Functional Limits and Continuity

4.1. Functional Limits

Definition 4.1 (**ϵ-δ definition of functional limits**)**.** Let $f\colon A \to \mathbf{R}$, and let c be a limit point of the domain A. We say the limit of f as x approaches c is a number L and write $\lim_{x\to c} f(x) = L$ provided that, for each $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ and $x \in A$ it follows that $|f(x) - L| < \epsilon$.

Note that the condition $0 < |x - c|$ is simply saying $x \neq c$. Therefore, as known from Calculus, the limit value L has nothing to do with whether f is defined at c or not; even $f(c)$ is defined (meaning $c \in A$), L may not have any relation with it.

Example 4.1. (i) Let $f(x) = 3x + 1$. In this case the domain A of this formula-defined function is considered to be all real numbers, that certainly makes sense of $f(x)$. Show $\lim_{x\to 2} f(x) = 7$.

Proof. Let $\epsilon > 0$. We need to produce a $\delta > 0$ with the property that $|f(x) - 7| < \epsilon$ holds for all x satisfying $0 < |x - 2| < \delta$. The ending requirement is the inequality $|f(x) - 7| < \epsilon$, which can be rewritten as

$$|f(x) - 7| = |(3x + 1) - 7| = |3x - 6| = 3|x - 2| < \epsilon.$$

Hence the requirement $|f(x) - 7| < \epsilon$ is equivalent to $|x - 2| < \epsilon/3$; that is, whenever $|x - 2| < \epsilon/3$, it follows that $|f(x) - 7| < \epsilon$. Therefore, we can select simply $\delta = \epsilon/3 > 0$ to satisfy the definition. □

(ii) Show $\lim_{x\to 2} x^2 = 4$.

Proof. Given $\epsilon > 0$, our goal is to produce a $\delta > 0$ such that $|x^2 - 4| < \epsilon$ for all x with $0 < |x-2| < \delta$. As above the domain set $A = \mathbf{R}$. We start to analyze the ending requirement $|x^2 - 4| < \epsilon$, which can be rewritten as

$$|x^2 - 4| = |x + 2||x - 2| < \epsilon.$$

Unlike the previous example, in front of $|x - 2|$ there is a function $|x + 2|$, not simply a constant number; we cannot divide by $|x + 2|$ to simply select $\delta = \epsilon/|x + 2|$ since this δ

depends on x. The idea is to first choose one fixed δ to control the term $|x+2|$. For example, let $|x-2|<1$ (with $\delta_1 = 1$). For all such x's, we have $-1 < x-2 < 1$ and hence $1 < x < 3$ and so $3 < x+2 < 5$; that is, $|x+2| < 5$. Hence, if $|x-2| < 1$ then $|x+2| < 5$ and so

$$|x^2-4| = |x+2||x-2| \le 5|x-2| \quad \text{(note that this inequality holds when } |x-2|<1).$$

Therefore, for all such x's, to make $|x^2-4| < \epsilon$, it suffices to require $|x-2| < \epsilon/5 = \delta_2$.

Now, choose $\delta = \min\{1, \epsilon/5\}$. If $0 < |x-2| < \delta$, then both $|x-2| < \delta_1 = 1$ and $|x-2| < \delta_2 = \epsilon/5$ hold and hence

$$|x^2-4| = |x+2||x-2| \le 5|x-2| < 5 \times \frac{\epsilon}{5} = \epsilon,$$

and the limit is proved. □

Topological Version of Functional Limits. Since the statement $|f(x)-L| < \epsilon$ is equivalent to $f(x) \in V_\epsilon(L)$ and the statement $|x-c| < \delta$ is equivalent to $x \in V_\delta(c)$ and hence the statement $0 < |x-c| < \delta$ and $x \in A$ simply means $x \in (V_\delta(c) \setminus \{c\}) \cap A$, we can rephrase the ϵ-δ definition above by using the topological terminologies (of neighborhoods).

Definition 4.2 (**Topological Definition of Functional Limits**)**.** Let $f\colon A \to \mathbf{R}$, and let c be a limit point of the domain A. We say $\lim_{x\to c} f(x) = L$ provided that

$$\forall\, \epsilon > 0 \;\; \exists\, \delta > 0, \;\; f(\hat{V}_\delta(c) \cap A) \subseteq V_\epsilon(L),$$

where $\hat{V}_\delta(c) = V_\delta(c) \setminus \{c\}$ denotes the **punctured neighborhood** of c.

