Chapter 5

Integrability on R

5.1. The Riemann Integral

Partition, Upper and Lower Sums.

Definition 5.1. Let $a, b \in \mathbf{R}$ and $a < b$.

(1) A **partition** P of interval $[a,b]$ is a set of (ordered) points $P = \{x_0, x_1, \dots, x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

(2) The **norm** $\|P\|$ of a partition P is the longest length of subintervals $[x_{j-1}, x_j]$, $j = 1, 2, \dots, n$; that is,

$$\|P\| = \max_{1 \le j \le n} |x_j - x_{j-1}|.$$

(3) A **refinement** of partition P is a partition Q such that $P \subseteq Q$. In this case, we also say that Q is **finer** than P.

Clearly, if P, Q are partitions of $[a,b]$ then $P \cup Q$ is also a partition of $[a,b]$ and is finer than both P and Q. And if Q is a refinement of P then $\|Q\| \le \|P\|$.

Example 5.1. **(The equal partitions.)** Let $a < b$ in $\mathbf{R}$ and $n \in N$. Let $\delta_n = (b-a)/n$ and $P = \{a + j\delta_n : j = 0, 1, \dots, n\}$. Then P is a partition of $[a,b]$ with all n subintervals having the same length $\delta_n = (b-a)/n$; so $\|P\| = \delta_n = (b-a)/n$.

Example 5.2. **(The dyadic partitions.)** For each $n \in \mathbf{N}$, $P_n = \{j/2^n : j = 0, 1, \dots, 2^n\}$ is a partition of interval $[0,1]$, and that P_m is finer than P_n when $m > n$ in $\mathbf{N}$.

Definition 5.2. Let f be a bounded function on the finite interval $[a,b]$. Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$ and set $\Delta x_j = x_j - x_{j-1}$ for $j = 1, 2, \dots, n$.

(1) The **upper Riemann sum** of f over P is defined to be the number

$$U(f,P) = \sum_{j=1}^{n} M_j(f)\Delta x_j, \quad \text{where } M_j(f) = \sup_{x \in [x_{j-1}, x_j]} f(x).$$

(2) The **lower Riemann sum** of f over P is defined to be the number

$$L(f,P)=\sum_{j=1}^{n} m_j(f)\Delta x_j, \quad \text{where } m_j(f)=\inf_{x\in[x_{j-1},x_j]} f(x).$$

Clearly, from the definition, $L(f,P)\le U(f,P)$.

Remark 5.3. When dealing with the sums, the following **telescoping** technique is usually very useful: If $g\colon \mathbf{N}\to\mathbf{R}$, then

$$\sum_{k=m}^{n}(g(k+1)-g(k))=g(n+1)-g(m)$$

for all $n\ge m$ in $\mathbf{N}$.

Proof. Write out all terms of the sum and cancel the same terms. □

EXAMPLE 5.3. If $f(x)=C$ is constant on $[a,b]$, then $U(f,P)=L(f,P)=C(b-a)$ for all partitions P of $[a,b]$.

Proof. Note that $M_j(f)=m_j(f)=C$. So

$$U(f,P)=L(f,P)=\sum_{j=1}^{n} C\Delta x_j = C\sum_{j=1}^{n}(x_j-x_{j-1})=C(x_n-x_0)=C(b-a).$$

□

EXAMPLE 5.4. Let $g(x)$ be the **Dirichlet function**:

$$g(x)=\begin{cases}1 & x\in\mathbf{Q}\\ 0 & x\notin\mathbf{Q}.\end{cases}$$

Then, for all partitions P of an interval $[a,b]$, we have

$$U(g,P)=b-a, \quad L(g,P)=0.$$

Lemma 5.1. *Let $f\colon[a,b]\to\mathbf{R}$ be bounded. If P,Q are partitions of $[a,b]$ and Q is finer than P, then*

$$L(f,P)\le L(f,Q)\le U(f,Q)\le U(f,Q).$$

Proof. Since Q is obtained from P by adding finitely many points, by induction, we only need to prove the case when Q is obtained from P by adding one extra point. So let

$$P=\{x_0,x_1,\dots,x_{k-1},x_k,\dots,x_n\}, \quad Q=\{x_0,x_1,\dots,x_{k-1},y,x_k,\dots,x_n\},$$

where $x_{k-1}<y<x_k$. Then

$$L(f,P=\sum_{j=1}^{n} m_j(f)\Delta x_j, \quad \text{where} \quad m_j(f)=\inf_{x\in[x_{j-1},x_j]} f(x),$$

$$L(f,Q)=\sum_{j=1}^{k-1} m_j(f)\Delta x_j+\left(\inf_{x\in[x_{k-1},y]} f(x)\right)(y-x_{k-1})+\left(\inf_{x\in[y,x_k]} f(x)\right)(x_k-y)+\sum_{j=k+1}^{n} m_j(f)\Delta x_j.$$

Note that

$$\inf_{x\in[x_{k-1},y]} f(x)\ge \inf_{x\in[x_{k-1},x_k]} f(x) \quad \text{and} \quad \inf_{x\in[y,x_k]} f(x)\ge \inf_{x\in[x_{k-1},x_k]} f(x).$$

Hence

$$\left(\inf_{x\in[x_{k-1},y]} f(x)\right)(y-x_{k-1}) + \left(\inf_{x\in[y,x_k]} f(x)\right)(x_k-y)$$
$$\geq \left(\inf_{x\in[x_{k-1},x_k]} f(x)\right)(y-x_{k-1}+x_k-y) = m_k(f)\Delta x_k.$$

Consequently,

$$L(f,Q) \geq \sum_{j=1}^{k-1} m_j(f)\Delta x_j + m_k(f)\Delta x_k + \sum_{j=k+1}^{n} m_j(f)\Delta x_j = L(f,P).$$

Similarly, we have $U(f,Q) \leq U(f,P)$. □

Lemma 5.2. *Let $f\colon [a,b]\to \mathbf{R}$ be bounded. If P, Q are any two partitions of $[a,b]$, then*

$$L(f,P) \leq U(f,Q).$$

Proof. Note that $P\cup Q$ is a refinement of both P and Q. Hence, by the previous lemma,

$$L(f,P) \leq L(f,P\cup Q) \leq U(f,P\cup Q) \leq U(f,Q).$$

□

Upper and Lower Integrals.

Definition 5.4. Let $f\colon [a,b]\to \mathbf{R}$ be bounded.

(1) The **upper integral** of f on $[a,b]$ is defined to be the number

$$(U)\int_a^b f = \inf\{U(f,P) : P \text{ is partition of } [a,b]\}.$$

(2) The **lower integral** of f on $[a,b]$ is defined to be the number

$$(L)\int_a^b f = \sup\{L(f,P) : P \text{ is partition of } [a,b]\}.$$

Clearly, from the previous lemma, we have $(L)\int_a^b f \leq (U)\int_a^b f$.

Riemann Integrability. We now introduce the Riemann integrability using the different definition from the text, but later show that this integrability is equivalent to the one given in the text.

