{n-q}$ and thus $q = \frac{2n}{n+2} < 2$, it follows that

$$\left(\fint_{B_{R/2}} |Du|^2\, dx\right)^{\frac{1}{2}} \leq c_1 \left(\fint_{B_R} |Du|^q\, dx\right)^{\frac{1}{q}}. \tag{5.28}$$

Except for the fact that the integration is on different **increasing** sets, the inequality (5.28) can be seen as a *reverse Hölder inequality.* If the domains of integration were the same, then the reverse Hölder inequality would imply a *higher integrability* of Du in L^p for some $p > 2$; this is the well-known result of F. Gehring '73.

The following *local* higher integrability result based on the reverse Hölder inequality with *increasing supports* is due to Giaquinta and Modica '79. The proof we will give is mainly due to E. Stredulinsky '80.

Theorem 5.16. *Let $q > 1$ and $f \in L^q_{loc}(B)$, $f \geq 0$. Suppose that*

$$\left(\fint_{B_{R/2}} f^q \, dx\right)^{\frac{1}{q}} \leq b \fint_{B_R} f \, dx, \quad \forall \, B_R \subset B,$$

where $b > 0$ is a constant. Then there exists an $\epsilon = \epsilon(n, b, q) > 0$ such that $f \in L^p_{loc}(B)$ for all $p \in (q, \, q + \epsilon)$. Moreover, for all $B_R \subset B$, we have

$$\left(\fint_{B_{R/2}} f^p \, dx\right)^{\frac{1}{p}} \leq C(n, b, q, p) \left(\fint_{B_R} f^q \, dx\right)^{\frac{1}{q}}.$$

Before proving this theorem, we quickly give an application of this result to obtain a higher integrability for energy minimizers.

Theorem 5.17. *Assume that $F \colon \mathbf{M}^{N \times n} \to \mathbf{R}$ satisfies*

$$m \, |\xi|^p \leq F(\xi) \leq M \, |\xi|^p; \quad 1 \leq p < \infty.$$

Let $u \in W^{1,p}_{loc}(\Omega; \mathbf{R}^N)$ be a local **spherical quasi-minimizer** *of $I(v) = \int_\Omega F(Dv(x)) \, dx$ in the sense that, for any ball $B_R(a) \subset\subset \Omega$,*

$$\int_{B_R(a)} F(Du) \leq C \int_{B_R(a)} F(D(u + \phi)), \quad \forall \, \phi \in W^{1,p}_0(B_R(a); \mathbf{R}^N),$$

where $C > 0$ is a constant. Then $u \in W^{1,q}_{loc}(\Omega; \mathbf{R}^N)$ for some constant $q > p$.

Proof. Let $B_R = B_R(a) \subset\subset \Omega$ be fixed. For any $R/2 \leq s < r \leq R$, let $\zeta \in W^{1,\infty}_0(\Omega)$ be a cut-off function satisfying $\operatorname{supp} \zeta \subseteq B_r \equiv B_r(a)$ and

$$0 \leq \zeta \leq 1, \quad \zeta|_{B_s} \equiv 1, \quad |D\zeta(x)| \leq \frac{1}{r - s}.$$

Let $A \equiv B_r \setminus B_s$ and

$$\phi = -\zeta(u - \lambda) \in W^{1,p}_0(B_r; \mathbf{R}^N),$$

where $\lambda \in \mathbf{R}^N$ is a constant determined later. We have

$$\begin{aligned} m \int_{B_s} |Du|^p\,dx &\le m \int_{B_r} |Du|^p\,dx \le \int_{B_r} F(Du)\,dx \\ &\le C \int_{B_r} F(D(u+\phi))\,dx \\ &\le C_0 \int_{B_r} |D(u-\zeta(u-\lambda))|^p\,dx \\ &= C_0 \int_{A} |D(u-\zeta(u-\lambda))|^p\,dx \\ &= C_0 \int_{A} |(1-\zeta)\,Du-(u-\lambda)\otimes D\zeta|^p\,dx \\ &\le C_0 \int_{A} |Du|^p\,dx + \frac{C_0}{(r-s)^p} \int_{B_R} |u-\lambda|^p\,dx. \end{aligned}$$

Now *filling the hole* (Widman's technique), i.e., adding $C_0 \int_{B_s} |Du|^p\,dx$ to both sides, we have

$$\int_{B_s} |Du|^p\,dx \le \theta \int_{B_r} |Du|^p\,dx + \frac{C_1}{(r-s)^p} \int_{B_R} |u-\lambda|^p\,dx, \tag{5.29}$$

where $\theta = \frac{C_0}{C_0+m} < 1$. We need a lemma.

Lemma 5.18. *Let $f(t)$ be a nonnegative bounded function on $[\tau_0, \tau_1]$, where $\tau_0 \ge 0$. Suppose that for $\tau_0 \le s < r \le \tau_1$ we have*

$$f(s) \le \theta f(r) + \big(A(r-s)^{-\alpha} + B\big),$$

where A, B, α, θ are nonnegative constants and $0 < \theta < 1$. Then for all $\tau_0 \le \rho < R \le \tau_1$ we have

$$f(\rho) \le C\,\big(A(R-\rho)^{-\alpha} + B\big), \tag{5.30}$$

where C is a constant depending on α and θ.

Proof. For fixed $\tau_0 \le \rho < R \le \tau_1$ let us define a sequence $\{r_i\}$ by

$$r_0 = \rho, \quad r_{i+1} - r_i = (1-\tau)\,\tau^i\,(R-\rho),$$

where $0 < \tau < 1$ is to be selected. By iteration, we have

$$f(r_0) \le \theta^k\, f(r_k) + \left[\frac{A}{(1-\tau)^\alpha}(R-\rho)^{-\alpha} + B\right] \sum_{i=0}^{k-1} \theta^i\,\tau^{-i\alpha}.$$

If we choose τ in such a way that $\tau^{-\alpha}\theta < 1$ and pass to the limit for $k \to \infty$ in the above inequality, we get (5.30) with $C = (1-\tau)^{-\alpha}(1-\theta\tau^{-\alpha})^{-1}$. ☐

From (5.29) and the lemma above (with $B = 0$, $\rho = R/2$) we have

$$\int_{B_{R/2}} |Du|^p\,dx \le \frac{C_2}{R^p} \int_{B_R} |u-\lambda|^p\,dx.$$

Now we choose $\lambda = u_{B_R} = ⨍_{B_R} u$ and use the Sobolev-Poincaré inequality

$$\int_{B_R} |u-u_{B_R}|^p\,dx \le \sigma_n \left(\int_{B_R} |Du|^{\frac{np}{n+p}}\,dx\right)^{\frac{n+p}{p}}$$

to obtain

$$⨍_{B_{R/2}} |Du|^p \, dx \le C_3 \left(⨍_{B_R} |Du|^{\frac{np}{n+p}} \, dx \right)^{\frac{n+p}{p}}.$$

Let $f = |Du|^{\frac{np}{n+p}}$ and $q = \frac{p+n}{p} > 1$. Then we have

$$\left(⨍_{B_{R/2}} f^q \, dx \right)^{\frac{1}{q}} \le b ⨍_{B_R} f \, dx,$$

and therefore by the theorem above we have $f \in L^a_{loc}(\Omega)$ for some $a > q$. This implies $|Du| \in L^d_{loc}(\Omega)$ for some $d > p$. The theorem is proved. □

Remarks. 1) If $p = n$ we have $u \in W^{1,q}_{loc}(\Omega; \mathbf{R}^N)$ for some $q > n$ and hence the Sobolev embedding theorem implies that $u \in C^{0,\alpha}_{loc}(\Omega; \mathbf{R}^N)$ for some $0 < \alpha < 1$. This is first proved by Morrey in the case when $n = 2$.