Sequential Criterion for Functional Limits. Functional limits can be completely characterized by the convergence of all related sequences.

Theorem 4.1 (**Sequential Criterion for Functional Limits**)**.** *Let $f\colon A \to \mathbf{R}$ and c be a limit point of A. Then the following two conditions are equivalent:*

(i) $\lim\limits_{x\to c} f(x) = L$.

(ii) *For all sequences (x_n) satisfying $x_n \in A$, $x_n \ne c$ and $(x_n) \to c$, it follows that the sequence $(f(x_n)) \to L$.*

Proof. 1. First assume (i) and we prove (ii). Let (x_n) satisfy $x_n \in A$, $x_n \ne c$ and $(x_n) \to c$. Given each $\epsilon > 0$, $\exists\, \delta > 0$ such that $|f(x) - L| < \delta$ for all $x \in A$ with $0 < |x-c| < \delta$. For this $\delta > 0$, $\exists\, N \in \mathbf{N}$ such that $|x_n - c| < \delta$ for all $n \ge N$ in $\mathbf{N}$. Since $x_n \ne c$, we have $0 < |x_n - c| < \delta$ for all $n \ge N$ and thus $|f(x_n) - L| < \epsilon$. This proves $(f(x_n)) \to L$.

2. We now assume (ii) and prove (i). Suppose $\lim\limits_{x\to c} f(x) \ne L$. Then $\exists\, \epsilon_0 > 0$, $x_n \in A$ with $0 < |x_n - c| < 1/n$ but $|f(x_n) - L| \ge \epsilon_0$ for all $n = 1, 2, \cdots$. (This is the **negation** of $\lim\limits_{x\to c} f(x) = L$.) For this sequence (x_n), by (ii), $(f(x_n)) \to L$, a contradiction to $|f(x_n) - L| \ge \epsilon_0$ for all $n = 1, 2, \cdots$. □

Corollary 4.2 (Divergence Criterion for Functional Limits)**.** *Let $f\colon A \to \mathbf{R}$ and c be a limit point of A. If there exist two sequences (x_n) and (y_n) in A, with $x_n \ne c$ and $y_n \ne c$, satisfying the property*

$$\lim x_n = \lim y_n = c \quad \textit{but} \quad \lim f(x_n) \ne \lim f(y_n),$$

then the functional limit $\lim_{x\to c} f(x)$ does not exist.

Example 4.2. Show that $\lim_{x\to 0} \sin(1/x)$ does not exist.

Proof. Let $x_n = 1/2n\pi$ and $y_n = 1/(2n\pi + \pi/2)$ for all $n \in \mathbf{N}$. Then $x_n \neq 0$ and $y_n \neq 0$ and $\lim x_n = \lim y_n = 0$. But $\sin(1/x_n) = \sin(2n\pi) = 0$ while $\sin(1/y_n) = \sin(2n\pi + \pi/2) = \sin(\pi/2) = 1$. By the corollary above, the functional limit $\lim_{x\to 0} \sin(1/x)$ does not exist. □

Theorem 4.3 (Algebraic Limit Theorem for Functional Limits). *Let $f, g\colon A \to \mathbf{R}$ and c be a limit point of A. Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ exist. Then*

(i) $\lim_{x\to c}[af(x) + bg(x)] = aL + bM$ *for all* $a, b \in \mathbf{R}$,

(ii) $\lim_{x\to c}[f(x)g(x)] = LM$,

(iii) $\lim_{x\to c}[f(x)/g(x)] = L/M$, *provided* $M \neq 0$.

4.2. Combinations of Continuous Functions

Definition 4.3 (Continuous Functions). Let $f\colon A \to \mathbf{R}$ and $c \in A$.

(i) We say f is **continuous at a point** $c \in A$ if, for each $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ and $x \in A$ it follows that $|f(x) - f(c)| < \epsilon$.
If f is not continuous at c we say f is **discontinuous** at c.

(ii) We say f is a **continuous function** on A if it is continuous at every point in A.

Note that continuity at c is not defined if $f(c)$ is not defined, i.e., if $c \notin A$. If $c \in A$ is an *isolated point* of A, then f is *always* continuous at c since for some $\delta > 0$ the only point x satisfying $|x - c| < \delta$ and $x \in A$ is $x = c$ and hence the condition $|f(x) - f(c)| = 0 < \epsilon$ always holds. If $c \in A$ is a limit point of A, then continuity of f at c is simply equivalent to

$$\lim_{x\to c} f(x) = f(c).$$

This is the most interesting case.

