Chapter 6

Sequences and Series of Functions

6.1. Convergence of a Sequence of Functions

Pointwise Convergence.

Definition 6.1. Let, for each $n \in \mathbf{N}$, function $f_n\colon A \to \mathbf{R}$ be defined. If, for each $x \in A$, the sequence $(f_n(x))$ converges (to a limit $f(x)$); that is,

$$\lim f_n(x) = f(x) \quad \forall\ x \in A,$$

then we say that (f_n) **converges pointwise** to the limit function f on A. In this case, we write $f_n(x) \to f(x)$ or $f_n \to f$ pointwise on A, or $\lim_{n\to\infty} f_n(x) = f(x)$ to emphasize the limit is for $n \to \infty$.

The pointwise convergence means that, given each $x \in A$, $\forall\ \epsilon > 0$, $\exists\ N \in \mathbf{N}$ such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall\ n \geq N.$$

Note that the number N here depends on both x and ϵ.

Example 6.1. (i) Let $f_n(x) = \frac{x^2+nx}{n}$, $x \in \mathbf{R}$. For any given $x \in \mathbf{R}$, $f_n(x) = x + \frac{x^2}{n} \to x$ as $n \to \infty$. Hence $f_n(x) \to f(x) = x$ pointwise on $\mathbf{R}$.

(ii) Let $g_n(x) = x^n$ on $[0,1]$. Note that $g_n(1) = 1$ for all $n \in \mathbf{N}$; so $(g_n(1)) \to 1$. If $0 \leq x < 1$, then $(g_n(x)) = (x^n) \to 0$. Hence the pointwise limit function of $g_n(x)$ on $[0,1]$ is given by

$$g(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1. \end{cases}$$

Note that although each g_n is continuous on $[0,1]$, the pointwise limit function g is not continuous at $x = 1$.

(iii) Let $h_n(x) = x^{1+\frac{1}{2n-1}}$ on $[-1,1]$. Then $(h_n(x)) \to |x|$ pointwise on $[-1,1]$.

(iv) (Exercise 5(b)) Let

$$f_n(x) = \begin{cases} \frac{1}{|x|} & \frac{1}{n} \leq |x| \leq 5 \\ n^2|x| & |x| < \frac{1}{n}. \end{cases}$$

Then $f_n(x) \to \frac{1}{|x|}$ $(x \neq 0)$ and $f_n(0) \to 0$. The limit function on $[-5, 5]$ is unbounded.

(v) Let $h_n(x) = \cos(nx)$. Then $(h_n(x))$ converges if and only if $x = 2k\pi$ for $k = 0, \pm 1, \pm 2, \cdots$. **Proof:** Let $a_n = \cos(nx) \to a$. Then $a_{2n} = \cos(2nx) = 2\cos^2(nx) - 1 = 2a_n^2 - 1 \to 2a^2 - 1 = a$; so $a \neq 0$. From the formula $\cos[(n+1)x] + \cos[(n-1)x] = 2\cos(nx)\cos x$, it follows that $a_{n+1} + a_{n-1} = 2a_n \cos x$ and hence $2a = 2a\cos x$. Since $a \neq 0$, we must have $\cos x = 1$; that is, $x = 2k\pi$ with $k = 0, \pm 1, \pm 2, \cdots$. For all such x's, $h_n(x) = 1$ and hence $(h_n(x)) \to 1$ only for such x's.

Uniform Convergence. Assume that f_n is continuous on a set A for each $n \in \mathbf{N}$ and $(f_n(x)) \to f(x)$ pointwise on A. When can we conclude that f is continuous on A or at some point $c \in A$?

Given $c \in A$, and $\epsilon > 0$, we write by the triangle inequality,

$$|f(x) - f(c)| \leq |f_n(x) - f(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|.$$

The last term on the right-hand side can be made $< \epsilon/3$ by choosing n large enough by the convergence of $(f_n(c)) \to f(c)$ and, once n is fixed, by choosing x close to c from the continuity of f_n at c the second term can be made $< \epsilon/3$ as well. The first term, however, is the most troublesome since, when n is chosen, the term $|f_n(x) - f(x)|$ may not be small as $x \to c$. To deal with this term, we need the following definition.

Definition 6.2. (**Uniform Convergence**) Let $f_n \colon A \to \mathbf{R}$ and $f \colon A \to \mathbf{R}$ be given functions. We say that the sequence (f_n) **converges uniformly on** A to function f if, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $x \in A$ and $n \geq N$ it follows that $|f_n(x) - f(x)| < \epsilon$.