Definition 5.5. Let f be a bounded function on a finite interval $[a,b]$. We say that f is **(Riemann) integrable** on $[a,b]$ if the upper integral and lower integral of f on $[a,b]$ are equal; that is, $(U)\int_a^b f = (L)\int_a^b f$. In this case, this common value is defined to be the **(Riemann) integral** of f on $[a,b]$ and is denoted by

$$\int_a^b f(x)\,dx = \int_a^b f := (U)\int_a^b f = (L)\int_a^b f.$$

Example 5.5. The Dirichlet function $g(x) = \begin{cases} 1 & x\in\mathbf{Q} \\ 0 & x\notin\mathbf{Q}\end{cases}$ is not Riemann integrable on any interval $[a,b]$ with $a<b$ since $(U)\int_a^b g = b-a$ but $(L)\int_a^b g = 0$.

Theorem 5.3. *If $f(x) = C$ is constant on $[a,b]$, then f is integrable on $[a,b]$ and*

$$\int_a^b f(x)\,dx = C(b-a).$$

Proof. It is easy to see that $U(f,P) = L(f,P) = C(b-a)$ for all partitions P of $[a,b]$ and hence $(U)\int_a^b f = (L)\int_a^b f$, which proves the theorem. □

Connection of Riemann Integrals and Areas. Let f be nonnegative and bounded on $[a,b]$. Imagine the region under the graph of $y = f(x)$ defined by

$$R := \{(x,y) : x \in [a,b],\ 0 \le y \le f(x)\}.$$

Does R have an area, or how do we define that R has an area? Given a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$, look at the j-th slice $R_j := \{(x,y) : x \in [x_{j-1}, x_j],\ 0 \le y \le f(x)\}$ and the rectangles $U_j = \{(x,y) : x \in [x_{j-1}, x_j],\ 0 \le y \le M_j(f)\}$ and $L_j = \{(x,y) : x \in [x_{j-1}, x_j],\ 0 \le y \le m_j(f)\}$. Clearly $L_j \subseteq R_j \subseteq U_j$. So if R_j has an area $|R_j|$, then $m_j(f)\Delta x_j \le |R_j| \le M_j(f)\Delta x_j$. Consequently, if R has an area $|R|$, then

$$L(f,P) \le |R| \le U(f,P);$$

that is, an upper Riemann sum gives an **over-estimate** of $|R|$ and a lower Riemann sum gives an **under-estimate** of $|R|$. Hence, if the area $|R|$ is well-defined, we must have that

$$(L)\int_a^b f \le |R| \le (U)\int_a^b f.$$

However, if $(L)\int_a^b f < (U)\int_a^b f$, then we have no reasonable way to choose a number $|R|$ between these two numbers and define it to be the area for R.

The Riemann integrability of f exactly states that when and only when $(L)\int_a^b f = (U)\int_a^b f$ can the proper area of the region R be defined, which equals the Riemann integral of f on $[a,b]$. Therefore, if f is nonnegative and integrable on $[a,b]$, then we say the region has the area $|R|$ defined by $|R| = \int_a^b f$, namely,

$$\int_a^b f(x)\,dx = \text{Area of } R := \{(x,y) : x \in [a,b],\ 0 \le y \le f(x)\}. \tag{5.1}$$

Criterion for Integrability. The following theorem gives an equivalent condition for Riemann integrability.

Theorem 5.4 (Criterion for Integrability). *Let f be a bounded function on a finite interval $[a,b]$. Then f is integrable on $[a,b]$ if and only if for each $\varepsilon > 0$ there exists a partition P_ε of $[a,b]$ such that*

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon.$$

Proof. (Sufficiency for Integrability.) Let $\varepsilon > 0$. Assume that there exists a partition P_ε of $[a,b]$ such that $U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$. Then $U(f,P_\varepsilon) < L(f,P_\varepsilon) + \varepsilon$ and so

$$(U)\int_a^b f \le U(f,P_\varepsilon) < L(f,P_\varepsilon) + \varepsilon \le (L)\int_a^b f + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves that $(U)\int_a^b f \le (L)\int_a^b f$ and hence $(U)\int_a^b f = (L)\int_a^b f$. So, by definition, f is integrable on $[a,b]$.

(Necessity for Integrability.) Assume f is integrable on $[a,b]$; namely, $(U)\int_a^b f = (L)\int_a^b f$. Let $\varepsilon > 0$. Then there exist partitions P_1, P_2 of $[a,b]$ such that

$$U(f,P_1) < (U)\int_a^b f + \varepsilon/2, \quad L(f,P_2) > (L)\int_a^b f - \varepsilon/2.$$

Let $P_\varepsilon = P_1 \cup P_2$. Then P_ε is a partition of $[a,b]$. Keeping in mind that $(U)\int_a^b f = (L)\int_a^b f$, we have

$$\begin{aligned} U(f,P_\varepsilon) - L(f,P_\varepsilon) &\le U(f,P_1) - L(f,P_2) \\ &= \left(U(f,P_1) - (U)\int_a^b f\right) + \left((L)\int_a^b f - L(f,P_2)\right) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

□

Integrability of Continuous Functions.

Theorem 5.5. *If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.*

Proof. Let $\varepsilon > 0$. Since f is *uniformly continuous* on $[a,b]$, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a} \quad \text{for all } x, y \in [a,b] \text{ with } |x-y| < \delta.$$

Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[a,b]$ with norm $\|P\| < \delta$. Since f is continuous on each subinterval $[x_{j-1}, x_j]$, by the **Extreme Value Theorem**, there exist $c_j, d_j \in [x_{j-1}, x_j]$ such that

$$M_j(f) = f(c_j), \quad m_j(f) = f(d_j).$$

Since $c_j - d_j| \le \Delta x_j \le \|P\| < \delta$, we have

$$M_j(f) - m_j(f) = f(c_j) - f(d_j) < \frac{\varepsilon}{b-a}.$$

Therefore,

$$U(f,P) - L(f,P) = \sum_{j=1}^n (M_j(f) - m_j(f))\Delta x_j < \frac{\varepsilon}{b-a}\sum_{j=1}^n \Delta x_j = \varepsilon.$$

So, by the Criterion for Integrability, f is integrable on $[a,b]$. □

Example 5.6. Prove that the function

$$f(x) = \begin{cases} 0 & 0 \le x < 1/2 \\ 1 & 1/2 \le x \le 1 \end{cases}$$

is integrable on $[0,1]$. Note that this function is not continuous on $[0,1]$.

Proof. Let $\varepsilon > 0$. Choose $0 < x_1 < 1/2 < x_2 < 1$ such that $x_2 - x_1 < \varepsilon$. Let $P_\varepsilon = \{x_0, x_1, x_2, x_3\}$ with $x_0 = 0$ and $x_3 = 1$. Then P_ε is a partition of $[0,1]$, and

$$m_1(f) = M_1(f) = 0, \quad m_2(f) = 0, \;\; M_2(f) = 1, \quad m_3(f) = M_3(f) = 1.$$

We easily see that

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) = \sum_{j=1}^3 (M_j(f) - m_j(f))\Delta x_j = x_2 - x_1 < \varepsilon.$$

Therefore, by the Criterion for Integrability, f is integrable. □

5.2. Riemann Sums

Definition 5.6. Let $f\colon [a,b] \to \mathbf{R}$ and $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$.