2) Any weak solution $u \in W^{1,2}_{loc}(\Omega; \mathbf{R}^N)$ of a system

$$\mathrm{Div}(A(x, u, Du)\, Du) = 0,$$

with $A(x, u, \xi)$ satisfying

$$\nu |\xi|^2 \le A(x, u, \xi)\xi \cdot \xi, \quad |A(x, u, \xi)\xi| \le L\, |\xi|,$$

is a local spherical quasi-minimizer of the Dirichlet integral $I(v) = \int_\Omega |Dv|^2 \, dx$. Indeed, if

$$\int_\Omega A(x, u, Du)\, Du \cdot D\phi \, dx = 0, \quad \forall\, \phi \in W^{1,2}_0(\Omega; \mathbf{R}^N),$$

taking $v = u + \phi$ with $\operatorname{supp} \phi \subset B_R(a) \subset\subset \Omega$, we get

$$\begin{aligned} \nu \int_{B_R(a)} |Du|^2 \, dx &\le \int_{B_R(a)} A(x, u, Du)\, Du \cdot Du \, dx \\ &= \int_{B_R(a)} A(x, u, Du)\, Du \cdot Dv \, dx \\ &\le L \left(\int_{B_R(a)} |Du|^2 dx \right)^{1/2} \left(\int_{B_R(a)} |Dv|^2 dx \right)^{1/2} \end{aligned}$$

and hence

$$\int_{B_R(a)} |Du|^2 \, dx \le (\frac{L}{\nu})^2 \int_{B_R(a)} |Dv|^2 \, dx.$$

Therefore, by the theorem above, $u \in W^{1,p}_{loc}(\Omega; \mathbf{R}^N)$ for some $p > 2$. □

Proof of Theorem 5.16. The main idea of the proof is to use the method of *cube decompositions* of Calderón-Zygmund. First of all, we change balls to cubes.

Let $B_R \subset\subset B$ be given. We let Q^0, Q^1, Q^2 be the cubes with sides parallel to the coordinate axese and with the same center as B_R such that

$$B_{R/2} \subset Q^0 \subset\subset Q^1 \subset\subset Q^2 \subset B_R.$$

For any function $g \in L^1_{loc}(\mathbf{R}^n)$ and $r > 0$ we define a *local maximal function*

$$M_r(g)(x) = \sup_{0<\rho<r} ⨍_{B_\rho(x)} |g(y)| \, dy.$$

Select $r > 0$ sufficiently small so that $B_{2r}(x) \subset Q^2$ for all $x \in \bar{Q}^1$. Note that we can choose $r = c_n R$ and $|Q^0|/|Q^2| = d_n$, $|Q^0|/|Q^2| = e_n$, constants depending only on the dimension n.

Let $\tilde{f} = f \cdot \chi_{Q^2}$. By assumption, we have

$$M_r(\tilde{f}^q)(x) \le b\, M^q_\infty(\tilde{f})(x), \quad \forall\, x \in Q^1. \tag{5.31}$$

We want to show that

$$\left(\fint_{Q^0} \tilde{f}^p\, dx \right)^{\frac{1}{p}} \le C \left(\fint_{Q^2} \tilde{f}^q\, dx \right)^{\frac{1}{q}} \tag{5.32}$$

for $p \in (q,\, q + \epsilon)$, where $\epsilon = \epsilon(n, b, q) > 0$ and $C = C(n, b, q, p) > 0$ are some constants.

In the following, we assume $|Q^0| = 1$. In this case, the number $r = r_n$ above depends only on the dimension n. We follow E. Stredulinsky '80. Let

$$R(t) = \frac{1}{2} + \left(\frac{c_0\, k^{q/n}}{k^{q/n} - 1} \right) t^{-q/n},$$

where $k > 1$ is a constant to be determined later and $c_0 > 0$ is a constant making $R(1/k) = (\text{side}Q^1)/2$. Then $R(t)$ is decreasing and $R(t) \to 1/2$ as $t \to \infty$ and

$$R(t) - R(tk) = c_0\, t^{-q/n}.$$

Let Q_t be the cube concentric to Q^0 with side-length equal to $2R(t)$. So we have

$$Q^0 \subset Q_s \subset Q_t \subset Q^1, \quad \forall\, s > t > 1/k$$

and $Q_{1/k} = Q^1$ and $Q_t \to Q^0$ as $t \to \infty$. Let

$$E(t) = \{x \in Q^2 \mid \tilde{f}(x) > t\}, \quad E_t = E(t) \cap Q_t, \quad E^0_t = E(t) \cap Q^0.$$

Finally let $m = m_n$ be the least integer such that $\operatorname{diam} Q^1/2^m \le r = r_n$ and $\nu_n = 2^{nm}$; define a constant $\delta > 0$ by

$$\delta = \min \left\{ \nu_n^{-1/q},\ \left(c_0^n\, |B_1| \right)^{1/q} \right\}.$$

The remainder of the proof will be divided into several steps.

In what follows, without loss of generality, we assume $\|\tilde{f}\|_{L^q(Q^2)} = \delta$; otherwise, replace $\tilde{f}$ by $\delta \|\tilde{f}\|^{-1}_{L^q(Q^2)} \tilde{f}$.

Step 1 (Cube decomposition). Fix $s \ge 1$. We have

$$\fint_{Q_s} \tilde{f}^q \le \int_{Q^2} \tilde{f}^q = \delta^q \le \nu_n^{-1} \le \nu_n^{-1}\, s^q.$$

Subdivide the cube Q_s dyadically $m = m_n$ times so that the diameter of each subcubes $= \operatorname{diam} Q_s/2^m \le \operatorname{diam} Q^1/2^m \le r$. For each such subcube Q

$$\fint_Q \tilde{f}^q \le \frac{|Q_s|}{|Q|} \fint_{Q_s} \tilde{f}^q \le 2^{nm}\, (\nu_n^{-1}\, s^q) = s^q.$$

Now subdivide further each Q as in the Calderón-Zygmund decomposition to get $\{P_i\}$, disjoint subcubes of Q_s, such that

$$\tilde{f}^q(x) \le s^q, \quad a.e. \quad x \in Q_s \setminus \cup_i P_i,$$

$$s^q < \fint_{P_i} \tilde{f}^q \le 2^n\, s^q.$$

Let $G = \cup_i P_i$ and we have

$$E_s \subseteq G, \qquad \int_G \tilde f^q \le 2^n\, s^q\, |G|.$$

The initial subdivision guarantees that $\operatorname{diam} P_i \le r$. Given $x \in P_i$, consider ball $B_\rho(x)$ with $\rho = \operatorname{diam} P_i \le r$; then

$$s^q < \fint_{P_i} \tilde f^q \le \frac{|B_\rho(x)|}{|P_i|} \fint_{B_\rho(x)} \tilde f^q \le a_n\, M_r(\tilde f^q)(x).$$

Therefore, by (5.31)

$$s^q < a_n b\, M^q_\infty(\tilde f)(x), \quad \forall\, x \in G. \tag{5.33}$$

Step 2 (Basic estimates). We now select $k > 1$ such that $k^q = 3^q a_n b$. From (5.33) with $s = kt$ we have

$$3t < M_\infty(\tilde f)(x), \quad \forall\, x \in G.$$

Hence, given $x \in G$, there exists a ball $B_\rho(x)$ such that

$$3t < \fint_{B_\rho(x)} \tilde f. \tag{5.34}$$

We claim that $B_\rho(x) \subseteq Q_t$. It is sufficient to show $\rho \le c_0\, t^{-q/n}$; indeed, if we denote the center of Q_t by $\bar q = (\bar q_\alpha)$ then for any $y \in B_\rho(x)$, since $x \in Q_s$, we have

$$\begin{aligned} |y_\alpha - \bar q_\alpha| &\le |y - x| + |x_\alpha - \bar q_\alpha| \le \rho + R(s) \\ &\le c_0\, t^{-q/n} + R(kt) = R(t) \end{aligned}$$

and hence $B_\rho(x) \subseteq Q_t$. The definition of set $E(t)$ gives

$$\int_{B_\rho(x)} \tilde f \le \int_{B_\rho(x) \cap E(t)} \tilde f + t|B_\rho(x)|.$$

From this and (5.34) we get

$$t\, |B_\rho(x)| \le \int_{B_\rho(x) \cap E(t)} \tilde f \le t^{1-q} \int_{E(t)} \tilde f^q$$

and, since $\int \tilde f^q = \delta^q \le c_0^n\, |B_1|$ by the definition of δ, it follows that

$$|B_\rho(x)| \le t^{-q}\, \delta^q \le |B_1|\, c_0^n\, t^{-q},$$

and hence $\rho \le c_0\, t^{-q/n}$; so $B_\rho(x) \subseteq Q_t$. Using this, we have

$$|B_\rho(x)| \le \frac{1}{t} \int_{B_\rho(x) \cap E_t} \tilde f.$$

Since these balls $\{B_\rho(x)\}$ cover the set G, using a basic covering lemma (Stein's book), we obtain a disjoint collection of balls $\{B_i = B_{\rho_i}(x_i)\}$ such that $|G| \le 5^n \sum_i |B_i|$ and consequently

$$\begin{aligned}\int_{E_s} \tilde{f}^q &\le \int_G \tilde{f}^q \le 2^n\, s^q\, |G| \\ &\le 2^n\, s^q\, 5^n \sum_i |B_i| \\ &\le 10^n\, s^q\, t^{-1} \int_{E_t} \tilde{f} \\ &= 10^n\, k\, s^{q-1} \int_{E_{s/k}} \tilde{f}.\end{aligned}$$