Remark 6.3. 1. For pointwise convergence, given $\epsilon > 0$, the number N is to be found after $x \in A$ is given (so N depends on x), while for the uniform convergence, the number N is to be found that works for *every* $x \in A$ (so N is independent of x).

2. From the definitions, if (f_n) converges uniformly to f on A then $(f_n(x)) \to f(x)$ pointwise on A. Therefore, the uniform limit function must be the pointwise limit function.

Example 6.2. Consider $f_n(x) = \frac{x^2+nx}{n}$ and $f(x) = x$ on $\mathbf{R}$. We know $(f_n(x)) \to f(x)$ pointwise on $\mathbf{R}$. However, given $\epsilon > 0$, can we find an $N \in \mathbf{N}$ such that

$$|f_n(x) - f(x)| = \frac{x^2}{n} < \epsilon \quad \forall\, n \geq N,\ x \in \mathbf{R}?$$

If such an N existed, we would take $x = \sqrt{N}$ and $n = N$ to obtain $1 < \epsilon$, a contradiction if our ϵ is chosen < 1. Therefore, the sequence (f_n) does not converge uniformly to f on $\mathbf{R}$.

In general, by negating the definition, (f_n) does not converge uniformly on A to a function f if and only if there exists an $\epsilon_0 > 0$ and sequences (n_k) in $\mathbf{N}$ with $n_k \geq k$ and (x_k) in A such that

$$|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0.$$

Example 6.3. We also consider $f_n(x) = \frac{x^2+nx}{n}$ and $f(x) = x$ but on bounded interval $[-b, b]$. Since

$$|f_n(x) - f(x)| = \frac{x^2}{n} \leq \frac{b^2}{n}$$

holds for all $x \in [-b, b]$, in order to make this quantity $< \epsilon$ for all such x', we can choose $N \in \mathbf{N}$ such that $N > \frac{b^2}{\epsilon}$. Then, for all $n \geq N$ and $x \in [-b, b]$, it does follow that

$|f_n(x) - f(x)| \leq \frac{b^2}{n} < \epsilon$. Hence (f_n) uniformly converges to f on $[-b, b]$ (but not on whole $\mathbf{R}$, as seen above).

Cauchy Criterion for Uniform Convergence. Like other Cauchy criteria, this criterion gives the necessary and sufficient condition for a sequence of functions to converge uniformly *without* knowing the limit function.

Theorem 6.1 (**Cauchy Criterion for Uniform Convergence**). *Let $f_n \colon A \to \mathbf{R}$ for each $n \in \mathbf{N}$. Then (f_n) converges uniformly on A if and only if, for each $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that*

$$|f_n(x) - f_m(x)| < \epsilon \quad \forall\, x \in A, \quad \forall\, n,\ m \geq N. \tag{6.1}$$

Proof. First assume (f_n) converges uniformly on A to a limit function f. Then, for each $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that

$$|f_n(x) - f(x)| < \epsilon/2 \quad \forall\, x \in A, \quad \forall\, n \geq N.$$

Hence, whenever $n,\ m \geq N$ and $x \in A$, it follows that

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon,$$

proving (6.1).

We now assume condition (6.1) holds. Then, given each $x \in A$, the sequence $(f_n(x))$ is Cauchy; hence it converges to a limit $f(x) \in \mathbf{R}$. This defines a function $f \colon A \to \mathbf{R}$, which is the pointwise convergence limit of $f_n(x)$. The condition, used with ϵ replaced by $\epsilon/2$, implies that there exists an $N \in \mathbf{N}$ such that for all $x \in A$ and $n, m \geq N$ we have $|f_n(x) - f_m(x)| < \epsilon/2$. Hence

$$-\epsilon/2 < f_n(x) - f_m(x) < \epsilon/2 \quad \forall\, n,\ m \geq N.$$

We now fix $x \in A$ and $n \geq N$ and take the limit of sequence $(f_m(x))$ in this inequality. Use the order limit theorem, we have

$$-\epsilon/2 \leq f_n(x) - f(x) \leq \epsilon/2 \quad \forall\, n \geq N.$$

Hence we have proved that $|f_n(x) - f(x)| \leq \epsilon/2 < \epsilon$ holds for all $x \in A$ and $n \geq N$. This is nothing but the definition of the uniform convergence of (f_n) to f on A. □