(1) A **Riemann sum** of f with respect to P generated by a *sample* $\{t_j\}$ of points $t_j \in [x_{j-1}, x_j]$ is the sum

$$\mathcal{S}(f, P, \{t_j\}) = \sum_{j=1}^{n} f(t_j)\Delta x_j.$$

(2) Let $I(f) \in \mathbf{R}$ be a number. We say that the **Riemann sums of f converge to $I(f)$ as** $\|P\| \to 0$ if for each $\varepsilon > 0$ there exists a partition P_ε of $[a,b]$ such that

$$|\mathcal{S}(f, P, \{t_j\}) - I(f)| < \varepsilon$$

for every partition P finer than P_ε and every sample $\{t_j\}$ with t_j chosen in the j-th subinterval of P. In this case, we also say that the limit

$$I(f) = \lim_{\|P\|\to 0} \mathcal{S}(f, P, \{t_j\}) = \lim_{\|P\|\to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j$$

exists. (Note that in the definition, we could also require $\|P_\varepsilon\| < \varepsilon$.)

Theorem 5.6 (**Second Criterion for Integrability**)**.** *Let f be a bounded function on a finite interval $[a,b]$. Then f is integrable on $[a,b]$ if and only if the limit*

$$I(f) = \lim_{\|P\|\to 0} \mathcal{S}(f, P, \{t_j\}) = \lim_{\|P\|\to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j$$

exists. In this case, we have $I(f) = \int_a^b f(x)\,dx$.

Proof. (Sufficiency for Integrability.) Assume that the limit

$$I(f) = \lim_{\|P\|\to 0} \mathcal{S}(f, P, \{t_j\}) = \lim_{\|P\|\to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j$$

exists. Let $\varepsilon > 0$. Then there exists a partition $P_\varepsilon = \{x_0, \dots, x_n\}$ of $[a,b]$ such that

$$|\sum_{j=1}^{n} f(t_j)\Delta x_j - I(f)| < \varepsilon/3$$

for every sample $\{t_j\}$ with $t_j \in [x_{j-1}, x_j]$. By the definition of $m_j(f), M_j(f)$, we select $t_j,\ s_j \in [x_{j-1}, x_j]$ such that

$$f(t_j) > M_j(f) - \frac{\varepsilon}{6(b-a)}, \quad f(s_j) < m_j(f) + \frac{\varepsilon}{6(b-a)},$$

and hence $M_j(f) - m_j(f) < f(t_j) - f(s_j) + \varepsilon/(3(b-a))$. So we have

$$\begin{aligned}
U(f,P_\varepsilon) - L(f,P_\varepsilon) &= \sum_{j=1}^{n}(M_j(f) - m_j(f))\Delta x_j \\
&\leq \sum_{j=1}^{n}(f(t_j) - f(s_j))\Delta x_j + \frac{\varepsilon}{3(b-a)}\sum_{j=1}^{n}\Delta x_j \\
&\leq \left|\sum_{j=1}^{n} f(t_j)\Delta x_j - I(f)\right| + \left|I(f) - \sum_{j=1}^{n} f(s_j)\Delta x_j\right| + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

By the criterion for integrability, f is integrable on $[a,b]$.

(Necessity for Integrability.) Suppose that f is integrable on $[a,b]$ and that $\varepsilon > 0$. There exist partitions P_1, P_2 of $[a,b]$ such that

$$L(f,P_1) > \int_a^b f - \varepsilon, \quad U(f,P_2) < \int_a^b f + \varepsilon.$$

Let $P_\varepsilon = P_1 \cup P_2$ be the refinement of P_1, P_2 on $[a,b]$. Then

$$L(f,P_\varepsilon) \geq L(f,P_1) > \int_a^b f - \varepsilon, \quad U(f,P_\varepsilon) \leq U(f,P_2) < \int_a^b f + \varepsilon.$$

Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[a,b]$ finer than P_ε and $\{t_j\}$ be any sample in P. Then $m_j(f) \leq f(t_j) \leq M_j(f)$ and

$$L(f,P) \geq L(f,P_\varepsilon) > \int_a^b f - \varepsilon, \quad U(f,P) \leq U(f,P_\varepsilon) < \int_a^b f + \varepsilon.$$

Hence

$$\int_a^b f - \varepsilon < L(f,P) \leq \sum_{j=1}^{n} f(t_j)\Delta x_j \leq U(f,P) < \int_a^b f + \varepsilon.$$

We conclude that

$$\left|\mathcal{S}(f,P,\{t_j\}) - \int_a^b f\right| < \varepsilon$$

for every partition P finer than P_ε and every sample $\{t_j\}$ chosen in P. This proves that the Riemann sums of f converge to $I(f) = \int_a^b f$ as $\|P\| \to 0$. □

Properties of Integrable Functions.

Theorem 5.7 (Linear Property). *If f, g are integrable on $[a,b]$ and $\alpha, \beta \in \mathbf{R}$, then $\alpha f + \beta g$ is integrable on $[a,b]$, and*

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

Proof. Use Riemann sums and the **Triangle Inequality**. □

Theorem 5.8 (Additivity Property). *Let $a, b \in \mathbf{R}$ with $a < b$, and f be integrable on $[a,b]$. Then f is integrable on each subinterval $[c,d]$ of $[a,b]$. Moreover,*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{5.2}$$

for all $c \in (a,b)$*. Conversely, if* $a < c < b$*,* $f\colon [a,b] \to \mathbf{R}$ *and* f *is integrable on* $[a,c]$ *and* $[c,b]$*, then* f *is integrable on* $[a,b]$*, and* (5.2) *holds.*

Proof. 1. Assume that f is integrable on $[a,b]$. Let $[c,d]$ be a subinterval of $[a,b]$. Let $\varepsilon > 0$. Choose a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Let $P' = P \cup \{c,d\}$ and $P_1 = P' \cap [c,d]$. Then P' is a refinement of P on $[a,b]$ and P_1 is a partition of $[c,d]$, which is part of partition P' of $[a,b]$. Therefore, we have

$$U(f,P_1) - L(f,P_1) \le U(f,P') - L(f,P') \le U(f,P) - L(f,P) < \varepsilon,$$

where $U(f,P_1), L(f,P_1)$ are defined with P_1 being a partition on $[c,d]$. Hence, by the **Criterion for Integrability**, f is integrable on $[c,d]$.

2. Assume that $a < c < b$, $f\colon [a,b] \to \mathbf{R}$ and f is integrable on $[a,c]$ and $[c,b]$. Let $\varepsilon > 0$. Choose partitions P_1 of $[a,c]$ and P_2 of $[c,b]$ such that

$$U(f,P_1) - L(f,P_1) < \varepsilon/2, \quad U(f,P_2) - L(f,P_2) < \varepsilon/2.$$

Let $P = P_1 \cup P_2$. Then P is a partition of $[a,b]$, and we have

$$U(f,P) - L(f,P) = (U(f,P_1) + U(f,P_2)) - (L(f,P_1) + L(f,P_2)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, by the **Criterion for Integrability**, f is integrable on $[a,b]$.