Step 3 (Reduction to Stieltjes integral form). Let $h(s) = \int_{E_s} \tilde{f}$ then h is nonincreasing, right continuous and $h(t) \to 0$ as $t \to \infty$. Since

$$t^{q-1}\, h(t) \le \int_{E_t} \tilde{f}^q \to 0 \quad \text{as } t \to \infty,$$

we can integrate by parts to get

$$\begin{aligned}-\int_s^\infty t^{q-1}\, dh(t) &= (q-1) \int_s^\infty t^{q-2}\, h(t)\, dt + s^{q-1}\, h(s) \\ &= (q-1) \int_s^\infty t^{q-2} \int_{E_t} \tilde{f}(y)\, dy\, dt + s^{q-1}\, h(s) \\ &\le (q-1) \int_{E_s} \tilde{f}(y) \int_s^{\tilde{f}} t^{q-2}\, dt\, dy + s^{q-1}\, h(s) \\ &= \int_{E_s} \tilde{f}^q(y)\, dy \le k\, 10^n\, s^{q-1} \int_{E_{s/k}} \tilde{f}(y)\, dy;\end{aligned}$$

hence, with $a = k\, 10^n$ depending only on n, b, q, we have

$$-\int_s^\infty t^{q-1}\, dh(t) \le a\, s^{q-1}\, h(s/k), \quad \forall\, s \ge 1. \tag{5.35}$$

Note that we also have for any $p > 1$

$$-\int_s^\infty t^{p-1}\, dh(t) \ge \int_{E_s^0} \tilde{f}^p(y)\, dy, \quad \forall\, s \ge 1. \tag{5.36}$$

We need a lemma mainly due to F. Gehring '73.

Lemma 5.19 (Gehring '73)**.** *Suppose* (5.35) *holds. Let* $p \ge q$ *satisfy* $1 > a\, k^{p-1}\, (p-q)/(p-1)$. *Then*

$$-\int_1^\infty t^{p-1}\, dh(t) \le c_1 \left(-\int_1^\infty t^{q-1}\, dh(t) \right) + c_2\, h(1/k).$$

Proof. Let $I_p^j = -\int_1^j t^{p-1}\, dh(t)$. Integration by parts yields $I_p^j = I_q^j + (p-q)\, J$, where

$$\begin{aligned} J &= \int_1^j t^{p-q-1} \left(- \int_t^j s^{q-1}\, dh(s) \right) dt \\ &\leq a \int_1^j t^{p-2}\, h(t/k)\, dt \\ &= a\, k^{p-1} \int_{1/k}^{j/k} t^{p-2}\, h(t)\, dt \\ &= \frac{a\, k^{p-1}}{p-1} \left[(\frac{j}{k})^{p-1} h(\frac{j}{k}) - (\frac{1}{k})^{p-1} h(\frac{1}{k}) - \int_{1/k}^{j/k} t^{p-1}\, dh(t) \right] \\ &\leq \frac{a\, k^{p-1}}{p-1} \left[- \int_{1/k}^{j} t^{p-1}\, dh(t) - \frac{j^{p-q}}{k^{p-1}} \int_j^\infty t^{q-1}\, dh(t) \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} I_p^j &\leq I_q^j + \frac{p-q}{p-1}\, a\, k^{p-1} \left(- \int_{1/k}^{j} t^{p-1}\, dh(t) \right) \\ &+ \left[\left(\frac{p-q}{p-1} \right) a - 1 \right] j^{p-q} \left(- \int_j^\infty t^{q-1}\, dh(t) \right) \\ &- \int_j^\infty s^{q-1}\, dh(s). \end{aligned}$$

Note that the third term on the righthand side of this inequality is ≤ 0 and let $j \to \infty$ we arrive at

$$\left[1 - \frac{p-q}{p-1}\, a\, k^{p-1} \right] I_p^\infty \leq I_q^\infty + \frac{p-q}{p-1}\, a\, k^{p-1} \left(- \int_{1/k}^{1} t^{p-1}\, dh(t) \right)$$

and the lemma is proved since $-\int_{1/k}^1 t^{p-1}\, dh(t) \leq h(1/k)$. □

Step 4 (Completion of the proof). We use (5.36) with $s = 1$ and the lemma above to get, for $p \in (q,\, q+\epsilon)$ with some $\epsilon = \epsilon(n, b, q) > 0$,

$$\int_{E_1^0} \tilde{f}^p \leq c_1 \int_{E_1} \tilde{f}^q + c_2\, h(1/k).$$

Since $\tilde{f}^p \leq \tilde{f}^q$ when $\tilde{f} \leq 1$ it follows that

$$\begin{aligned} \int_{Q^0} \tilde{f}^p &\leq (c_1 + 1) \int_{Q^2} \tilde{f}^q + c_2\, k^{q-1} \int_{Q^2} \tilde{f}^q \\ &\leq C\, \delta^q = C(n, b, q, p)\, \delta^p. \end{aligned}$$

After renormalization we have

$$\left(\int_{Q^0} \tilde{f}^p \right)^{1/p} \leq C(n, b, q, p) \left(\int_{Q^2} \tilde{f}^q \right)^{1/q}.$$

A dilation argument can also remove the assumption $|Q^0| = 1$.

The proof of Theorem 5.16 is now complete.

5.6. Singular set of solutions of quasilinear systems

From Theorem 5.14, we see that the singular set $S_u \subseteq \Sigma_1 \cup \Sigma_2$, where

$$\begin{aligned} \Sigma_1 &= \left\{ a \in \Omega \,\Big|\, \liminf_{R \to 0^+} R^{-n} \int_{B_R(a)} |u - u_{B_R(a)}|^2 \, dx > 0 \right\}, \\ \Sigma_2 &= \{ a \in \Omega \mid \sup_{R>0} |u_{B_R(a)}| = \infty \}. \end{aligned}$$

By Caccioppoli's inequality (Lemma 5.10) and Poincaré's inequality, we know

$$\Sigma_1 = \left\{ a \in \Omega \,\Big|\, \liminf_{R \to 0^+} R^{2-n} \int_{B_R(a)} |Du|^2 \, dx > 0 \right\}.$$

Note that by Hölder's inequality for any $p > 2$

$$R^{2-n} \int_{B_R(a)} |Du|^2 \, dx \le C \left(R^{p-n} \int_{B_R(a)} |Du|^p \, dx \right)^{\frac{2}{p}}.$$

Therefore

$$\Sigma_1 \cup \Sigma_2 \subseteq E_p \cup G, \tag{5.37}$$

where

$$\begin{aligned} E_p &= \left\{ a \in \Omega \mid \liminf_{R \to 0^+} R^{p-n} \int_{B_R(a)} |Du|^p \, dx > 0 \right\}, \\ G &= \{ a \in \Omega \mid \sup_{R>0} |u_{B_R(a)}| = \infty \} \cup \{ a \in \Omega \mid \nexists \lim_{R \to 0^+} u_{B_R(a)} \}. \end{aligned}$$

By the higher integrability theorem above, there exists a number $p > 2$ such that $u \in W^{1,p}_{loc}(\Omega; \mathbf{R}^N)$; therefore, we need to estimate the set E_p and G for such functions u.