Theorem 6.2 (**Continuity of uniform limit function**). *Let (f_n) and f be functions on A and let (f_n) converge uniformly to f on A. Assume for each $n \in \mathbf{N}$ the function f_n is continuous at a point $c \in A$. Then f is continuous at c as well.*

Proof. Given $\epsilon > 0$, by the uniform convergence of (f_n) to f on A, there exists an $N \in \mathbf{N}$ such that

$$|f_n(x) - f(x)| < \epsilon/3 \quad \forall\, x \in A, \quad \forall\, n \geq N.$$

We fix $n = N$ and consider function f_N, which is continuous at $c \in A$; hence, there exists a number $\delta > 0$ such that

$$|f_N(x) - f_N(c)| < \epsilon/3 \quad \forall\, x \in A, \quad |x - c| < \delta.$$

Then, whenever $x \in A$ and $|x - c| < \delta$, it follows that

$$\begin{aligned} |f(x) - f(c)| &\leq |f_N(x) - f(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

This proves the continuity of f at $c \in A$. □

6.2. Uniform Convergence and Differentiation

Theorem 6.3. *Let $f_n(x) \to f(x)$ pointwise on $[a,b]$ and assume each f_n is differentiable on an open interval containing $[a,b]$. If f_n' converges uniformly on $[a,b]$ to a function g, then f is differentiable and $f' = g$ on $[a,b]$.*

Proof. Let $c \in [a,b]$ be given. We want to show

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = g(c).$$

Note that

$$\begin{aligned}\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| &\le \left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right| \\ &\quad + \left|\frac{f_n(x)-f_n(c)}{x-c} - f_n'(c)\right| + |f_n'(c) - g(c)|.\end{aligned} \tag{6.2}$$

Given $\epsilon > 0$, we want to find a $\delta > 0$ such that each of the three terms is $< \epsilon/3$ for all $|x - c| < \delta$ and $x \in [a,b]$. The third term is independent of x, but depends on n. Since $f_n'(c) \to g(c)$, there exists an $N_1 \in \mathbf{N}$ such that

$$|f_n'(c) - g(c)| < \epsilon/3 \quad \forall\, n \ge N_1.$$

The second term for fixed n can be easily handled. The first term requires the most work. Let's do it now. Note that by applying the MVT to function $f_m(x) - f_n(x)$ we have, for $x \neq c$,

$$\begin{aligned}\frac{f_m(x)-f_m(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c} &= \frac{(f_m(x)-f_n(x)) - (f_m(c)-f_n(c))}{x-c} \\ &= f_m'(\alpha) - f_n'(\alpha)\end{aligned} \tag{6.3}$$

for some α between x and c (such a number α depends on many things: m, n, x, c). However, since (f_n') converges uniformly on $[a,b]$, by the Cauchy Criterion for Uniform Convergence, there exists an $N_2 \in \mathbf{N}$ such that

$$|f_n'(x) - f_m'(x)| < \epsilon/4 \quad \forall\, x \in [a,b], \quad \forall\, m, n \ge N_2.$$

We now use (6.3) to conclude that

$$\left|\frac{f_m(x)-f_m(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right| = |f_m'(\alpha) - f_n'(\alpha)| < \epsilon/4$$

for all $x \neq c$ and all n, $m \ge N_2$. Take the limit as $m \to \infty$ and use the order limit theorem, and we have

$$\left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right| \le \epsilon/4 < \epsilon/3 \quad \forall\, x \neq c, \quad \forall\, n \ge N_2.$$

Now let $N = \max\{N_1, N_2\}$. Since f_N is differentiable at c with derivative $f_N'(c)$, there exists a $\delta > 0$ such that

$$\left|\frac{f_N(x)-f_N(c)}{x-c} - f_N'(c)\right| < \epsilon/3 \quad \forall\, x \in [a,b],\ |x - c| < \delta.$$

For this $\delta > 0$, whenever $x \in [a, b]$ and $|x - c| < \delta$, it follows from (6.2) with $n = N$ that

$$\begin{aligned}\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| &\le \left|\frac{f(x)-f(c)}{x-c} - \frac{f_N(x)-f_N(c)}{x-c}\right| \\ &\quad + \left|\frac{f_N(x)-f_N(c)}{x-c} - f_N'(c)\right| + |f_N'(c) - g(c)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

So $f'(c) = g(c)$. □

We also have a stronger result under a weaker assumption.