3. To verify the additivity property (5.2), suppose that P is a partition of $[a,b]$. Let $P_0 = P \cup \{c\}$, $P_1 = P_0 \cap [a,c]$, and $P_2 = P_0 \cap [c,b]$. Then $P_0 = P_1 \cup P_2$ and

$$U(f,P) \ge U(f,P_0) = U(f,P_1) + U(f,P_2) \ge \int_a^c f + \int_c^b f,$$

$$L(f,P) \le L(f,P_0) = L(f,P_1) + L(f,P_2) \le \int_a^c f + \int_c^b f.$$

Hence $(U)\int_a^b f \ge \int_a^c f + \int_c^b f$ and $(L)\int_a^b f \le \int_a^c f + \int_c^b f$; so $\int_a^b f = \int_a^c f + \int_c^b f$. □

Theorem 5.9 (Order Property). *Let* $a,b \in \mathbf{R}$ *with* $a < b$*. If* f,g *are integrable on* $[a,b]$ *and* $f(x) \le g(x)$ *for all* $x \in [a,b]$*, then*

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

In particular, if $m \le f(x) \le M$ *for all* $x \in [a,b]$*, then*

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Proof. For all partitions P,Q of $[a,b]$, we have $L(f,Q) \le U(f,P) \le U(g,P)$. So

$$\int_a^b f(x)\,dx = (L)\int_a^b f \le (U)\int_a^b g = \int_a^b g(x)\,dx.$$

The special case follows easily. □

Theorem 5.10. *If* f *is (Riemann) integrable on* $[a,b]$*, then* $|f|$ *is (Riemann) integrable on* $[a,b]$*, and*

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

Proof. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. We claim that

$$M_j(|f|) - m_j(|f|) \le M_j(f) - m_j(f) \quad \forall\, j = 1, 2, \ldots, n. \tag{5.3}$$

Indeed, let $\varepsilon > 0$ and let $x, y \in [x_{j-1}, x_j]$ be such that

$$M_j(|f|) < |f(x)| + \varepsilon/2, \quad m_j(|f|) > |f(y)| - \varepsilon/2.$$

Then $M_j(|f|) - m_j(|f|) < |f(x)| - |f(y)| + \varepsilon$. If $f(x), f(y)$ are both ≥ 0 then

$$|f(x)| - |f(y)| = f(x) - f(y) \le M_j(f) - m_j(f);$$

if $f(x), f(y)$ are both ≤ 0 then

$$|f(x)| - |f(y)| = f(y) - f(x) \le M_j(f) - m_j(f);$$

if $f(x) \le 0 \le f(y)$, then $M_j(f) \ge 0 \ge m_j(f)$ and

$$|f(x)| - |f(y)| = -f(x) - f(y) \le -f(x) \le -m_j(f) \le M_j(f) - m_j(f);$$

finally, if $f(x) \ge 0 \ge f(y)$, then $M_j(f) \ge 0 \ge m_j(f)$ and

$$|f(x)| - |f(y)| = f(x) + f(y) \le f(x) \le M_j(f) \le M_j(f) - m_j(f).$$

Therefore, in any cases, we have $|f(x)| - |f(y)| \le M_j(f) - m_j(f)$. This proves

$$M_j(|f|) - m_j(|f|) < |f(x)| - |f(y)| + \varepsilon \le M_j(f) - m_j(f) + \varepsilon$$

for all $\varepsilon > 0$; hence (5.3) follows. Let $\varepsilon > 0$ and choose a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. Then by (5.3) we have

$$U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P) < \varepsilon,$$

which proves that $|f|$ is integrable on $[a, b]$. Using the inequality $-|f| \le f \le |f|$ and the **Linear and Order Properties** above, we easily have that

$$-\int_a^b |f(x)|\, dx \le \int_a^b f(x)\, dx \le \int_a^b |f(x)|\, dx,$$

which completes the proof. □

Remark 5.7. The converse of the previous theorem is false. Indeed, consider the function

$$h(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ -1 & x \notin \mathbf{Q}. \end{cases}$$

Then $|h(x)| = 1$ is constant and hence is Riemann integrable on an interval $[a, b]$; however, it is easily seen that

$$(U)\int_a^b h = b - a > 0, \quad (L)\int_a^b h = a - b < 0,$$

and hence h is not Riemann integrable on $[a, b]$.

Corollary 5.11. *If f is (Riemann) integrable on $[a, b]$, then f^2 is (Riemann) integrable on $[a, b]$.*

Proof. Since f is bounded on $[a, b]$, assume $|f(x)| \le M$ for all $x \in [a, b]$. Let P be any partition of $[a, b]$. Then $M_j(|f|) \le M$ and $m_j(|f|) \le M$. Note that $M_j(f^2) = M_j(|f|^2) = (M_j(|f|))^2$ and similarly $m_j(f^2) = (m_j(|f|))^2$ (**Verify!**) and hence

$$\begin{aligned} M_j(f^2) - m_j(f^2) &= (M_j(|f|))^2 - (m_j(|f|))^2 \\ &= (M_j(|f|) + m_j(|f|))\,(M_j(|f|) - m_j(|f|)) \le 2M(M_j(|f|) - m_j(|f|)). \end{aligned}$$

This implies that

$$U(f^2, P) - L(f^2, P) \le 2M(U(|f|, P) - L(|f|, P)),$$

from which the integrability of f^2 is a consequence of the integrability of $|f|$ proved above. □

Theorem 5.12. *If f, g are integrable on $[a, b]$, then so is fg.*

Proof. Use the Corollary above and the identity

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2}.$$

□

Mean Value Theorems for Integrals. We continue to assume that $a, b \in \mathbf{R}$ with $a < b$.

Theorem 5.13 (**First Mean Value Theorem for Integrals).** *Suppose that f and g are integrable on $[a, b]$ with $g(x) \ge 0$ for all $x \in [a, b]$. Let*

$$m = \inf_{x\in[a,b]} f(x), \quad M = \sup_{x\in[a,b]} f(x).$$

Then there exists a number $c \in [m, M]$ such that

$$\int_a^b f(x)g(x)\,dx = c\int_a^b g(x)\,dx.$$

In particular, if f is continuous on $[a, b]$, then there exists an $x_0 \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(x_0)\int_a^b g(x)\,dx.$$

Proof. Since $m \le f(x) \le M$, $g(x) \ge 0$ on $[a, b]$, we have $mg(x) \le f(x)g(x) \le Mg(x)$ for all $x \in [a, b]$, and hence, by the **Linear and Order Properties**,

$$m\int_a^b g \le \int_a^b fg \le M\int_a^b g.$$

If $\int_a^b g = 0$, then $\int_a^b fg = 0$ and so any $c \in [m, M]$ will prove the result. So assume $\int_a^b g > 0$ and let $c = \frac{\int_a^b fg}{\int_a^b g}$. This c will prove the theorem since $m \le c \le M$. In particular, if f is continuous, then by the **Intermediate Value Theorem**, there exists an $x_0 \in [a, b]$ such that $f(x_0) = c$. □

Remark 5.8. The **First Mean Value Theorem for Integrals** above states that if f is integrable on $[a, b]$ with $a < b$, then

$$\inf_{x\in[a,b]} f(x) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \sup_{x\in[a,b]} f(x).$$

In particular, if f is continuous on $[a, b]$, then there exists an $x_0 \in [a, b]$ such that

$$f(x_0) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

In general, the number $\frac{1}{b-a}\int_a^b f(x)\,dx$ is called the **mean-value** or the **average** of integrable function f on $[a, b]$.