We need to recall the definition and some properties of **Hausdorff measures.**

Let X be a metric space and $\mathcal{F}$ a family of subsets of X containing the empty set $\emptyset$. Let $h\colon \mathcal{F} \to [0, \infty]$ be a function such that $h(\emptyset) = 0$. For any $\epsilon > 0$ and subset E of X we define

$$\mu_\epsilon(E) = \inf \left\{ \sum_{j \in \mathbb{Z}^+} h(F_j) \,\Big|\, E \subseteq \bigcup_{j \in \mathbb{Z}^+} F_j, \;\; F_j \in \mathcal{F}, \;\; \operatorname{diam} F_j < \epsilon \right\}$$

and, since $\mu_\epsilon \ge \mu_\delta$ if $0 < \epsilon < \delta$,

$$\mu(E) = \lim_{\epsilon \to 0^+} \mu_\epsilon(E) = \sup_{\epsilon > 0} \mu_\epsilon(E).$$

In this way, $\mu(E)$ is called the *Carathéodory construction* for $\mathcal{F}$ and h. Note that the set function $\mu(E)$ is an *outer measure* with respect to which all Borel sets are measurable.

To define the k-dimensional Hausdorff measure in $\mathbf{R}^n$ we let $X = \mathbf{R}^n$, $\mathcal{F}$ the family of all open sets in $\mathbf{R}^n$ and for $k = 0, 1, \cdots$

$$h(F) = h_k(F) = \omega_k \, 2^{-k} (\operatorname{diam} F)^k,$$

where ω_k is the Lebesgue measure of unit ball of $\mathbf{R}^k$. The Carathéodory construction μ for this choice of $\mathcal{F}$ and h_k is called the **k-dimensional Hausdorff measure** in $\mathbf{R}^n$ and denoted by $\mathcal{H}^k$.

Note that $\mathcal{H}^0(E) =$ number of points of E and that if $\mathcal{H}^k(E) < \infty$ then $\mathcal{H}^{k+\epsilon}(E) = 0$ for all $\epsilon > 0$.

We define the **Hausdorff dimension** of E by

$$\dim_{\mathcal{H}} E = \inf\{k > 0 \,|\, \mathcal{H}^k(E) = 0\}.$$

Remark. If we choose the family $\mathcal{F}$ to be the set of all open balls, using the same h_k, we obtain the so-called k-dimensional *spherical* Hausdorff measure $\mathcal{S}^k$. But, in general, $\mathcal{H}^k \neq \mathcal{S}^k$. However, we can easily verify that

$$\mathcal{H}^k(E) \leq \mathcal{S}^k(E) \leq 2^k\, \mathcal{H}^k(E).$$

Therefore, the family of subsets of $\mathcal{H}^k$-measure zero coincides with that of $\mathcal{S}^k$-measure zero; hence, in particular, the Hausdorff dimension is the same from both constructions. □

Theorem 5.20. *Let Ω be an open set in $\mathbf{R}^n$. Let $v \in L^1_{loc}(\Omega)$, $0 \leq \alpha < n$ and*

$$E = \left\{ a \in \Omega \,\middle|\, \limsup_{R\to 0^+} R^{-\alpha} \int_{B_R(a)} |v|\, dx > 0 \right\}.$$

Then $\mathcal{H}^\alpha(E) = 0$ and hence $\dim_{\mathcal{H}} E \leq \alpha$.

Proof. It is sufficient to show that for each compact set $K \subset \Omega$

$$\mathcal{H}^\alpha(E \cap K) = 0.$$

Let $F = E \cap K$ and

$$F^k = \left\{ a \in F \,\middle|\, \limsup_{R\to 0^+} R^{-\alpha} \int_{B_R(a)} |v|\, dx > \frac{1}{k} \right\}.$$

Then $F = \cup_{k=1}^{\infty} F^k$. It is sufficient to prove

$$\mathcal{H}^\alpha(F^k) = 0, \quad \forall\, k = 1, 2, \cdots .$$

Let $K \subset Q \subset \bar{Q} \subset \Omega$ and let $d = \operatorname{dist}(K; \partial\Omega)$. Then for any fixed $\epsilon \in (0, d)$ there exists for each $a \in F^k$ a number $r_a \in (0,\, \epsilon)$ such that

$$r_a^{-\alpha} \int_{B_{r_a}(a)} |v|\, dx > \frac{1}{2k}.$$

From *Besicovitch's theorem* (Stein's book) there exists a countable family of disjoint balls $\{B_i\}$ with

$$B_i = B_{r_{a_i}}(a_i); \quad a_i \in F^k$$

such that

$$F^k \subseteq \bigcup_{i=1}^{\infty} B_{5r_{a_i}}(a_i).$$

Letting $r_i = r_{a_i}$, we get

$$\begin{aligned} \sum_i r_i^\alpha &\leq 2k \sum_i \int_{B_{r_i}(a_i)} |v|\, dx \\ &\leq 2k \int_{\cup_i B_{r_i}(a_i)} |v|\, dx. \end{aligned}$$

Now since $\alpha < n$ we have

$$\begin{aligned} \left|\bigcup_i B_{r_i}(a_i)\right| = \omega_n \sum_i r_i^n &\leq \omega_n\, \epsilon^{n-\alpha} \sum_i r_i^\alpha \\ &\leq 2k\omega_n\, \epsilon^{n-\alpha} \int_Q |v|\, dx, \end{aligned}$$

which goes to zero as $\epsilon \to 0^+$. This implies

$$\lim_{\epsilon \to 0^+} \sum_i r_i^\alpha = 0.$$

Note that by definition, since $F^k \subseteq \cup_i B_{5r_i}(a_i)$ and $\operatorname{diam}(B_{5r_i}(a_i)) < \delta = 10\epsilon$, it follows that

$$\mathcal{H}^\alpha_\delta(F^k) \le C \sum_i r_i^\alpha$$

and thus we have

$$\mathcal{H}^\alpha(F^k) = \lim_{\delta \to 0^+} \mathcal{H}^\alpha_\delta(F^k) = 0,$$

as desired. The proof is complete. □

From this theorem, for the set E_p in (5.37), we have $\mathcal{H}^{n-p}(E_p) = 0$. For the set G in (5.37), we now prove

$$\mathcal{H}^{n-p+\epsilon}(G) = 0, \quad \forall\, \epsilon > 0. \tag{5.38}$$

(Note that, in general, it is not true that $\mathcal{H}^{n-p}(G) = 0$.) By Theorem 5.20, this will be proved if we show that

$$G \subseteq E \equiv \left\{ a \in \Omega \,\Big|\, \limsup_{R \to 0^+} R^{-(n-p+\epsilon)} \int_{B_R(a)} |Du|^p \, dx > 0 \right\}, \quad \forall\, \epsilon > 0.$$

To see this, we assume $a \notin E$ for some $\epsilon > 0$. Since the function $R \mapsto u_{B_R(a)}$ is absolutely continuous in the interval $(0, \operatorname{dist}(a; \partial\Omega))$ and

$$\left| \frac{d}{dR} u_{B_R(a)} \right| \le \left(⨍_{B_R(a)} |Du|^p \, dx \right)^{1/p},$$

using $a \notin E$, we have

$$\sup_{0<R<R_0} \left\{ R^{-(n-p+\epsilon)} \int_{B_R(a)} |Du|^p \, dx \right\}^{1/p} = L < \infty,$$

where $R_0 = \min\{1, \frac{1}{2}\operatorname{dist}(a; \partial\Omega)\}$. This implies

$$\left| \frac{d}{dR} u_{B_R(a)} \right| \le L\, R^{-1+\frac{\epsilon}{p}}$$

and hence, for $0 < s < r < R_0$

$$|u_{B_r(a)} - u_{B_s(a)}| \le \int_s^r \left| \frac{d}{dR} u_{B_R(a)} \right| dR \le \frac{Lp}{\epsilon} \left| r^{\frac{\epsilon}{p}} - s^{\frac{\epsilon}{p}} \right|,$$

which implies that $a \notin G$, by the definition of set G; this shows $G \subseteq E$, as claimed.

Now (5.38) together with (5.37) and $\mathcal{H}^{n-p}(E_p) = 0$ shows that

$$\mathcal{H}^{n-p+\epsilon}(S_u) = 0, \quad \forall \epsilon > 0.$$

Obviously, $p - \epsilon > 2$ for all sufficiently small $\epsilon > 0$; thus we have proved the following result, modifying Theorem 5.14.