Theorem 6.4. *Let (f_n) be a sequence of differentiable functions defined on $[a, b]$, and assume (f_n') converges uniformly to a function g on $[a, b]$. If there exists a point $c \in [a, b]$ for which the sequence $(f_n(c))$ converges, then (f_n) must converge uniformly on $[a, b]$ and the limit function f is differentiable on $[a, b]$ and satisfies $f'(x) = g(x)$ on $[a, b]$.*

Proof. Let us prove the uniform convergence of (f_n) on $[a, b]$; then the differentiation results follow from the previous theorem. We use the Cauchy criterion to show that (f_n) converges uniformly on $[a, b]$.
Given any $\epsilon > 0$, by the uniform convergence of (f_n'), there exists an $N_1 \in \mathbf{N}$ such that

$$|f_n'(x) - f_m'(x)| < \epsilon/2(b-a) \quad \forall\, x \in [a, b], \quad \forall\, n,\, m \ge N_1.$$

Also, by the convergence of sequence $(f_n(c))$, there exists an $N_2 \in \mathbf{N}$ such that

$$|f_n(c) - f_m(c)| < \epsilon/2 \quad \forall\, n,\, m \ge N_2.$$

We use the MVT to function $f_n(x) - f_m(x)$ to obtain

$$(f_n(x) - f_m(x)) - (f_n(c) - f_m(c)) = (f_n'(\alpha) - f_m'(\alpha))(x - c)$$

for some α between x and c and hence $\alpha \in [a, b]$. Therefore, whenever $n,\, m \ge N_1$,

$$\begin{aligned}|(f_n(x) - f_m(x)) - (f_n(c) - f_m(c))| &= |(f_n'(\alpha) - f_m'(\alpha))||x - c| \\ &\le |f_n'(\alpha) - f_m'(\alpha)|(b-a) < \frac{\epsilon}{2(b-a)}(b-a) = \epsilon/2.\end{aligned}$$

Finally, let $N = \max\{N_1, N_2\}$. Then, whenever $n,\, m \ge N$ and $x \in [a, b]$, it follows that

$$\begin{aligned}|f_n(x) - f_m(x)| &\le |(f_n(x) - f_m(x)) - (f_n(c) - f_m(c))| + |(f_n(c) - f_m(c))| \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

Hence, by the Cauchy Criterion, (f_n) converges uniformly on $[a, b]$ to some function f. This proves the theorem. □

Example 6.4. Let $g_n(x) = \frac{x^n}{n}$ on $x \in [0, 1]$. Again (g_n) uniformly converges to $g(x) = 0$, but $g_n'(x) = x^{n-1}$ only pointwise converges to $h(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$, not uniformly.

Example 6.5. Let $h_n(x) = (-1)^n \frac{\sin(nx)}{n}$ on $\mathbf{R}$. Then (h_n) converges uniformly to $h(x) = 0$ on $\mathbf{R}$. However $h_n'(x) = (-1)^n \cos(nx)$ does not converge for any $x \in \mathbf{R}$.

The previous examples indicate that for a sequence of functions (f_n) to converge uniformly it is not necessary that (f_n') converge uniformly (even pointwise).

6.3. Series of Functions

A series of functions is an infinite series of the form

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots ,$$

where $f_1, f_2, \cdots$ are functions defined on a common set $A \subseteq \mathbf{R}$. Let the sequence of partial sums be defined by

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \quad n \in \mathbf{N}.$$

Definition 6.4. The series of functions $\sum_{n=1}^{\infty} f_n(x)$ is said to **converge pointwise on** A if the sequence $(s_n(x))$ converges pointwise on A; if $(s_n(x)) \to f(x)$ pointwise on A, then we write

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \quad x \in A.$$

If (s_n) converges uniformly on A then we say that the series $\sum_{n=1}^{\infty} f_n(x)$ **converges uniformly on** A.