Theorem 5.14. *If f is integrable on $[a,b]$ and define*

$$F(x) = \int_a^x f(t)\,dt \quad \text{and} \quad G(x) = \int_x^b f(t)\,dt$$

for all $x \in [a,b]$, where we define $F(a) = 0$ and $G(b) = 0$, then F and G are well-defined and are continuous on $[a,b]$.

Proof. Since f is bounded on $[a,b]$, assume $|f(x)| \le M$ for all $x \in [a,b]$. For any $x < y$ in $[a,b]$ we have that

$$F(y) = \int_a^y f(t)\,dt = \int_a^x f(t)\,dt + \int_x^y f(t)\,dt = F(x) + \int_x^y f(t)\,dt$$

and so

$$|F(y) - F(x)| = \left|\int_x^y f(t)\,dt\right| \le \int_x^y |f(t)|\,dt \le M(y-x),$$

which shows that $|F(x) - F(y)| \le M|x-y|$ for all $x, y \in [a,b]$. Similarly, we also show that $|G(x) - G(y)| \le M|x-y|$ for all $x, y \in [a,b]$. Hence F and G are both well-defined and continuous on $[a,b]$. □

Remark 5.9. Sometimes functions defined by $F(x) = \int_a^x f(t)\,dt$ are also considered for $x < a$. For such a purpose, we extend the integral $\int_a^b f(x)\,dx$ to the case $a \ge b$ as follows. We always define

$$\int_a^a f(x)\,dx = 0$$

for all functions f defined and bounded in some interval containing a. If $b < a$ and f is integrable on interval $[b,a]$, then we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Example 5.7. If $f(x) = \begin{cases} 1 & x \ge 0 \\ -1 & x < 0, \end{cases}$ then the function $F(x) = \int_0^x f(t)\,dt$ is defined for all $x \in \mathbf{R}$ and is given by $F(x) = |x|$ for all $x \in \mathbf{R}$.

Theorem 5.15 (Second Mean Value Theorem for Integrals). *Suppose that f and g are integrable on $[a,b]$ with $g(x) \ge 0$ for all $x \in [a,b]$. If m, M are two numbers satisfying*

$$m \le \inf_{x\in[a,b]} f(x), \quad M \ge \sup_{x\in[a,b]} f(x),$$

then there exists a number $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = m\int_a^c g(x)\,dx + M\int_c^b g(x)\,dx.$$

In particular, if $f(x) \ge 0$ for all $x \in [a,b]$, then there exists a number $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = M\int_c^b g(x)\,dx.$$

Proof. Let

$$F(x) = m\int_a^x g(t)\,dt + M\int_x^b g(t)\,dt \quad \forall\, x \in [a,b].$$

Then F is continuous on $[a,b]$ and by the **Order Property**,

$$F(b) = m\int_a^b g \le \int_a^b fg \le M\int_a^b g = F(a).$$

Hence by the **Intermediate Value Theorem** again, there exists $c \in [a,b]$ such that $F(c) = \int_a^b fg$. □

5.3. The Fundamental Theorem of Calculus

Theorem 5.16 (Fundamental Theorem of Calculus). *Let $a, b \in \mathbf{R}$ with $a < b$ and $f\colon [a,b] \to \mathbf{R}$.*

(1) *If f is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F \in \mathcal{C}^1[a,b]$ and*

$$F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x) \quad \forall\, x \in [a,b].$$

(2) *If f is Riemann integrable on $[a,b]$ and G is a continuous function on $[a,b]$ which is differentiable in (a,b) with $G'(x) = f(x)$ for all $x \in (a,b)$, then*

$$\int_a^x f(t)\,dt = G(x) - G(a) \quad \forall\, x \in [a,b].$$

Proof. Let us prove the second part only. Let $\varepsilon > 0$. Since f is integrable on $[a,x]$, choose a partition $P = \{x_0, \ldots, x_n\}$ of $[a,x]$ (so $x_0 = a, x_n = x$) such that

$$\left|\sum_{j=1}^n f(t_j)\Delta x_j - \int_a^x f(t)dt\right| < \varepsilon$$

for any choice of sample $\{t_j\}$ with $t_j \in [x_{j-1}, x_j]$. Since G is continuous on $[x_{j-1}, x_j]$ and differentiable in (x_{j-1}, x_j), by the **Mean Value Theorem**, we choose $t_j \in (x_{j-1}, x_j)$ such that $G(x_j) - G(x_{j-1}) = G'(t_j)\Delta x_j = f(t_j)\Delta x_j$. Therefore

$$\left|G(x) - G(a) - \int_a^x f(t)\,dt\right| = \left|\sum_{j=1}^n (G(x_j) - G(x_{j-1})) - \int_a^x f(t)dt\right|$$

$$= \left|\sum_{j=1}^n f(t_j)\Delta x_j - \int_a^x f(t)dt\right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $G(x) - G(a) - \int_a^x f(t)\,dt = 0$. □

Theorem 5.17 (Integration by Parts). *Suppose that f, g are differentiable on $[a,b]$ with f', g' Riemann integrable on $[a,b]$. Then*

$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx,$$

which can also be written as

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)\big|_a^b - \int_a^b f(x)g'(x)\,dx,$$

with the notation $h(x)|_a^b := h(b) - h(a)$.

Proof. Let $G(x) = f(x)g(x)$ for $x \in [a, b]$. Then G is differentiable on $[a, b]$ and $G'(x) = f'(x)g(x) + f(x)g'(x) := h(x)$ for all $x \in [a, b]$. Since f, g, f', g' are all integrable on $[a, b]$, it follows that h, $f'g$ and fg' are all integrable on $[a, b]$. Hence by the second part of **FTC**,

$$\int_a^b h(x)\,dx = G(b) - G(a).$$

However, by the **Linear Property**,

$$\int_a^b h(x)\,dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx.$$

So regrouping, we prove the theorem. □

EXAMPLE 5.8. Find $\int_0^{\pi/2} x \sin x\,dx$.

Solution. Let $f(x) = -\cos x$ and $g(x) = x$. Then $f'(x)g(x) = x \sin x$ and so

$$\int_0^{\pi/2} x \sin x\,dx = f(x)g(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} f(x)g'(x)\,dx$$
$$= (0 - 0) + \int_0^{\pi/2} \cos x\,dx = \sin x|_0^{\pi/2} = 1.$$

□

Inverse Function Theorem on R. We have the following result.