Theorem 4.4 (Characterizations of Continuity). *Let $f\colon A \to \mathbf{R}$ and $c \in A$ be a limit point of A. Then the following conditions are equivalent:*

(i) *f is continuous at c.*

(ii) $\lim_{x\to c} f(x) = f(c)$.

(iii) $\forall\, \epsilon > 0 \;\; \exists\, \delta > 0, \;\; f(V_\delta(c) \cap A) \subseteq V_\epsilon(f(c))$.

(iv) *Whenever $x_n \in A$ and $(x_n) \to c$ it follows that $(f(x_n)) \to f(c)$.*

Proof. (i), (ii), and (iii) are simply a different way to describe the definition of the continuity; the condition (iv) with $x_n \neq c$ would be already equivalent to the convergence $\lim_{x\to c} f(x) = f(c)$. Details are omitted. □

Corollary 4.5 (Criterion for Discontinuity). *Let $f\colon A \to \mathbf{R}$ and $c \in A$ be a limit point of A. Then f is not continuous at c if and only if for some number $\epsilon_0 > 0$ and sequence (x_n) in A with $(x_n) \to c$ it follows that $|f(x_n) - f(c)| \geq \epsilon_0$ for all $n \in \mathbf{N}$.*

Proof. Use $\neg(ii) \iff \neg(iv)$. □

Theorem 4.6 (Algebraic Continuity Theorem). *Let $f, g\colon A \to \mathbf{R}$ be continuous at a point $c \in A$. Then*

(i) $af(x) + bg(x)$ *is continuous at c for all* $a, b \in \mathbf{R}$;

(ii) $f(x)g(x)$ *is continuous at c*;

(iii) $f(x)/g(x)$ *is continuous at c, provided the quotient is well defined.*

EXAMPLE 4.3. Polynomials are all continuous functions on $\mathbf{R}$. Hence all rational functions (quotients of polynomials) are continuous at points where the denominator is not zero.

EXAMPLE 4.4. Let

$$g(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then g is continuous at 0.

Proof. Note that $|g(x)| \le |x|$ for all $x \in \mathbf{R}$; this is clearly true if $x = 0$ and is also true if $x \neq 0$ by the formula of $g(x)$ since $|\sin\theta| \le 1$ for all θ. Given each $\epsilon > 0$, let $\delta = \epsilon$. Then, whenever $|x - 0| < \delta$, since $g(0) = 0$,

$$|g(x) - g(0)| = |g(x)| \le |x| < \delta = \epsilon.$$

So g is continuous at 0. □

EXAMPLE 4.5. Function $f(x) = \sqrt{x}$ is continuous on $A = \{x \in \mathbf{R} : \ x \ge 0\} = [0, \infty)$. This can be shown by the definition directly or using the sequential criterion.

Theorem 4.7 (**Composition of Continuous Functions**). *Let $f\colon A \to \mathbf{R}$ and $g\colon B \to \mathbf{R}$, where $B \supseteq f(A)$ and so that the* **composition function** *$g \circ f\colon A \to \mathbf{R}$ is defined by $g \circ f(x) = g(f(x))$ for $x \in A$. Then, if f is continuous at a point $c \in A$ and g is continuous at $f(c) \in B$, $g \circ f$ is continuous at $c \in A$ as well.*

Proof. Given $\epsilon > 0$, there exists a $\tau > 0$ such that

$$|g(y) - g(f(c))| < \epsilon \quad \forall\, y \in B,\ |y - f(c)| < \tau.$$

With this $\tau > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(c)| < \tau \quad \forall\, x \in A,\ |x - c| < \delta.$$

Therefore, whenever $x \in A$ and $|x - c| < \delta$, it follows that $f(x) \in B$ and $|f(x) - f(c)| < \tau$, and hence

$$|g \circ f(x) - g \circ f(c)| = |g(f(x)) - g(f(c))| < \epsilon.$$

Hence, by definition, $g \circ f$ is continuous at c. □

4.3. Continuous Functions on Compact Sets and Uniform Continuity

Definition 4.4. A function $f\colon A \to \mathbf{R}$ is said to be **bounded on a set** $B \subseteq A$ if the set $f(B)$ is a bounded set. If $f(A)$ is a bounded set, we say $f\colon A \to \mathbf{R}$ is a **bounded function**.

