Basic Topology of R

3.1. Open and Closed Sets

We have already defined the **ϵ-neighborhood** $V_\epsilon(a)$ of a number $a \in \mathbf{R}$; namely, $V_\epsilon(a) = (a-\epsilon, a+\epsilon)$, the open interval around a.

Definition 3.1. (i) Let $a \in A$. We say a is an **interior point** of A if there exists a neighborhood $V_\epsilon(a)$ of a that is completely contained in A.

(ii) A set $O \subseteq \mathbf{R}$ is said to be **open** if every element $a \in O$ is an interior point of O.

Open intervals $(a,b), (a,\infty)$ and $(-\infty, b)$ are open sets in $\mathbf{R}$. However, intervals of the form $[a,b], (a,b]$ or $[a,b)$ are not open sets.

Theorem 3.1. (i) *If $O_\alpha \subseteq \mathbf{R}$ is open for each $\alpha \in I$, so is the union $\bigcup_{\alpha\in I} O_\alpha$.*

(ii) *If $O_1, O_2, \cdots, O_n$ are open sets, so is the intersection $\bigcap_{i=1}^n O_i$.*

Part (ii) may be false for infinite number of open sets:

$$\bigcap_{i=1}^{\infty} \left(-1 - \frac{1}{i}, 1 + \frac{1}{i}\right) = [-1, 1].$$

Proof. To prove (i), let $O = \bigcup_{\alpha\in I} O_\alpha$. Take any point $a \in O$. Then $a \in O_\alpha$ for some $\alpha \in I$. For this α, since O_α is open, there exists a neighborhood $V_\epsilon(a) \subseteq O_\alpha$ since $a \in O_\alpha$. Clearly this neighborhood $V_\epsilon(a)$ is also contained in the union O. This proves O is open.

For (ii), let $O = \bigcap_{i=1}^n O_i$. Let $a \in O$. Then $a \in O_i$ for each $i = 1, 2, \cdots, n$. Since O_i is open, there exists a neighborhood $V_{\epsilon_i}(a) \subseteq O_i$ for $i = 1, 2, \cdots, n$, where $\epsilon_i > 0$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$. Then $\epsilon > 0$ and $V_\epsilon(a) \subseteq V_{\epsilon_i}(a)$. Hence $V_\epsilon(a) \subseteq O_i$ for all $i = 1, 2, \cdots, n$. So $V_\epsilon(a) \subseteq \cap_{i=1}^n O_i = O$. Hence O is open. □

Definition 3.2. (i) A point x is called a **limit point** of a set A if, for every ϵ-neighborhood $V_\epsilon(x)$ of x, there exists a point $y \neq x$ such that $y \in V_\epsilon(x) \cap A$, that is,

$$\forall\, \epsilon > 0, \quad (V_\epsilon(x) \setminus \{x\}) \cap A \neq \emptyset.$$

A limit point is also called a **cluster point** or an **accumulation point**. The set of all limit points of A will be denoted by $L(A)$.

(ii) A point $a \in A$ is called an **isolated point** of A if a is not a limit point of A.

(iii) A set F is called a **closed set** if it contains all its limit points; that is, $L(F) \subseteq F$.

Theorem 3.2. *A point x is a limit point of set A if and only if there exists a sequence (a_n) contained in A such that $a_n \neq x$ for all $n \in \mathbf{N}$ and $(a_n) \to x$.*

Proof. Assume x is a limit point of A. For each $n \in \mathbf{N}$, there exists an element $a_n \in V_{1/n}(x)$ such that $a_n \in A$ and $a_n \neq x$. This sequence (a_n) has the required property.

Now assume (a_n) is a sequence such that $a_n \in A$, $a_n \neq x$ for all $n \in \mathbf{N}$ and $(a_n) \to x$. Given any $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that $|a_N - x| < \epsilon$. Hence $a_N \in V_\epsilon(x)$ and clearly, $a_N \neq x$. By the definition, x is a limit of A. □

Theorem 3.3. *A set F is closed if and only if the limit of every Cauchy sequence (or convergent sequence) contained in F is also an element of F.*

Proof. Let F be closed. Let (x_n) be a Cauchy sequence with $x_n \in F$. By the CC, $(x_n) \to x$. We show $x \in F$. Suppose not: $x \notin F$. Then $x_n \neq x$ for all $n \in \mathbf{N}$. By the theorem above, x is a limit point of F and hence $x \in F$, a contradiction. So $x \in F$.