Theorem 6.5. *Let f_n be continuous functions defined on a set $A \subseteq \mathbf{R}$, and assume $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A to a function f. Then f is continuous on A.*

Proof. Easy consequence of the case for sequence. □

Theorem 6.6 (Term-by-Term Differentiation Theorem). *Let (f_n) be a sequence of differentiable functions defined on $[a, b]$, and assume $\sum_{n=1}^{\infty} f_n'$ converges uniformly to a function g on $[a, b]$. If there exists a point $c \in [a, b]$ for which the series $\sum_{n=1}^{\infty} f_n(c)$ converges, then the series $\sum_{n=1}^{\infty} f_n$ must converge uniformly to a differentiable function f on $[a, b]$ satisfying $f'(x) = g(x)$ on $[a, b]$. In other words, if $\sum_{n=1}^{\infty} f_n(c)$ converges at one point $c \in [a, b]$, and $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[a, b]$, then*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

is well-defined for all $x \in [a, b]$ and is differentiable with

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \quad \forall\, x \in [a, b].$$

Proof. Use Theorem 6.4. □

Theorem 6.7 (Cauchy Criterion for Uniform Convergence of Series). *A series $\sum_{n=1}^{\infty} f_n$ converges uniformly on a set $A \subseteq \mathbf{R}$ if and only if for every $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that for all $n > m \geq N$,*

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \epsilon \quad \forall\, x \in A.$$

Proof. Use Theorem 6.1. □

Corollary 6.8 (Weierstrass M-Test). *For each $n \in \mathbf{N}$, let f_n be a function defined on A satisfying $|f_n(x)| \leq M_n$ for all $x \in A$, where $M_n > 0$ is a real number. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.*

Proof. Use the previous Cauchy's Criterion. □

Example 6.6. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^3}.$$

(a) Show f and is differentiable and f' is continuous on $\mathbf{R}$.

(b) Can we determine if f is twice-differentiable?

Example 6.7. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots .$$

Then $f(x)$ converges on $[0,1)$, but $f(1) = \infty$. This is a special case of the power series to be learned next.

6.4. Power Series

A **power series** is an infinite series of power functions:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots .$$

The point c is called the **center** of the power series and sequence (a_n) is called the **sequence of coefficients** of the power series.

The power series also converges at its center c. In what follows, we always assume $c = 0$.

Theorem 6.9. *If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \neq 0$, then it converges absolutely for all x satisfying $|x| < |x_0|$.*

Proof. If $\sum_{n=0}^{\infty} a_n x_0^n$ converges, then the sequence $(a_n x_0^n)$ is bounded. Let $M > 0$ satisfy $|a_n x_0^n| \le M$ for all $n \in \mathbf{N}$. Assume $x \in \mathbf{R}$ satisfies $|x| < |x_0|$. Then

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \le M \left|\frac{x}{x_0}\right|^n .$$

Since $|x/x_0| < 1$, by comparison with the geometric series, it follows easily that $\sum_{n=0}^{\infty} |a_n x^n|$ converges; hence $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. □

Radius of Convergence. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Let

$$S = \{x \in \mathbf{R} \mid \ \textstyle\sum_{n=0}^{\infty} a_n x^n \text{ converges}\}.$$

There are three cases for this set S:

(a) $S = \{0\}$. In this case, the power series converges only at center 0.

(b) $S = \mathbf{R}$. In this case, the power series converges at every $x \in \mathbf{R}$.

(c) $S \neq \{0\}$ and $S \neq \mathbf{R}$.

In Case (c), there exist two points $x_0, x_1 \in \mathbf{R}$ such that $x_0 \neq 0$, $x_0 \in S$ and $x_1 \notin S$. By Theorem 6.9, for all $x \in \mathbf{R}$ with $|x| > |x_1|$, the power series diverges at x; otherwise, if it converges at some x with $|x| > |x_1|$ then by the theorem, $x_1 \in S$. Therefore, the set S is included in the interval $[-|x_1|, |x_1|]$ and hence S is a nonempty bounded set in $\mathbf{R}$. By the **AoC**, let

$$R = \sup S.$$

This R satisfies $|x_0| \le R \le |x_1|$; hence $R \in (0, \infty)$.

Lemma 6.10. *In Case (c), the power series converges at all x with $|x| < R$ and diverges at all x with $|x| > R$.*

Proof. 1. If $|x| < R = \sup S$, then there exists $x' \in S$ such that $x' > |x|$. Since the power series converges at x' and $|x| < |x'| = x'$, by Theorem 6.9, the power series converges at x.