Theorem 4.8 (**Preservation of Compact Sets**). *Let $f\colon A \to \mathbf{R}$ be continuous on A. If $K \subseteq A$ is a compact set, then $f(K)$ is compact as well.*

Proof. Given any sequence (y_n) in $f(K)$, we show that there exists a subsequence (y_{n_k}) converging to some limit $y \in f(K)$. This will prove that $f(K)$ is compact. Since $y_n \in f(K)$, we have $y_n = f(x_n)$ for some $x_n \in K$. Now (x_n) is sequence contained in K. Since K is compact, there exists a subsequence (x_{n_k}) converging to a limit $x \in K$. By the continuity, $(f(x_{n_k})) \to f(x)$. Since $y_{n_k} = f(x_{n_k})$, we have $(y_{n_k}) \to y = f(x) \in f(K)$. This proves that $f(K)$ is compact. □

Remark 4.1. In contrast to this theorem, continuous functions do not preserve bounded sets, open sets or closed sets. Consider the following examples.

(i) $f(x) = \frac{1}{x}$ is continuous in $(0, 1)$, but f is not bounded in $(0, 1)$. This function also maps the closed interval $A = [1, \infty)$ to the set $f(A) = (0, 1]$, which is not closed.

(ii) $f(x) = x^2$ maps the open set $A = (-1, 1)$ to the set $f(A) = [0, 1)$, which is not open.

Theorem 4.9 (**Extreme Value Theorem**). *If $f : K \to \mathbf{R}$ is continuous on a compact set $K \subseteq \mathbf{R}$, then f attains its maximum and minimum values; namely, there exist numbers $x_*, x^* \in K$ such that $f(x_*) \le f(x) \le f(x^*)$ for all $x \in K$.*

Proof. We only prove the maximum case. Let $A = f(K)$. Then A is compact; hence A is bounded and closed. Let $M = \sup A$. Then $f(x) \le M$ for all $x \in K$. We now show that there exists a number $x^* \in K$ such that $f(x^*) = M$, which proves that the maximum is attained. Since $M = \sup A$, for each $n \in \mathbf{N}$, there exists a number $y_n \in A$ such that $y_n > M - \frac{1}{n}$ (which just states that $M - \frac{1}{n}$ is not an upper-bound for A). Since $y_n \le M$, this implies $(y_n) \to M$. Now, since $y_n \in A = f(K)$ and A is compact, by definition of compact sets, (y_n) has a convergent subsequence whose limit is in A, but this subsequence also converges to M; therefore, $M \in A = f(K)$, which means that there exists a number $x^* \in K$ such that $M = f(x^*)$. □

Uniform Continuity.

Definition 4.5. A function $f : A \to \mathbf{R}$ is said to be **uniformly continuous on** A if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x, y \in A$ and $|x - y| < \delta$ it follows that $|f(x) - f(y)| < \epsilon$.

Remark 4.2. We only talk about the uniform continuity of a function *on a given set* not *at a point*. From the definition, we see that every uniformly continuous function on a set A must be continuous at every point of A and so must be a continuous function on A.

Example 4.6. Show that $f(x) = 3x + 1$ is uniformly continuous on $\mathbf{R}$.

Proof. Since $f(x) - f(y) = 3(x - y)$, given each $\epsilon > 0$, letting $\delta = \epsilon/3$, it follows that whenever $|x - y| < \delta$,

$$|f(x) - f(y)| = 3|x - y| < 3\delta = \epsilon.$$

So $f(x) = 3x + 1$ is **uniformly continuous** on $\mathbf{R}$. □

Example 4.7. Is function $g(x) = x^2$ uniformly continuous on $\mathbf{R}$?

Solution. Let us suppose that $g(x) = x^2$ is uniformly continuous on $\mathbf{R}$. Then $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $|g(x) - g(y)| < \epsilon$ for all $x, y \in \mathbf{R}$ with $|x - y| < \delta$. However, if we choose (large numbers) $x = N + \delta/2$ and $y = N$ with $N \in \mathbf{N}$ and $N > 2\epsilon/\delta$, then

$$|g(x) - g(y)| = |x - y||x + y| = \frac{\delta}{2}(2N + \frac{\delta}{2}) > \delta N > 2\epsilon,$$

a contradiction; therefore, $g(x) = x^2$ is **not uniformly continuous on R**. □

By negating the definition of uniform continuity, we have the following criterion for nonuniform continuity.