Theorem 5.21. *Assume $A(x, u)$ is continuous on $(x, u) \in \Omega \times \mathbf{R}^N$. Let $u \in W^{1,2}(\Omega; \mathbf{R}^N)$ be a weak solution of* (5.17). *Then there exists an open set $\Omega_0 \subset \Omega$ such that $u \in C^{0,\mu}_{loc}(\Omega_0; \mathbf{R}^N)$ for all $0 < \mu < 1$ and $\mathcal{H}^{n-p}(\Omega \setminus \Omega_0) = 0$ for some $p > 2$.*

Remarks. 1) If $n = 2$ then we have $\mathcal{H}^0(\Omega \setminus \Omega_0) = 0$ and thus $\Omega_0 = \Omega$; so, we have the full regularity for weak solutions in two dimension. This implies that every weak solution to a smooth 2×2 strongly (Legendre) elliptic system is smooth; compare with Müller and Šverák's example for smooth 2×2 strongly Legendre-Hadamard elliptic systems.

2) It has been conjectured that

$$\dim_{\mathcal{H}} S_u = n - 3; \quad \mathcal{H}^{n-3+\epsilon}(S_u) = 0, \ \forall\, \epsilon > 0.$$

This conjecture is still open in general. □

5.7. Quasiconvexity and partial regularity

In this section, we prove a result of Evans concerning the partial regularity of energy minimizers in the calculus of variations.

We assume $F\colon \mathbf{M}^{N\times n} \to \mathbf{R}$ is a C^2 function satisfying

$$\begin{cases} 0 \le F(\xi) \le |\xi|^2 + 1, \\ |DF(\xi)| \le L\,(|\xi| + 1), \\ |D^2F(\xi)| \le L. \end{cases} \tag{5.39}$$

Note that we do not assume the *Legendre ellipticity* condition on $F(\xi)$.

Recall that F is **uniformly strict quasiconvex** if for any ball $B \subset \mathbf{R}^n$,

$$⨍_B F(\xi + D\phi(x))\,dx \ge F(\xi) + \gamma ⨍_B |D\phi|^2\,dx \tag{5.40}$$

for all $\xi \in \mathbf{M}^{N\times n}$ and $\phi \in W^{1,2}_0(B;\mathbf{R}^N)$, where $\gamma > 0$ is a constant.

Theorem 5.22 (Evans '86)**.** *Assume $F(\xi)$ is uniformly strict quasiconvex. Suppose $u \in W^{1,2}_{loc}(\Omega;\mathbf{R}^N)$ is a local minimizer of the functional*

$$I(v) = \int_\Omega F(Dv(x))\,dx$$

in the sense that

$$I(u) \le I(u+\phi); \quad \forall\, \phi \in W^{1,2}_0(\Omega';\mathbf{R}^N), \quad \Omega' \subset\subset \Omega. \tag{5.41}$$

Then there exists an open set $\Omega_0 \subset \Omega$ such that $|\Omega \setminus \Omega_0| = 0$ and $u \in C^{1,\mu}_{loc}(\Omega_0;\mathbf{R}^N)$ for each $\mu \in (0,\,1)$.

We prove several lemmas before proving this theorem.

Lemma 5.23. *Let $F(\xi)$ be uniformly strict quasiconvex. Then F satisfies the Legendre-Hadamard condition; that is,*

$$F_{\xi^i_\alpha \xi^j_\beta}(\xi)\, p_\alpha p_\beta\, q^i q^j \ge 2\gamma\, |p|^2\, |q|^2.$$

Proof. For any $\xi \in \mathbf{M}^{N\times n}$ and $\phi \in W^{1,2}_0(B;\mathbf{R}^N)$, let us consider function

$$f(t) = \int_B F(\xi + tD\phi(x))\,dx - t^2\gamma \int_B |D\phi(x)|^2\,dx.$$

Then we have $f(t) \ge f(0)$ for all $t \in \mathbf{R}$ and f is of C^2 and hence $f''(0) \ge 0$; this implies

$$\int_B F_{\xi^i_\alpha \xi^j_\beta}(\xi)\, D_\alpha\phi^i(x) D_\beta\phi^j(x)\,dx \ge 2\gamma \int_B |D\phi(x)|^2\,dx,$$

which, being valid for all $\phi \in W_0^{1,2}(B;\mathbf{R}^N)$, implies that the constants $A_{ij}^{\alpha\beta} = F_{\xi_\alpha^i \xi_\beta^j}(\xi)$ define a coercive bilinear form on $H = W_0^{1,2}(B;\mathbf{R}^N)$ and thus, as before using the "sawtooth" like test function ϕ we can prove

$$F_{\xi_\alpha^i \xi_\beta^j}(\xi)\, p_\alpha p_\beta\, q^i q^j \geq 2\gamma\, |p|^2\, |q|^2;$$

that is, the Legendre-Hadamard condition must hold. This proves the lemma. □

In the following we always assume F satisfies the assumptions stated in the theorem and u is a local minimizer of $I(v)$ defined before.

Lemma 5.24. *There exists a constant $C_1 = C_1(n, L, \gamma)$ such that for all $A \in \mathbf{M}^{N\times n}$ and $\lambda \in \mathbf{R}^N$ we have*

$$\fint_{B_{R/2}(a)} |Du - A|^2\, dx \leq \frac{C_1}{R^2} \fint_{B_R(a)} |u - \lambda - Ax|^2\, dx$$

for all balls $B_R(a) \subset\subset \Omega$.

Proof. We denote $B_\rho = B_\rho(a)$. Let $B_R \subset\subset \Omega$ be fixed. For any $R/2 \leq s < r \leq R$, let $\zeta \in W_0^{1,\infty}(\Omega)$ be a cut-off function used before which satisfies $\operatorname{supp}\zeta \subseteq B_r \equiv B_r(a)$ and

$$0 \leq \zeta \leq 1, \quad \zeta|_{B_s} \equiv 1, \quad |D\zeta(x)| \leq \frac{1}{r-s}.$$

Define

$$\phi(x) = \zeta(u - \lambda - Ax), \quad \psi(x) = (1-\zeta)\,(u - \lambda - Ax);$$

then $\phi \in W_0^{1,2}(B_r;\mathbf{R}^N)$, $\psi \in W^{1,2}(B_r;\mathbf{R}^N)$, and $D\phi + D\psi = Du - A$. We have, by the uniform strict quasiconvexity,

$$\begin{aligned}
\int_{B_r} [F(A) + \gamma\, |D\phi|^2]\, dx \;&\leq\; \int_{B_r} F(A + D\phi)\, dx \\
&=\; \int_{B_r} F(Du - D\psi))\, dx \\
&\leq\; \int_{B_r} \Big[F(Du) - DF(Du)\, D\psi + C\, |D\psi|^2\Big]\, dx.
\end{aligned}$$

Since u is a local minimizer, we have

$$\begin{aligned}
\int_{B_r} F(Du)\, dx \;&\leq\; \int_{B_r} F(Du - D\phi)\, dx = \int_{B_r} F(A + D\psi)\, dx \\
&\leq\; \int_{B_r} \Big[F(A) + DF(A)\, D\psi + C\, |D\psi|^2\Big]\, dx.
\end{aligned}$$

Combining the previous two inequalities, canceling the term $\int_{B_r} F(A)\, dx$, we have

$$\gamma \int_{B_r} |D\phi|^2\, dx \leq \int_{B_r} \Big[\big(DF(A) - DF(Du)\big)\, D\psi + C|D\psi|^2\Big]\, dx.$$

From the definition of ϕ and the quardratic growth of F we have

$$\int_{B_s} |Du - A|^2\, dx \leq C \int_{B_r} \Big[|Du - A|\, |D\psi| + |D\psi|^2\Big]\, dx. \tag{5.42}$$

Note that $\psi = 0$ on B_s, and

$$|D\psi| \leq |Du - A| + \frac{1}{r-s}\, |u - \lambda - Ax|;$$

hence from (5.42) we have

$$\int_{B_s} |Du - A|^2\,dx \le C \int_{B_r\setminus B_s} |Du - A|^2\,dx + \frac{C}{(r-s)^2}\int_{B_R} |u - \lambda - Ax|^2\,dx$$

and by filling the hole again we have

$$\int_{B_s} |Du - A|^2\,dx \le \theta \int_{B_r} |Du - A|^2\,dx + \frac{C}{(r-s)^2}\int_{B_R} |u - \lambda - Ax|^2\,dx,$$

where $\theta = \frac{C}{C+1} < 1$. This inequality is valid for all $R/2 \le s < r \le R$. We thus use Lemma 5.18 to derive

$$\int_{B_{R/2}} |Du - A|^2\,dx \le \frac{C_1}{R^2}\int_{B_R} |u - \lambda - Ax|^2\,dx.$$