2. If $|x| > R$, we show that the power series diverges at x. If not, assume the power series converges at x. Let $y = \frac{|x|+R}{2}$. Then $R < y < |x|$. Since the power series converges at x and $y < |x|$, by Theorem 6.9 again, the power series converges at y; hence $y \in S$. But $R = \sup S$; so $y \le R$, a contradiction. $\square$

Definition 6.5 (Radius of Convergence). Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with the set S defined as above. The **radius of convergence** of this power series is the number $R \in [0,\infty]$ defined as follows: $R = 0$ in Case (a); $R = \infty$ in Case (b); $R = \sup S \in (0,\infty)$ in Case (c).

Remark 6.6. In fact, there is a formula for the radius of convergence given by

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}}$$

in all cases. However, this formula will not be needed for our lecture and the homework.

Example 6.8. For the power series

$$\sum_{n=0}^{\infty} n!\, x^n,$$

the nth-term is $a_n = n!x^n$. If $x \neq 0$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = (n+1)|x| \to \infty,$$

and hence $|a_n| \to \infty$. Therefore, $\sum a_n$ diverges. So the given power series only converges at $x = 0$ and diverges for all $x \neq 0$. The radius of convergence for this power series is $R = 0$.

Example 6.9. For power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

the nth-term is $a_n = \frac{x^n}{n!}$. Hence

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0$$

for all $x \neq 0$; thus, the power series converges absolutely for all $x \neq 0$, and certainly it always converges for $x = 0$. Therefore the given series converges absolutely for all x. The radius of convergence for this power series is $R = \infty$.

Example 6.10. In Case (c) above, the set S could only be one of the intervals

$$[-R,R],\ \ [-R,R),\ \ (-R,R],\ \ (-R,R),$$

and all cases can happen.

(a) The power series $\sum_{n=1}^{\infty} x^n/n^2$ has the radius of convergence $R = 1$ and converges exactly on $[-1,1]$.

(b) The power series $\sum_{n=1}^{\infty} x^n/n$ has the radius of convergence $R = 1$ and converges exactly on $[-1,1)$.

(c) The power series $\sum_{n=1}^{\infty}(-x)^n/n$ has the radius of convergence $R = 1$ and converges exactly on $(-1, 1]$.

(d) The power series $\sum_{n=1}^{\infty} x^{2n}/n$ has the radius of convergence $R = 1$ and converges exactly on $(-1, 1)$.

Theorem 6.11. *If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point $x_0 \neq 0$, then it converges uniformly on the interval $[-|x_0|, |x_0|]$.*

Proof. Use Weierstrass's M-Test. □

Abel's Theorem.

Lemma 6.12 (Abel's Lemma). *Let sequence (b_n) satisfy $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$ and sequence (a_n) satisfy*

$$|a_1 + a_2 + \cdots + a_n| \leq A, \quad \forall\, n \in \mathbf{N},$$

for a constant $A > 0$. Then, for all $n \in \mathbf{N}$,

$$|a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n| \leq A b_1.$$

Proof. Let $s_0 = 0$ and $s_k = a_1 + a_2 + \cdots + a_k$ for $k \in \mathbf{N}$. Then $a_k = s_k - s_{k-1}$ for all $k \in \mathbf{N}$. Hence

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (s_k - s_{k-1}) b_k = \sum_{k=1}^{n} s_k b_k - \sum_{k=1}^{n} s_{k-1} b_k$$

$$= \sum_{k=1}^{n} s_k b_k - \sum_{k=0}^{n-1} s_k b_{k+1} = \sum_{k=1}^{n-1} s_k b_k + s_n b_n - \sum_{k=1}^{n-1} s_k b_{k+1} = \sum_{k=1}^{n-1} s_k (b_k - b_{k+1}) + s_n b_n.$$

Since $|s_k| \leq A$ and $b_k - b_{k+1} \geq 0$, it follows that

$$\left| \sum_{k=1}^{n} a_k b_k \right| \leq \left| \sum_{k=1}^{n-1} s_k (b_k - b_{k+1}) + s_n b_n \right|$$

$$\leq \sum_{k=1}^{n-1} |s_k| (b_k - b_{k+1}) + |s_n| b_n \leq A \sum_{k=1}^{n-1} (b_k - b_{k+1}) + A b_n = A b_1.$$

□

Theorem 6.13 (Abel's Theorem). *If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = R > 0$, then it converges uniformly on the interval $[0, R]$.*

If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = -R < 0$, then it converges uniformly on the interval $[-R, 0]$.