Theorem 5.18 (Inverse Function Theorem on R). *Let I be an open interval and $f\colon I \to \mathbf{R}$ be one-to-one and continuous. Then $J := f(I)$ is an open interval and the inverse function f^{-1} is continuous on J. If $f'(a)$ exists and is nonzero for some $a \in I$, then the inverse function f^{-1} is differentiable at $b = f(a)$ and $(f^{-1})'(b) = 1/f'(a)$.*

Proof. 1. We first show that f must be strictly monotone on I. Suppose not. Then there exist points $h, k, l \in I$ such that $h < k < l$ but $f(k)$ does not lie between $f(h)$ and $f(l)$. Since $f(h) \neq f(l)$, with loss of generality, we may assume $f(h) < f(l)$; then either $f(k) < f(h) < f(l)$ or $f(h) < f(l) < f(k)$. Hence by the one-dimensional **Intermediate Value Theorem**, either there exists $x_1 \in (k, l)$ such that $f(x_1) = f(h)$ or there exists $x_2 \in (h, k)$ such that $f(x_2) = f(l)$. In either cases, since f is one-to-one, we have contradiction: $x_1 = h \in (k, l)$ or $x_2 = l \in (h, k)$. Therefore, f must be strictly monotone on I. Without loss of generality, in the following, we assume that f is strictly increasing on I; that is, $f(x) < f(y)$ for all $x < y$ in I.

2. Let $I = (\alpha, \beta)$. We claim that $J := f(I) = (f(\alpha^+), f(\beta^-))$ is an open interval. It is clear that if $a \in (\alpha, \beta)$, then $a \in (\alpha', \beta')$ for some α', β' with $\alpha < \alpha' < \beta' < \beta$ and thus $f(\alpha') < f(a) < f(\beta')$. Since $f(\alpha^+) < f(\alpha') < f(\beta') < f(\beta^-)$, it follows that $f(a) \in (f(\alpha^+), f(\beta^-))$, which proves that $f(I) \subseteq (f(\alpha^+), f(\beta^-))$. Conversely, let $b \in (f(\alpha^+), f(\beta^-))$; namely, $f(\alpha^+) < b < f(\beta^-)$. Then there exist α', β' with $\alpha < \alpha' < \beta' < \beta$ such that $f(\alpha') < b < f(\beta')$. Since f is continuous, by the **IVT**, there exists $a \in (\alpha', \beta')$ such that $f(a) = b$ and thus $b \in f(I)$, which proves that $(f(\alpha^+), f(\beta^-)) \subseteq f(I)$. Therefore, $J := f(I) = (f(\alpha^+), f(\beta^-))$ and is thus an open interval.

3. We now prove that f^{-1} is continuous on J. Suppose that f^{-1} is not continuous at some point $y_0 \in J$. Then there exists a sequence $\{y_n\}$ in J and a number $\varepsilon_0 > 0$ with $(y_n) \to y_0$ but

$$|f^{-1}(y_n) - f^{-1}(y_0)| > \varepsilon_0 \quad \forall\, n \in \mathbf{N}.$$

Since $y_n \neq y_0$ for all $n \in \mathbf{N}$, without loss of generality, we assume that there exists a subsequence (y_{n_j}) with $y_{n_j} > y_0$ for all $j \in \mathbf{N}$. Hence

$$f^{-1}(y_{n_j}) > f^{-1}(y_0) + \varepsilon_0 > f^{-1}(y_0).$$

Since $f^{-1}(y_{n_j}), f^{-1}(y_0) \in I$ and I is an open interval, we have $f^{-1}(y_0) + \varepsilon_0 \in I = f^{-1}(J)$ and thus $f^{-1}(y_0) + \varepsilon_0 = f^{-1}(z_0)$ for some $z_0 \in J$. Then

$$f^{-1}(y_{n_j}) > f^{-1}(z_0) > f^{-1}(y_0).$$

This implies $y_{n_j} > z_0 > y_0$ for all $j \in \mathbf{N}$, which is a contradiction to the convergence $(y_n) \to y_0$. So f^{-1} is continuous on $J = f(I)$.

4. Now assume $f'(a)$ exists and is nonzero at some $a \in I$, and we prove that f^{-1} is differentiable at $b = f(a) \in J$ and $(f^{-1})'(b) = 1/f'(a)$. We choose an interval $(c,d) \subset I$ such that $a \in (c,d)$. Then $f(c) < b = f(a) < f(d)$. Choose $\delta > 0$ so small that $b+h \in (f(c), f(d))$ for all $|h| < \delta$. Fix such an $h \neq 0$ and set $x = f^{-1}(b+h)$. Then $f(x) - f(a) = b + h - b = h$. By the continuity of f^{-1}, we have that $x \to a$ if and only if $h \to 0$. Therefore,

$$\lim_{h\to 0} \frac{f^{-1}(b+h) - f^{-1}(b)}{h} = \lim_{x\to a} \frac{x-a}{f(x)-f(a)} = \frac{1}{f'(a)}.$$

□

Theorem 5.19 (Change of Variables). *Let* $\phi \in \mathcal{C}^1[a,b]$ *with* $\phi' \neq 0$ *on* $[a,b]$. *Let* $\phi([a,b]) = [c,d]$. *If* f *is integrable on* $[c,d]$, *then* $f(\phi(x))|\phi'(x)|$ *is integrable on* $[a,b]$, *and*

$$\int_c^d f(t)\,dt = \int_a^b f(\phi(x))|\phi'(x)|\,dx.$$

Proof. By the assumption, ϕ' is either positive on $[a,b]$ or negative on $[a,b]$. We consider only the case when $\phi'(x) > 0$ for all $x \in [a,b]$. In this case ϕ is strictly increasing on $[a,b]$ and $\phi([a,b]) = [c,d] = [\phi(a), \phi(b)]$. Also the inverse function $\phi^{-1} \colon [c,d] \to [a,b]$ is also differentiable. Assume $|f(x)| \leq M$ for all $x \in [a,b]$. Let $\varepsilon > 0$. Since ϕ' is uniformly continuous on $[a,b]$, choose $\delta > 0$ such that

$$|\phi'(s) - \phi'(t)| < \frac{\varepsilon}{2M(b-a)} \quad \forall\, s,t \in [a,b], \ \ |s-t| < \delta.$$

Since ϕ^{-1} is uniformly continuous on $[c,d]$, choose $\eta > 0$ such that

$$|\phi^{-1}(y) - \phi^{-1}(z)| < \delta \quad \forall\, y,z \in [c,d], \ \ |y-z| < \eta.$$