Theorem 4.10 (**Sequential Criterion for Nonuniform Continuity**). *A function $f : A \to \mathbf{R}$ fails to be uniformly continuous on A if and only if there exist $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \ge \epsilon_0$.*

Example 4.8. The function $h(x) = \sin(1/x)$ is continuous at every point in the open interval $(0,1)$. Let

$$x_n = \frac{1}{2n\pi}, \quad y_n = \frac{1}{2n\pi + \frac{\pi}{2}} \quad \text{in } (0,1);$$

then $x_n - y_n \to 0$ but $|h(x_n) - h(y_n)| = 1$. So $h(x)$ is not uniformly continuous on $(0,1)$.

Theorem 4.11 (**Uniform continuity on compact sets**). *Every continuous function on a compact set K is uniformly continuous on K.*

Proof. Suppose f is not uniformly continuous on K. Then $\exists\,\epsilon_0 > 0\,\exists\, x_n, y_n \in K$ such that $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$. Since (x_n) is in K and K is compact, by definition of compact sets, there exists a convergent subsequence (x_{n_k}) converging to a number $x \in K$ as $n_k \to \infty$. We also have $(y_{n_k}) \to x$ from $|y_{n_k} - x| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x| \to 0$ as $n_k \to \infty$. Hence, by continuity, $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$ and so $f(x_{n_k}) - f(y_{n_k}) \to 0$; this contradicts with $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon_0$. □

Remark 4.3. By Theorem 4.11, the function $g(x) = x^2$ is uniformly continuous on all closed intervals $[a,b]$ but is not uniformly continuous on $\mathbf{R}$ (see Example 4.7). So, a function may be uniformly continuous on one set but not uniformly continuous on another set.

Example 4.9. Let f be continuous on $[0,\infty)$ and uniformly continuous on $[b,\infty)$ for some $b > 0$. Show that f is uniformly continuous on $[0,\infty)$.

Proof. Use the uniform continuity of f on both $[0,b]$ and $[b,\infty)$. Details are left as **Homework!** □

4.4. The Intermediate Value Theorem

Theorem 4.12 (**Intermediate Value Theorem (IVT)**). *Let $f\colon [a,b] \to \mathbf{R}$ be continuous and $f(a) \neq f(b)$. Assume L is a number between $f(a)$ and $f(b)$. Then there exists a number $c \in (a,b)$ where $f(c) = L$.*

Proof. We discuss the proof based on the AoC; other different proofs are also given in the text. To fix the idea, we assume $f(a) < L < f(b)$ (the other case $f(b) < L < f(a)$ can be treated similarly). Consider the set

$$K = \{x \in [a,b] \,:\, f(x) \leq L\}.$$

Since $a \in K$, K is nonempty; since $K \subseteq [a,b]$, K is bounded. Hence by the AoC, let $c = \sup K$. Since b is an upper-bound for K, we have $c \leq b$; since $a \in K$, we have $a \leq c$. Hence $c \in [a,b]$. We prove that $f(c) = L$. Suppose $f(c) \neq L$; then the following two cases would both reach a contradiction.

Case 1: $f(c) < L$. In this case, $c \neq b$ and hence $c < b$. Take $\epsilon = L - f(c)$ and $V_\epsilon(f(c)) = (f(c) - \epsilon, f(c) + \epsilon)$. By continuity at c, there exists a $\delta > 0$ such that $f(V_\delta(c) \cap [a,b]) \subseteq V_\epsilon(f(c))$. Take a number $\delta_0 > 0$ such that $\delta_0 < \min\{b - c, \delta\}$. Then $c + \delta_0 \in [a,b] \cap V_\delta(c)$ and so $f(c + \delta_0) \in V_\epsilon(f(c))$; that is, $f(c + \delta_0) < f(c) + \epsilon = L$. Hence $c + \delta_0 \in K$, a contradiction with $c = \sup K$.