Proof. Let us assume $\sum_{n=0}^{\infty} a_n R^n$ converges. We want to use the Cauchy criterion to show that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, R]$. Given $\epsilon > 0$, from the convergence of $\sum_{n=0}^{\infty} a_n R^n$, there exists a number $N \in \mathbf{N}$ such that

$$|a_{m+1} R^{m+1} + a_{m+2} R^{m+2} + \cdots + a_n R^n| < \epsilon/2, \quad \forall\, n > m \geq N.$$

Let $x \in [0, R]$. For any fixed $m \geq N$, consider sequence $B_n = (\frac{x}{R})^{m+n}$ and sequence $A_n = a_{m+n} R^{m+n}$ with $n \in \mathbf{N}$. Then the sequences (B_n) and (A_n) satisfy the conditions in Abel's Lemma above with $A = \epsilon/2$. Hence

$$\left| a_{m+1} R^{m+1} (\frac{x}{R})^{m+1} + a_{m+2} R^{m+2} (\frac{x}{R})^{m+2} + \cdots + a_n R^n (\frac{x}{R})^n \right| \leq (\frac{x}{R})^{m+1} \epsilon/2 < \epsilon.$$

Therefore,

$$|a_{m+1}x^{m+1} + a_{m+2}x^{m+2} + \cdots + a_n x^n| < \epsilon, \quad \forall\, x \in [0, R], \quad n > m \geq N.$$

This proves the uniform convergence of $\sum_{n=0}^{\infty} a_n x^n$ on $[0, R]$. □

The Success of Power Series. Given a power series $\sum_{n=0}^{\infty} a_n x^n$, the **differentiated series** $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is also a power series.

Theorem 6.14. *If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges for all $x \in (-R, R)$ as well.*

Proof. Let t be such that $|x| < t < R$. The conclusion follows from the identity

$$|n a_n x^{n-1}| = \frac{1}{t} \left(n \left| \frac{x^{n-1}}{t^{n-1}} \right| \right) |a_n t^n|.$$

Since $r = |x|/t < 1$, it follows that $(n r^{n-1}) \to 0$. This can be shown by the convergence of series $\sum n r^n$ using the ratio-test. □

Theorem 6.15. *If a power series converges pointwise on a set A, then it converges uniformly on any compact subset of A.*

Proof. When A contains one of R or $-R$, where R is the radius of convergence, we need the Abel's theorem. Other cases, the theorem can be proved without it. □

Theorem 6.16. *Assume the power series*

$$g(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges on a set A. Then, g is continuous on A and differentiable on any open interval $(-R, R) \subseteq A$ with the derivative given by the term-by-term differentiation

$$g'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad x \in (-R, R).$$

Moreover, g is infinitely differentiable on $(-R, R)$, and the higher-order derivatives can be obtained via the term-by-term differentiation of the previous differentiated power series.

Proof. Only for the continuity of g at possibly the end-points of the interval of convergence is the previous theorem needed. All other conclusions can be proved without using the Abel's theorem. □

Example 6.11. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (x \in \mathbf{R}).$$

Then, from the term-by-term differentiation,

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = f(x).$$

Also $f(0) = 1$. Let $g(x) = f(x)e^{-x}$. Then $g(0) = 1$ and $g'(x) = f'(x)e^{-x} - f(x)e^{-x} = 0$ for all $x \in \mathbf{R}$. So $g(x) \equiv 1$; that is, $f(x) = e^x$ for all $x \in \mathbf{R}$. In particular, with $x = 1$,

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e = 2.718281828459045\ldots.$$

Example 6.12. (Exercise 6.5.1.) Note that the power series

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

has radius of convergence $R = 1$ and converges **exactly** for $x \in (-1, 1]$. We can prove this by just using the fact $g(-1)$ diverges and $g(1)$ converges. (Explain why.) Hence g is continuous on $(-1, 1]$ and is differentiable on $(-1, 1)$. The derivative is given by

$$g'(x) = \sum_{n=1}^{\infty} (-x)^{n-1}, \quad -1 < x < 1.$$

So, by the geometric series, $g'(x) = \frac{1}{1+x}$ for $-1 < x < 1$. But $g(0) = 0$. Hence

$$g(x) = \int_0^x g'(t)\, dt = \int_0^x \frac{1}{1+t}\, dt$$
$$= [\ln|1+t|]_0^x = \ln|1+x| = \ln(1+x);$$

that is,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \ln(1+x), \quad -1 < x < 1.$$

This identity also holds when $x = 1$ since g is continuous at $x = 1$. So we have that the value of the **alternative harmonic series** is given by

$$g(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

6.5. Taylor Series

The power series defines an infinitely differentiable function on its open interval of convergence. Given an infinitely differentiable function on an open interval, can we express the function as a power series centered at an interior point of the interval? Such a series is called the Taylor series of the function centered at the given point.