Since f is integrable on $[c,d]$, choose a partition P_ε of $[c,d]$ such that $\|P_\varepsilon\| < \eta$ such that

$$\left| \mathcal{S}(f, P, \{u_j\}) - \int_c^d f(t)\,dt \right| < \varepsilon/2$$

for every partition $P = \{y_0, y_1, \dots, y_n\}$ of $[c,d]$ finer than P_ε and every choice of sample $u_j \in [y_{j-1}, y_j]$. Let $\tilde{P}_\varepsilon = \phi^{-1}(P_\varepsilon)$. Then $\tilde{P}_\varepsilon$ is a partition of $[a,b]$ and $\|\tilde{P}_\varepsilon\| < \delta$ (by the choice of η). Now let $\tilde{P} = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a,b]$ finer than $\tilde{P}_\varepsilon$ and let $t_j \in [x_{j-1}, x_j]$ be any sample points. Let $P = \phi(\tilde{P}) = \{y_0, y_1, \dots, y_n\}$ with $y_j = \phi(x_j)$ and let $u_j = \phi(t_j)$. Then P is a partition of $[c,d]$ finer than P_ε. By the MVT, choose $c_j \in [x_{j-1}, x_j]$ such that $y_j - y_{j-1} = \phi(x_j) - \phi(x_{j-1}) = \phi'(c_j)\Delta x_j$. Then

$$f(\phi(t_j))\phi'(c_j)\Delta x_j = f(u_j)\Delta y_j.$$

Hence,

$$\left|\sum_{j=1}^{n} f(\phi(t_j))\phi'(t_j)\Delta x_j - \int_c^d f(t)\,dt\right|$$

$$\leq \left|\sum_{j=1}^{n} f(\phi(t_j))(\phi'(t_j)-\phi'(c_j))\Delta x_j\right| + \left|\sum_{j=1}^{n} f(\phi(t_j))\phi'(c_j)\Delta x_j - \int_c^d f(t)\,dt\right|$$

$$\leq M\sum_{j=1}^{n} |\phi'(t_j)-\phi'(c_j)|\Delta x_j + \left|\sum_{j=1}^{n} f(u_j)\Delta y_j - \int_c^d f(t)\,dt\right|$$

$$\leq M\sum_{j=1}^{n} \frac{\varepsilon \Delta x_j}{2M(b-a)} + \left|\sum_{j=1}^{n} f(u_j)\Delta y_j - \int_c^d f(t)\,dt\right|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This completes the proof. □

Remark 5.10. The difficult part of the part is to show that $f(\phi(x))|\phi'(x)|$ is integrable on $[a,b]$ when we only know $f(t)$ is integrable on $[c,d]$. These two functions are defined on different intervals and f is not assumed to be continuous. If we assume f is continuous then the proof is much easier (see next theorem).

Theorem 5.20 (Change of Variables for Continuous Integrands). *Suppose that $\phi \in \mathcal{C}^1[a,b]$ and f is continuous on some interval containing the set $\phi([a,b])$. Then*

$$\int_{\phi(a)}^{\phi(b)} f(t)\,dt = \int_a^b f(\phi(x))\phi'(x)\,dx.$$

Note that the integral on the lefthand side still makes sense even when $\phi(b) \leq \phi(a)$.

Proof. Define

$$G(x) = \int_a^x f(\phi(t))\phi'(t)\,dt \quad \forall\, x \in [a,b],$$

$$F(u) = \int_{\phi(a)}^u f(t)\,dt \quad \forall\, u \in \phi([a,b]),$$

where the integral even when $u \leq \phi(a)$ is well-defined as above. Then, by FTC, $G'(x) = f(\phi(x))\phi'(x)$ and $F'(u) = f(u)$. Hence by the Chain Rule,

$$(G(x) - F(\phi(x)))' = G'(x) - F'(\phi(x))\phi'(x) = 0$$

for all $x \in [a,b]$. Therefore, $G(x) - F(\phi(x))$ is constant on $[a,b]$, which can be evaluated to be zero by choosing $x = a$. So $G(x) = F(\phi(x))$ for all $x \in [a,b]$; in particular, $G(b) = F(\phi(b))$, which exactly proves the theorem. □

5.4. Improper Riemann Integration

In this section, we extend the Riemann integrals to unbounded intervals or unbounded functions or both.

We first make the following motivating fact about the Riemann integrals.

Lemma 5.21. *Let $a, b \in \mathbf{R}$ with $a < b$ and let f be integrable on $[a, b]$. Then*

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\left(\lim_{d\to b^-}\int_c^d f(x)\,dx\right).$$

Proof. Let $F(x) = \int_a^x f(t)\,dt$ for all $x \in [a, b]$. Then F is continuous on $[a, b]$ and hence

$$\int_a^b f(x)\,dx = F(b) - F(a) = \lim_{c\to a^+}(\lim_{d\to b^-}(F(d) - F(c))) = \lim_{c\to a^+}\left(\lim_{d\to b^-}\int_c^d f(x)\,dx\right).$$

□

Improper Integrability.

Definition 5.11. Let (a, b) be a nonempty, open (possibly unbounded) interval and $f\colon (a, b) \to \mathbf{R}$ be a function.

(1) We say that f is **locally integrable on** (a, b) if f is integrable on each finite closed interval $[c, d]$ of (a, b).

(2) We say that f is **improperly (Riemann) integrable** on (a, b) if f is locally integrable on (a, b) and the limit

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\left(\lim_{d\to b^-}\int_c^d f(x)\,dx\right) \tag{5.4}$$

exists and is finite. In this case, this limit is called the **improper Riemann integral** of f on (a, b). Sometimes we also use the notation

$$\int_a^b f(x)\,dx = \int_{a^+}^{b^-} f(x)\,dx$$

to distinguish the improper integrals from the Riemann integrals defined earlier.

Lemma 5.22. *The order of limits in* (5.4) *does not matter. In particular, if the limit in* (5.4) *exists and is finite, then the limit*

$$\lim_{d\to b^-}\left(\lim_{c\to a^+}\int_c^d f(x)\,dx\right)$$

exists and equals $\int_a^b f(x)\,dx$.

Proof. Let $x_0 \in (a, b)$. Then

$$\lim_{c\to a^+}\left(\lim_{d\to b^-}\int_c^d f(x)\,dx\right) = \lim_{c\to a^+}\left(\int_c^{x_0} f(x)\,dx + \lim_{d\to b^-}\int_{x_0}^d f(x)\,dx\right)$$

$$= \lim_{c\to a^+}\int_c^{x_0} f(x)\,dx + \lim_{d\to b^-}\int_{x_0}^d f(x)\,dx. \tag{5.5}$$

Since, for each c, $\lim_{d\to b^-}\int_c^d f(x)dx$ exists, we have

$$\begin{aligned}\lim_{x_0\to b^-}\left(\lim_{d\to b^-}\int_{x_0}^d f(x)\,dx\right) &= \lim_{x_0\to b^-}\left[\lim_{d\to b^-}\left(\int_c^d f(x)\,dx-\int_c^{x_0} f(x)\,dx\right)\right]\\ &= \lim_{x_0\to b^-}\left[\lim_{d\to b^-}\int_c^d f(x)\,dx-\int_c^{x_0} f(x)\,dx\right]\\ &= \lim_{d\to b^-}\int_c^d f(x)\,dx-\lim_{x_0\to b^-}\int_c^{x_0} f(x)\,dx=0.\end{aligned}$$

Therefore, in (5.5) letting $x_0\to b^-$, we obtain that

$$\lim_{x_0\to b^-}\left(\lim_{c\to a^+}\int_c^{x_0} f(x)\,dx\right)=\lim_{c\to a^+}\left(\lim_{d\to b^-}\int_c^d f(x)\,dx\right):=\int_{a^+}^{b^-} f(x)\,dx.$$

□

Remark 5.12. (i) If f is integrable on $[c,b]$ for all $c\in(a,b)$, then the improper Riemann integral of f on (a,b) is also given by

$$\int_a^b f(x)\,dx=\lim_{c\to a^+}\int_c^b f(x)\,dx:=\int_{a^+}^b f(x)\,dx.$$

If this limit exists and is finite, we also say that f is **improperly integrable on** $(a,b]$. The similar situation applies at the endpoint b, in which case we say that f is **improperly integrable on** $[a,b)$.