The lemma is proved. □

Define (compare with (5.19) before)

$$\Phi_u(a,r) = ⨍_{B_r(a)} |Du - (Du)_{a,r}|^2\,dx, \quad (Du)_{a,r} = ⨍_{B_r(a)} Du\,dx.$$

Theorem 5.25 (Main Lemma)**.** *For each $M > 0$ there exists a constant $C_2(M)$ with the property that for each $0 < \tau < \frac{1}{4}$ there exists $\epsilon(M,\tau) > 0$ such that for every ball $B_r(a) \subset\subset \Omega$ the conditions*

$$|(Du)_{a,r}| \le M, \quad |(Du)_{a,\tau r}| \le M$$

and $\Phi_u(a,r) \le \epsilon(M,\tau)$ imply

$$\Phi_u(a,\tau r) \le C_2(M)\,\tau^2\,\Phi_u(a,r). \tag{5.43}$$

Proof. As in Theorem 5.13, we prove by a contradiction method. Suppose there exists $M_0 > 0$ we cannot find $C_2(M_0)$ with the required property. Then, for some $\tau \in (0, \frac{1}{4})$, we would find balls $B_{r_m}(a_m) \subset\subset \Omega$ for each $m = 1, 2, ...$ such that

$$|(Du)_{a_m,r_m}| \le M_0, \quad |(Du)_{a_m,\tau r_m}| \le M_0$$

and

$$\Phi_u(a_m, r_m) \equiv \lambda_m^2 \to 0 \quad \text{as } m \to \infty,$$

but

$$\Phi_u(a_m, \tau r_m) > m\,\tau^2\,\lambda_m^2 \quad \forall\, m = 1, 2, \cdots. \tag{5.44}$$

For simplicity, we denote

$$b^m = u_{a_m,r_m}, \quad c^m = u_{a_m,2\tau r_m}, \quad e^m = u_{a_m,\tau r_m};$$
$$B^m = (Du)_{a_m,r_m}, \quad E^m = (Du)_{a_m,\tau r_m}.$$

By Lemma 5.24, we have

$$⨍_{B_{\tau r_m}(a_m)} |Du - E^m|^2\,dx \le \frac{C_1}{(2\tau\,r_m)^2} ⨍_{B_{2\tau r_m}(a_m)} |u - c^m - E^m(x - a_m)|^2\,dx. \tag{5.45}$$

We now use the *blowing up* technique. Let

$$v_m(z) = \frac{u(a_m + r_m\,z) - b^m - r_m\,B^m\,z}{\lambda_m\,r_m}, \quad z \in B_1(0) \equiv B_1.$$

Then

$$Dv_m(z) = \frac{Du(a_m + r_m\,z) - B^m}{\lambda_m}, \quad (v_m)_{B_1} = 0, \quad (Dv_m)_{B_1} = 0.$$

Let

$$f^m = (v_m)_{B_{2\tau}}, \quad g^m = (v_m)_{B_\tau}, \quad G^m = (Dv_m)_{B_\tau}.$$

Now we have

$$\fint_{B_1} |Dv_m|^2 \, dz = 1$$

and hence, since $(v_m)_{B_1} = 0$, by Sobolev-Poincaré's inequality

$$\fint_{B_1} |v_m|^2 \, dz \le c_n.$$

By (5.45) we also have

$$\fint_{B_\tau} |Dv_m - G^m|^2 \, dz \le \frac{C_1}{\tau^2} \fint_{B_{2\tau}} |v_m - f^m - G^m z|^2 \, dz. \tag{5.46}$$

Since $|B^m| \le M_0$ we can have a subsequence relabeled the same such that

$$\begin{cases} B^m \to B & \text{in } \mathbf{M}^{N\times n}, \\ v_m \to v & \text{strongly in } L^2(B_1; \mathbf{R}^N), \\ Dv_m \rightharpoonup Dv & \text{weakly in } L^2(B_1; \mathbf{M}^{N\times n}). \end{cases} \tag{5.47}$$

Lemma 5.26. *$v = v(z)$ is a weak solution in B_1 of the linear system*

$$D_{z_\alpha}(F_{\xi^i_\alpha \xi^j_\beta}(B)\, D_{z_\beta} v^j(z)) = 0, \quad i = 1, 2, \cdots, N.$$

Proof. It is not difficult to see that $u = u(x)$ is a weak solution in Ω of the *Euler-Lagrange* equations

$$\frac{\partial}{\partial x_\alpha}\left(\frac{\partial F}{\partial \xi^i_\alpha}(Du(x))\right) = 0, \quad i = 1, 2, \cdots, N.$$

Let $\phi \in C_0^\infty(B_1; \mathbf{R}^N)$. We use $\psi(x) = \phi(\frac{x - a_m}{r_m})$ as a test function in the system for u and then change variables to obtain

$$\begin{aligned} 0 &= \int_{B_1} \left[F_{\xi^i_\alpha}(\lambda_m Dv_m + B^m) - F_{\xi^i_\alpha}(B^m)\right] D_{z_\alpha}\phi^i(z)\, dz \\ &= \int_{B_1} \left[\int_0^1 F_{\xi^i_\alpha \xi^j_\beta}(s\lambda_m Dv_m + B^m)\, ds\right] D_{z_\beta} v^j_m \, D_{z_\alpha}\phi^i \, dz. \end{aligned}$$

By (5.47) we can also assume $\lambda_m Dv_m(z) \to 0$ for almost every $z \in B_1$ as $m \to \infty$. Therefore, letting $m \to \infty$, by Lebesgue's dominated convergence theorem we have

$$\int_{B_1} F_{\xi^i_\alpha \xi^j_\beta}(B)\, D_{z_\beta} v^j_m \, D_{z_\alpha}\phi^i \, dz = 0$$

for all $\phi \in C_0^\infty(B_1; \mathbf{R}^N)$ and the lemma is proved. □

Using this lemma, since the coefficients $F_{\xi^i_\alpha \xi^j_\beta}(B)$ of this system satisfy the hypothesis $(H2)$ considered before, we have

$$\sup_{B_{1/2}} |D^2 v|^2 \le C \fint_{B_1} |Dv|^2 \, dz \le C.$$

Using a Poincaré type inequality (see Gilbarg-Trudinger, P.164)

$$\|u - u_S\|_{L^p(D)} \le c_n \, |S|^{\frac{1}{n}-1} \, (\operatorname{diam} D)^n \, \|Du\|_{L^p(D)}$$

for any convex domain D, subset $S \subset D$ and $u \in W^{1,p}(D)$, we obtain

$$\fint_{B_{2\tau}} |Dv - (Dv)_{0,\tau}|^2 \, dz \le C\,\tau^2. \tag{5.48}$$

Therefore

$$\begin{aligned} \lim_{m\to\infty} \fint_{B_{2\tau}} |v_m - f^m - G^m z|^2 \, dz &= \fint_{B_{2\tau}} |v - (v)_{0,2\tau} - (Dv)_{0,\tau}\, z|^2 \, dz \\ \text{(by Poincaré type inequality)} \quad &\le C\,\tau^2 \fint_{B_{2\tau}} |Dv - (Dv)_{0,\tau}|^2 \, dz \\ \text{(by (5.48))} \quad &\le C\,\tau^4. \end{aligned}$$

From this and (5.46), we have

$$\limsup_{m\to\infty} \fint_{B_\tau} |Dv_m - G^m|^2 \, dz \le C_3\,\tau^2.$$

However if we scale (5.44) we would get

$$\fint_{B_\tau} |Dv_m - G^m|^2 \, dz > m\,\tau^2.$$

This contradicts with the previous estimate since $C_3 > 0$ is independent of m. The proof is complete. □

Lemma 5.27. *Let $M > 0$ and τ satisfy*

$$0 < \tau < \min\{1/4,\ (C_2(2M))^{-1/2}\}, \tag{5.49}$$

where $C_2(2M)$ is the constant from the main lemma above, with $2M$ replacing M. Then there exists a number $\eta(M,\tau) > 0$ such that, for every ball $B_r(a) \subset\subset \Omega$, the validity of three inequalities

$$|(Du)_{a,r}| \le M, \quad |(Du)_{a,\tau r}| \le M, \quad \Phi_u(a,r) \le \eta(M,\tau)$$

implies

$$\Phi_u(a,\tau^l\, r) \le [C_2(2M)\,\tau^2]^l\, \Phi_u(a,r), \quad \forall\, l = 1, 2, \cdots. \tag{5.50}$$