Let f be an infinitely differentiable function defined on $(-R, R)$. Suppose f equals a power series centered at 0 as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots, \quad x \in (-R, R).$$

Then $a_0 = f(0)$. By the term-by-term differentiation, we have, for all $k = 1, 2, 3, \cdots$,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n x^{n-k}, \quad x \in (-R, R).$$

Hence $f^{(k)}(0) = k! a_k$. Therefore, with $0! = 1$, we have

$$a_k = \frac{f^{(k)}(0)}{k!}, \quad k = 0, 1, 2, \cdots.$$

This shows that if a function f equals a power series near 0 then the coefficients a_k of the power series must be given by the above formula.

Definition 6.7. Let f be an infinitely differentiable function near a point $a \in \mathbf{R}$. Then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{near } a \tag{6.4}$$

is called the **Taylor series** of f about a. When $a = 0$, the Taylor series iis also called the **Maclaurin series**; but we will still call it the Taylor series.

Lagrange's Remainder Theorem. Does the Taylor series of f converge to f near a? We assume $a = 0$. The following result is useful to answer such a question.

Theorem 6.17 (Lagrange's Remainder Theorem)**.** *Let f be infinitely differentiable on $(-R, R)$ and define*

$$S_N(x) = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!}x^k \quad \forall x \in (-R, R).$$

Then, given any $0 < |x| < R$, there exists a number c with $|c| < |x|$ such that the **error term** *$E_N(x) = f(x) - S_N(x)$ satisfies*

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}.$$

Proof. Without loss of generality, assume $0 < x < R$. Consider function $E_N(t) = f(t) - S_N(t)$. Then

$$E_N^{(n)}(0) = 0 \quad \text{for all } n = 0, 1, 2, \cdots, N,$$

and $E_N^{(N+1)}(t) = f^{(N+1)}(t)$ for all $t \in (-R, R)$. So, by the **Generalized Mean-Value Theorem** repeatedly, we have $x > x_1 > x_2 > \cdots > x_{N+1} > 0$ such that

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N'(x_1)}{(N+1)x_1^N} = \frac{E_N''(x_2)}{(N+1)Nx_2^{N-1}} = \cdots = \frac{E_N^{(N+1)}(x_{N+1})}{(N+1)N\cdots 2\cdot 1} = \frac{f^{(N+1)}(c)}{(N+1)!},$$

and this proves the theorem with $c = x_{N+1}$. □

Example 6.13. Show

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} \quad \forall\, x \in \mathbf{R}.$$

Proof. Let $f(x) = \sin x$. Then

$$f^{(2n)}(x) = (-1)^n \sin x, \quad f^{(2n+1)}(x) = (-1)^n \cos x$$

for all $n = 0, 1, 2, \ldots$. Therefore the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

is exactly the Taylor series of $\sin x$ about 0. Let $S_N(x)$ be the partial sum up to power x^N. Then

$$S_{2k}(x) = S_{2k+1}(x) = \sum_{n=0}^{k} \frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

By Lagrange's Remainder Theorem,

$$|\sin x - S_{2k+1}(x)| = |E_{2k+1}(x)| = \left| \frac{f^{(2k+2)}(c)}{(2k+2)!} x^{2k+2} \right| \leq \frac{1}{(2k+2)!} |x|^{2k+2} \to 0$$

as $k \to \infty$ for all $x \in \mathbf{R}$. This proves the convergence; moreover it gives the **error estimate**

$$\left| \sum_{n=0}^{k} \frac{(-1)^n}{(2n+1)!} x^{2n+1} - \sin x \right| \leq \frac{|x|^{2k+2}}{(2k+2)!}$$

for all $x \in \mathbf{R}$ and all $k = 0, 1, \cdots$. □

A Counterexample. The Taylor series of f may not be equal to f near a.

Example 6.14. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Then f is infinitely differentiable at all $x \neq 0$. It is a good exercise to show that f has all orders of derivatives at 0 and $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \cdots$. Therefore, the Taylor series of f about 0 is identically zero; obviously $f(x) \neq 0$ whenever $x \neq 0$. This shows that f is not equal to its Taylor series near 0.