(ii) It is easily seen that f is improperly integrable on (a,b) if and only if f is improperly integrable on $(a,c]$ and also improperly integrable on $[c,b)$, where $c\in(a,b)$ is some number. In this case, we have that

$$\int_{a^+}^{b^-} f(x)\,dx=\int_{a^+}^c f(x)\,dx+\int_c^{b^-} f(x)\,dx.$$

Example 5.9. Show that function $f(x)=1/\sqrt{x}$ is improperly integrable on $(0,1]$.

Proof. Exercise! □

Example 5.10. Show that function $f(x)=1/x^2$ is improperly integrable on $[1,\infty)$.

Proof. Exercise! □

Properties of Improper Integrals.

Theorem 5.23 (Linear Property). *If f,g are improperly integrable on (a,b) and $\alpha,\beta\in\mathbf{R}$, then $\alpha f+\beta g$ is improperly integrable on (a,b), and*

$$\int_a^b(\alpha f(x)+\beta g(x))\,dx=\alpha\int_a^b f(x)\,dx+\beta\int_a^b g(x)\,dx.$$

Proof. Use the **Linear Property** of integrals on each subinterval $[c,d]$ of (a,b). □

Theorem 5.24 (Comparison Theorem for Improper Integrals). *Suppose that f,g are locally integrable on (a,b) and $0\le f(x)\le g(x)$ for all $x\in(a,b)$. If g is improperly integrable on (a,b), then f is also improperly integrable on (a,b) and*

$$\int_a^b f(x)\,dx\le\int_a^b g(x)\,dx.$$

Proof. Fix $c \in (a, b)$. Let $F(d) = \int_c^d f(x)\,dx$ and $G(d) = \int_c^d g(x)dx$ for $d \in [c, b)$. Then by the **Order Property**, $F(d) \le G(d)$. Note that F and G are increasing on $[c, b)$ and $G(b^-)$ exists. Hence F is bounded above by $G(b^-)$ and so $F(d^-)$ exists and is finite. This shows that f is improperly integrable on $[c, b)$. By the similar argument, we also show that f is improperly integrable on $(a, c]$; thus f is improperly integrable on (a, b). The order property

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

follows easily from the order property of the Riemann integrals of f and g on each subinterval $[c, d]$ of (a, b). □

EXAMPLE 5.11. Show that $f(x) = (\sin x)/x^{3/2}$ is improperly integrable on $(0, 1]$.

Proof. Since $0 \le \sin x \le x$ for all $x \in [0, 1]$ (use elementary calculus to prove it!), it follows that

$$0 \le f(x) \le x \cdot x^{-3/2} = x^{-1/2} \quad \forall\, x \in (0, 1].$$

Since $x^{-1/2}$ is improperly integrable on $(0, 1]$, by the theorem above, f is improperly integrable on $(0, 1]$. □

EXAMPLE 5.12. Show that $f(x) = (\ln x)/x^{5/2}$ is improperly integrable on $[1, \infty)$.

Proof. Since $0 \le \ln x \le x$ for all $x \ge 1$ (use elementary calculus to prove it!), it follows that

$$0 \le f(x) \le x \cdot x^{-5/2} = x^{-3/2} \quad \forall\, x \ge 1.$$

Since $x^{-3/2}$ is improperly integrable on $[1, \infty)$, by the theorem above, f is improperly integrable on $[1, \infty)$. □

Lemma 5.25. *If f is bounded and locally integrable on (a, b) and $|g|$ is improperly integrable on (a, b), then $|fg|$ is improperly integrable on (a, b).*

Proof. Use $0 \le |fg| \le M|g|$ and the **Comparison Theorem** above. □

Absolute and Conditional Improper Integrability.

Definition 5.13. Let $f\colon (a, b) \to \mathbf{R}$.

(1) We say that f is **absolutely integrable** on (a, b) if f is locally integrable on (a, b) and $|f|$ is improperly integrable on (a, b).

(2) We say that f is **conditionally integrable** on (a, b) if f is improperly integrable on (a, b) but $|f|$ is not improperly integrable on (a, b).

Theorem 5.26. *If f is absolutely integrable on (a, b), then f is improperly integrable on (a, b) and*

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

Proof. Since $0 \le |f| + f \le 2|f|$, by the **Comparison Theorem**, $f + |f|$ is improperly integrable on (a, b). Hence, by the Linear Property, $f = (|f| + f) - |f|$ is also improperly integrable on (a, b). Moreover, for all $c < d$ in (a, b),

$$\left| \int_c^d f(x)\,dx \right| \le \int_c^d |f(x)|\,dx.$$

We then complete the proof by taking the limit as $c \to a^+$ and $d \to b^-$. □

The converse of Theorem 5.26 is false.

EXAMPLE 5.13. Prove that $f(x) = \dfrac{\sin x}{x}$ is conditionally integrable on $[1, \infty)$.

Proof. Integrating by parts, we have for all $d > 1$,

$$\int_1^d \frac{\sin x}{x}\, dx = -\frac{\cos x}{x}\Big|_1^d - \int_1^d \frac{\cos x}{x^2}\, dx.$$

Since $1/x^2$ is absolutely integrable on $[1, \infty)$, we have $(\cos x)/x^2$ is absolutely integrable on $[1, \infty)$; hence $(\cos x)/x^2$ is improperly integrable on $[1, \infty)$. Taking the limit as $d \to \infty$ above, we have

$$\int_1^\infty \frac{\sin x}{x}\, dx = \cos(1) - \int_1^\infty \frac{\cos x}{x^2}\, dx$$

exists and is finite. This proves that $(\sin x)/x$ is improperly integrable on $[1, \infty)$.

We now show that $|\sin x|/x$ is not improperly integrable on $[1, \infty)$, which proves that $(\sin x)/x$ is conditionally integrable on $[1, \infty)$. Note that if $n \in \mathbf{N}$ and $n \geq 2$ then

$$\int_1^{n\pi} \frac{|\sin x|}{x}\, dx \geq \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\, dx \geq \sum_{k=2}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x|\, dx = \frac{2}{\pi} \sum_{k=2}^{n} \frac{1}{k}.$$

Hence

$$\lim_{n\to\infty} \int_1^{n\pi} \frac{|\sin x|}{x}\, dx = \infty.$$

So $|\sin x|/x$ is not improperly integrable on $[1, \infty)$. $\square$