Proof. For M, τ satisfying the given condition, define

$$\eta(M,\tau) = \min\left\{\epsilon(2M,\tau),\ \frac{\tau^{2n} M^2}{4}\Big(1 - \sqrt{C_2(2M)}\,\tau\Big)^2\right\}, \tag{5.51}$$

where $\epsilon(2M,\tau)$ is the constant from Main Lemma (Theorem 5.25), with $2M$ replacing M. We prove (5.50) by induction on l. The case $l = 1$ is immediate from the main lemma since $\Phi_u(a,r) \le \eta(M,\tau) \le \epsilon(2M,\tau)$. Now assume (5.50) holds for all $l = 1, 2, \cdots, k$. We claim

$$|(Du)_{a,\tau^k r}| \le 3M/2, \tag{5.52}$$

$$|(Du)_{a,\tau^{k+1} r}| \le 2M, \tag{5.53}$$

$$\Phi_u(a,\tau^k r) \le \epsilon(2M,\tau). \tag{5.54}$$

We prove these relations below. Once these relations are proved, the Main Lemma (with $2M$ replacing M and $\tau^k r$ replacing r) and the induction assumption will yield

$$\begin{aligned} \Phi_u(a,\tau^{k+1}\, r) &\le C_2(2M)\,\tau^2\,\Phi_u(a,\tau^k\, r) \\ &\le [C_2(2M)\,\tau^2]^{k+1}\,\Phi_u(a,r), \end{aligned}$$

which proves (5.50) for $l = k+1$ and hence it holds for all $l = 1, 2, \cdots$; the lemma will be proved. Therefore, we need only to prove (5.52)-(5.54) under the assumption that (5.50) holds for $l = 1, 2, \cdots, k$.

Proof of (5.52). For all $l = 0, 1, ...$ we have

$$\begin{aligned}
|(Du)_{a,\tau^{l+1}r} - (Du)_{a,\tau^l r}| &\le \fint_{B(a,\tau^{l+1}r)} |Du - (Du)_{a,\tau^l r}|\,dx \\
&\le \frac{1}{\tau^n}\left(\fint_{B(a,\tau^l r)} |Du - (Du)_{a,\tau^l r}|^2\,dx\right)^{\frac{1}{2}} \\
&\le \frac{1}{\tau^n}\,\Phi_u(a,\tau^l r)^{\frac{1}{2}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|(Du)_{a,\tau^k r}| &\le |(Du)_{a,r}| + \sum_{l=1}^{k-1} |(Du)_{a,\tau^{l+1}r} - (Du)_{a,\tau^l r}| \\
&\le M + \frac{1}{\tau^n}\sum_{l=1}^{k-1} \Phi_u(a,\tau^l r)^{1/2} \\
&\le M + \frac{1}{\tau^n}\sum_{l=1}^{k-1} [(C_2(2M)\,\tau^2\,)^l\,\Phi_u(a,r)]^{1/2} \\
&\le M + \frac{1}{\tau^n}\eta(M,\tau)^{1/2}\,(1-\sqrt{C_2(2M)}\,\tau)^{-1} \\
&\le 3M/2.
\end{aligned}$$

Proof of (5.53). By the estimates above we have

$$\begin{aligned}
|(Du)_{a,\tau^{k+1}r} - (Du)_{a,\tau^k r}| &\le \frac{1}{\tau^n}\,\Phi_u(a,\tau^k r)^{\frac{1}{2}} \\
&\le \frac{1}{\tau^n}[C_2(2M)\,\tau^2]^{k/2}\,\Phi_u(a,r)^{1/2} \\
&\le \frac{\eta(M,\tau)^{1/2}}{\tau^n} \\
&\le M/2,
\end{aligned}$$

and hence

$$|(Du)_{a,\tau^{k+1}r}| \le |(Du)_{a,\tau^{k+1}r} - (Du)_{a,\tau^k r}| + |(Du)_{a,\tau^k r}| \le 2M.$$

Proof of (5.54). We easily have

$$\begin{aligned}
\Phi_u(a,\tau^k r) &\le (C_2(2M)\,\tau^2)^k\,\Phi_u(a,r) \\
&\le \Phi_u(a,r) \le \eta(M,\tau) \\
&\le \epsilon(2M,\tau).
\end{aligned}$$

Therefore, (5.52)-(5.54) and hence the lemma are proved. $\square$

Proof of Theorem 5.22. Set

$$\Omega_0 = \left\{a \in \Omega \,\middle|\, \exists \lim_{r\to 0^+} |(Du)_{a,r}| < \infty,\ \lim_{r\to 0^+} \Phi_u(a,r) = 0\right\}. \tag{5.55}$$

Since $u \in W^{1,2}_{loc}(\Omega;\mathbf{R}^N)$, we easily see $|\Omega \setminus \Omega_0| = 0$. We shall prove Ω_0 is open and that $u \in C^{1,\mu}(\Omega_0;\mathbf{R}^N)$ for each $\mu \in (0,\,1)$. Indeed, for each $a \in \Omega_0$, there exists a number $M = M(a)$ such that

$$|(Du)_{a,s}| \le M, \quad \forall\ 0 < s < \operatorname{dist}(a;\partial\Omega).$$

For each $\mu \in (0,\,1)$ we select τ, $0 < \tau < \min\{1/4,\, C_2(2M)^{-1/2}\}$, so that

$$C_2(2M)\,\tau^{2-2\mu} \le 1.$$

Next, from the definition of Ω_0, we choose r, $0 < r < \frac{1}{2}\operatorname{dist}(a;\partial\Omega)$, such that

$$\Phi_u(a,r) < \eta(M,\tau).$$

Since the mappings

$$a \mapsto \Phi_u(a,r), \quad (Du)_{a,r}, \quad (Du)_{a,\tau r}$$

are continuous, we have a ball $B_R(a) \subset B_r(a) \subset\subset \Omega$ such that

$$\Phi_u(x,r) < \eta(M,\tau), \quad |(Du)_{x,r}| < M, \quad |(Du)_{x,\tau r}| < M$$

for all $x \in B_R(a)$. Consequently, Lemma 5.27 implies

$$\Phi_u(x,\tau^l r) \le \left(C_2(2M)\,\tau^2\right)^l \Phi_u(x,r), \quad \forall\ l = 1, 2, \ldots$$

and hence in view of the choice of τ we have

$$\Phi_u(x,\tau^l r) \le \tau^{2\mu\, l}\,\Phi_u(x,r) \le (\tau^l\, r)^{2\mu}\, r^{-2\mu}\,\eta(M,\tau)$$

for all $l = 1, 2, \cdots$ and $x \in B_R(a)$. This implies

$$\int_{B_\rho(x)} |Du - (Du)_{x,\rho}|^2\, dy \le C(M,\tau,r)\,\rho^{n+2\mu}$$

for all $x \in B_R(a)$ and $0 < \rho < \operatorname{dist}(x; B_R(a))$. Therefore by the local Campanato estimate we have

$$Du \in C^{0,\mu}(B_{R/2}(a);\mathbf{M}^{N\times n}).$$

This in turn implies $B_{R/2}(a) \subset \Omega_0$ by the definition of Ω_0 and so Ω_0 is an open set and $u \in C^{1,\mu}_{loc}(\Omega_0;\mathbf{R}^N)$ for all $0 < \mu < 1$. Theorem 5.22 is proved.