Case 2: $f(c) > L$. In this case, $c \neq a$ and hence $c > a$. Take $\epsilon = f(c) - L$ and $V_\epsilon(f(c)) = (f(c) - \epsilon, f(c) + \epsilon)$. By continuity at c, there exists a $\delta > 0$ such that $f(V_\delta(c) \cap [a,b]) \subseteq V_\epsilon(f(c))$. Take a number $\delta_0 > 0$ such that $\delta_0 < \min\{c - a, \delta\}$; so $c - \delta_0 > a$. Hence $[c - \delta_0, c] \subseteq [a,b] \cap V_\delta(c)$ and so $f(x) \in V_\epsilon(f(c))$ for all $x \in [c - \delta_0, c]$; that is,

$f(x) > f(c) - \epsilon = L$ and hence $x \notin K$ if $x \in [c - \delta_0, c]$. This implies for all $x \in K$ it must follow that $x < c - \delta_0$ and hence $c - \delta_0$ is an upper-bound for K, which, by $c = \sup K$, would imply that $c \leq c - \delta_0$, again a contradiction. □

4.5. Sets of Discontinuity for Monotone Functions

One-sided Limits. Let $f\colon (a,b) \to \mathbf{R}$ and $c \in (a,b)$. We say $\lim\limits_{x\to c^+} f(x) = L$ if, for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever } c < x < \min\{c+\delta, b\}.$$

Similarly, we say $\lim\limits_{x\to c^-} f(x) = L$ if, for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever } \max\{a, c-\delta\} < x < c.$$

Theorem 4.13. *Let $f\colon (a,b) \to \mathbf{R}$ and $c \in (a,b)$. Then $\lim\limits_{x\to c} f(x)$ exists if and only if both $\lim\limits_{x\to c^+} f(x)$ and $\lim\limits_{x\to c^-} f(x)$ exist and are equal. In this case, all these limits are the same.*

Type of Discontinuity. Let $f\colon (a,b) \to \mathbf{R}$ and $c \in (a,b)$. If f is discontinuous at c, then we have the following three cases:

(a) (**Removable discontinuity**) $\lim\limits_{x\to c} f(x)$ exists but is not equal to $f(c)$.

(b) (**Jump discontinuity**) $\lim\limits_{x\to c^+} f(x)$ and $\lim\limits_{x\to c^-} f(x)$ both exist but are not equal.

(c) (**Essential discontinuity**) None of case (a) or (b) holds.

Definition 4.6. A function $f\colon (a,b) \to \mathbf{R}$ is said to be **increasing on** (a,b) (or **decreasing on** (a,b)) if $f(t) \leq f(s)$ (or $f(t) \geq f(s)$) for all $a < t < s < b$. A function $f\colon (a,b) \to \mathbf{R}$ is called **monotone on** (a,b) if it is either increasing on (a,b) or decreasing on (a,b).

Theorem 4.14. *The set of discontinuity of a monotone function on (a,b) is at most countable.*

Proof. Without loss of generality, assume $f\colon (a,b) \to \mathbf{R}$ is increasing. Let

$$S = \{c \in (a,b) \;:\; f \text{ is discontinuous at } c\}.$$

Assume $S \neq \emptyset$. We show that at every $c \in S$ the function f has a jump discontinuity. First show that both $\lim\limits_{x\to c^-} f(x)$ and $\lim\limits_{x\to c^+} f(x)$ exist and satisfy that

$$\lim_{x\to c^-} f(x) \leq f(c) \leq \lim_{x\to c^+} f(x).$$

(**Homework!**) Therefore, if $\lim\limits_{x\to c} f(x)$ exists, then $\lim\limits_{x\to c} f(x) = f(c)$; that is f is continuous at c. So f can not have a removable discontinuity. Hence every point $c \in S$ is a jump discontinuity of f. We now define a function $h\colon S \to \mathbf{Q}$ as follows. Given $c \in S$, since $\lim\limits_{x\to c^-} f(x) < \lim\limits_{x\to c^+} f(x)$, there exists a rational number r between $\lim\limits_{x\to c^-} f(x)$ and $\lim\limits_{x\to c^+} f(x)$. Take any such rational number r and define $h(c) = r$. This defines a function $h\colon S \to \mathbf{Q}$. We show h is one-to-one. Let $c, d \in S$ and $c < d$. We show that $h(c) < h(d)$. This follows from the inequalities:

$$h(c) < \lim_{x\to c^+} f(x) \leq \lim_{x\to d^-} f(x) < h(d).$$

(**Homework!**) Since $h\colon S \to \mathbf{Q}$ is one-to-one and $\mathbf{Q}$ is countable, it follows that S is at most countable. □