Remark. The $C^{1,\mu}$-partial regularity for minimizers has been extended to functionals of type

$$I(u) = \int_\Omega F(x,u,Du)\,dx,$$

where $F(x,u,\xi)$ is uniformly strictly quasiconvex in ξ and Hölder continuous in (x,u); see Acerbi-Fusco '87, Fusco-Hutchinson '85, and Giaquinta-Modica '86. The following theorem is the most general one in this direction. □

Theorem 5.28 (Acerbi-Fusco '87). *Let $F\colon \Omega \times \mathbf{R}^N \times \mathbf{M}^{N\times n} \to \mathbf{R}$ satisfy that $F_{\xi\xi}(x,u,\xi)$ is continuous and, for some $p \ge 2$,*

$$\begin{aligned} |F(x,u,\xi)| &\le L(1+|\xi|^p), \\ |F(x,u,\xi) - F(y,v,\xi)| &\le L(1+|\xi|^p)\,\omega(|x-y|^p + |u-v|^p), \end{aligned}$$

where $0 \leq \omega(t) \leq t^\sigma$, $0 < \sigma < 1/p$ *and* ω *is bounded, concave and increasing. Assume there exist constants* $\gamma > 0$, C_0 *such that for all* (x_0, u_0, ξ_0)

$$\int_B F(x_0, u_0, \xi_0 + D\phi)\,dx \geq \int_B \Big[F(x_0, u_0, \xi_0) + \gamma\big(|D\phi|^2 + |D\phi|^p\big)\Big]\,dx, \tag{5.56}$$

$$\int_B F(x_0, u_0, D\phi(x))\,dx \geq \int_B \big(C_0 + \gamma\,|D\phi(x)|^p\big)\,dx \tag{5.57}$$

for all balls B *and* $\phi \in C_0^\infty(B; \mathbf{R}^N)$. *Let* $u \in W^{1,p}_{loc}(\Omega; \mathbf{R}^N)$ *be a local minimizer of the functional* I *defined by* F *as above. Then there exists an open set* Ω_0 *of* Ω *such that* $|\Omega \setminus \Omega_0| = 0$ *and* $u \in C^{1,\mu}_{loc}(\Omega_0; \mathbf{R}^N)$ *for some* $0 < \mu < 1$.

Bibliography

[1] Acerbi, E. and N. Fusco, *Semicontinuity problems in the calculus of variations,* Arch. Rational Mech. Anal., **86** (1984), 125–145.

[2] Acerbi, E. and N. Fusco, *A regularity theorem for minimizers of quasiconvex integrals,* Arch. Rational Mech. Anal., **99** (1987), 261–281.

[3] Adams, R. A., "Sobolev Spaces," Academic Press, 1975.

[4] Ball, J. M., *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal., **63** (1977), 337–403.

[5] Ball, J. M., J. C. Currie and P. J. Olver, *Null Lagrangians, weak continuity, and variational problems of arbitrary order,* J. Funct. Anal., **41** (1981), 135–174.

[6] Ball, J. M. and R. D. James, *Proposed experimental tests of a theory of fine microstructures and the two well problem,* Phil. Trans. Roy. Soc. London, A **338** (1992), 389–450.

[7] Ball, J. M. and F. Murat, $W^{1,p}$*-Quasiconvexity and variational problems for multiple integrals,* J. Funct. Anal., **58**, (1984), 225–253.

[8] Campanato, S., *Proprietá di Hölderianitá di alcune classi die funzioni,* Ann. SNS Pisa, **17** (1963), 175-188.

[9] Campanato, S., *Equazioni ellittiche del secondo ordine e spazi* $\mathcal{L}^{2,\lambda}$, Ann. Mat. Pura Appl., **69** (1965), 321–380.

[10] Dacorogna, B., "Direct Methods in the Calculus of Variations," Springer, New York, 1989.

[11] Dacorogna, B. and P. Marcellini, *A counterexample in the vectorial calculus of variations,* in "Material Instabilities in Continuum Mechanics," (J. M. Ball ed.), Oxford University Press, 1988.

[12] De Giorgi, E., *Sulla differenziabilitá e l'analiticitá delle estremali degli integrali multipli regolari,* Mem. Accad. Sci. Torino cl. Sci. Fis. Mat. Nat. (3), **3** (1957), 25-43.

[13] De Giorgi, E., *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico,* Boll. UMI, **4** (1968), 135–137.

[14] Evans, L. C., *Quasiconvexity and partial regularity in the calculus of variations,* Arch. Rational Mech. Anal., **95** (1986), 227–252.

[15] Evans, L. C., "Partial Differential Equations," AMS-GSM, Vol. **19**, 1998.

[16] Fusco, N. and J. Hutchinson, $C^{1,\alpha}$ *partial regularity of functions minimising quasiconvex integrals,* Manuscripta Math., **54** (1985), 121–143.

[17] Gehring, F. W., *The* L^p*-integrability of the partial derivatives of a quasiconformal mapping,* Acta Math., **130** (1973), 265–277.

[18] Giaquinta, M., "Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems," Princeton University Press, 1983.

[19] Giaquinta, M., "Introduction to Regularity Theory for Nonlinear Elliptic Systems," Birkhäuser, 1993.

[20] Giaquinta, M. and E. Giusti, *On the regularity of the minima of certain variational integrals,* Acta Math., **148** (1982), 31–46.

[21] Giaquinta, M. and E. Giusti, *Quasi-minima,* Ann. Inst. H. Poincaré, Analyse non linéaire, **1** (1984), 79–104.

[22] Giaquinta, M. and G. Modica, *Regularity results for some classes of higher order nonlinear elliptic systems,* J. Reine Angew. Math., **311** (1979), 145–169.

[23] Giaquinta, M. and G. Modica, *Partial regularity of minimizers of quasiconvex integrals,* Ann. Inst. H. Poincaré, Analyse Non Linéaire, **3** (1986), 185–208.

[24] Giusti, E. and M. Miranda, *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale del calcolo delle variazioni,* Boll. UMI, **2** (1968), 1–8.

[25] Gilbarg, D. and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order," 2nd ed., Springer-Verlag, Berlin, 1984.

[26] Hörmander, L., "The Analysis of Linear Partial Differential Operators," Vol. 1-4, Springer, New York, 1983-85.

[27] John, F. and L. Nirenberg, *On functions of bounded mean oscillation,* Comm. Pure Appl. Math., **14** (1961), 415–426.

[28] Le Dret, H., *An example of H^1-unboundedness of solutions to strongly elliptic systems of partial differential equations in a laminated geometry,* Proc. Roy. Soc. Edinburgh Sect. A, **105** (1987), 77–82.

[29] Meyers, N., *Quasi-convexity and lower semicontinuity of multiple variational integrals of any order,* Trans. Amer. Math. Soc., **119** (1965), 125–149.

[30] Meyers, N. and A. Elcrat, *Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions,* Duke Math. J., **42** (1975), 121–136.

[31] Morrey, C. B., Jr., *Quasiconvexity and the lower semicontinuity of multiple integrals,* Pacific J. Math., **2** (1952), 25–53.

[32] Morrey, C. B., Jr., "Multiple Integrals in the Calculus of Variations," Springer, New York, 1966.

[33] Moser, J., *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations,* Comm. Pure Appl. Math., **13** (1960), 457–468.

[34] Müller, S., *Higher integrability of determinants and weak convergence in L^1*, J. Reine Angew. Math., **412** (1990), 20–34.

[35] Müller, S., "Variational Models for Microstructure and Phase Transitions," Lecture Notes, Max-Planck-Inst. Math. Sc., Leipzig, 1998.

[36] Müller, S. and V. Šverák, *Unexpected solutions of first and second order partial differential equations,* Doc. Math. Extra Volume ICM 1998, II, 691-702.

[37] Nash, J., *Continuity of solutions of parabolic and elliptic equations,* Amer. J. Math., **8** (1958), 931–954.

[38] Nečas, J., *Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity,* in "Theory of Non Linear Operators," 1977.

[39] Nirenberg, L., *On elliptic partial differential equations,* Ann. S.N.S. Pisa, **13** (1959), 115–162.

[40] Schoen, R. and K. Uhlenbeck, *A regularity theory for harmonic maps,* J. Differential Geom., **17** (1982), 307–336.

[41] Simon, L., "Theorems on Regularity and Singularity of Energy Minimizing Maps," Birkhäuser, 1996.

[42] Stein, E. M., "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, 1970.

[43] Stredulinsky, E. W., *Higher integrability from reverse Hölder inequalities,* Indiana Univ. Math. J., **29** (1980), 408–417.

[44] Šverák, V., *Rank-one convexity does not imply quasiconvexity,* Proc. Roy. Soc. Edinburgh Sect. A, **120** (1992), 185–189.

[45] Yan, B., *On rank-one convex and polyconvex conformal energy functions with slow growth,* Proc. Roy. Soc. Edinburgh Sect. A, **127** (1997), 651–663.

[46] Yosida, K., "Functional Analysis," 6th edition, Springer, New York, 1980.

[47] Young, L. C., "Lectures on the Calculus of Variations and Optimal Control Theory," Saunders, 1969 (reprint by Chelsea, 1980).