Now assume that the limit of every Cauchy sequence (or convergent sequence) contained in F is also an element of F. We show F is closed. Let x be any limit point of F. Then, by the theorem above, there exists a sequence (x_n) with $x_n \in F$, $x_n \neq x$, such that $(x_n) \to x$. This implies (x_n) is a Cauchy sequence in F. Hence $x \in F$. □

EXAMPLE 3.1. (i) Each element in the set $A = \{\frac{1}{n} \,:\, n \in \mathbf{N}\}$ is an isolated point of A. Also 0 is the *only* limit point of A. Since $0 \notin A$, this set A is not closed.

(ii) **Closed intervals** $[a, \infty), (-\infty, b]$ and $[a, b]$ are closed sets.

(iii) The interval $[a, b) = \{x \in \mathbf{R} \,:\, a \leq x < b\}$ is neither open nor closed.

(iv) Every $x \in \mathbf{R}$ is a limit point of $\mathbf{Q}$; this follows from the density of $\mathbf{Q}$ in $\mathbf{R}$.

Definition 3.3. The **closure** of a set A is the union of A and the set $L(A)$ of all limit points of A. The closure of A is usually denoted by $\bar{A}$; namely $\bar{A} = A \cup L(A)$.

Theorem 3.4. *For any set A, the closure $\bar{A}$ is a closed set and is the smallest closed set containing A.*

Proof. 1. We first prove $\bar{A}$ is closed. Let a be a limit point of $\bar{A}$. We show $a \in \bar{A}$. If $a \in A$ then $a \in \bar{A}$. So assume $a \notin A$. Since a is a limit point of $\bar{A}$, there exists a sequence (x_n) with $(x_n) \to a$ and $x_n \in \bar{A}$ and $x_n \neq a$ for all $n \in \mathbf{N}$. For any $n \in \mathbf{N}$, if $x_n \in A$ define $y_n = x_n$ and hence $y_n \neq a$; if $x_n \notin A$, since $x_n \in \bar{A}$, then $x_n \in L(A)$, and in this case, define $y_n \in A$ such that $0 < |y_n - x_n| < |x_n - a|$ and hence $y_n \neq a$; such a y_n exists from the definition of limit point x_n with $\epsilon = |x_n - a| > 0$. Therefore, we obtain a sequence (y_n) with the property: $y_n \in A$, $y_n \neq a$ and

$$|y_n - a| \leq |y_n - x_n| + |x_n - a| \leq 2|x_n - a| \quad \forall\, n \in \mathbf{N}.$$

Hence $y_n \in A$, $(y_n) \to a$ and $y_n \neq a$ for all $n \in \mathbf{N}$. By the theorem above, this shows that a is a limit point of A; hence $a \in \bar{A}$. We have proved that $\bar{A}$ is closed.

2. Clearly $\bar{A}$ contains A. To show that $\bar{A}$ is the smallest closed set containing A, assume B is any closed set containing A and we want to show $\bar{A} \subseteq B$. Let $x \in \bar{A}$ and we show $x \in B$. If $x \in A$ then $x \in B$. Assume $x \in L(A)$. Then $\exists\, x_n \in A$, $x_n \neq x$ such that $(x_n) \to x$. Since $x_n \in B$, this shows that x is also a limit point of B (this actually shows that *if $A \subseteq B$, then $L(A) \subseteq L(B)$*). Since B is closed, we have $x \in B$. So $\bar{A} \subseteq B$. □

Corollary 3.5. *If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.*

Proof. Let $A \subseteq B$. Then $A \subseteq \bar{B}$. So $\bar{B}$ is a closed set containing A; hence, by Theorem 3.4, $\bar{A} \subseteq \bar{B}$. □

Complements. As above, given a set $A \subseteq \mathbf{R}$, denote its **complement** A^c by

$$A^c = \mathbf{R} \setminus A = \{x \in \mathbf{R} : \; x \notin A\}.$$

Theorem 3.6. *A set O is open if and only if its complement $O^c = \mathbf{R} \setminus O$ is closed. Likewise, a set F is closed if and only if its complement $F^c = \mathbf{R} \setminus F$ is open.*

Proof. 1. Assume O is open; we show that $F = O^c$ is closed. Let $x \in L(F)$. Suppose $x \notin F$. Then $x \in O$ and hence $V_\epsilon(x) \subset O$ for some $\epsilon > 0$. This implies that $V_\epsilon(x) \cap F = \emptyset$, contradicting $x \in L(F)$.

2. Assume F is closed; we show that $O = F^c$ is open. Let $a \in O$; then $a \notin F$. Since F is closed, $a \notin L(F)$; hence $\exists\, \epsilon > 0$ such that $(V_\epsilon(a) \setminus \{a\}) \cap F = \emptyset$. Since $a \notin F$, it follows that $V_\epsilon(a) \cap F = \emptyset$ and hence $V_\epsilon(a) \subset F^c = O$. Hence O is open. □

Theorem 3.7. (i) *The union of a finite collection of closed sets is closed.*

(ii) *The intersection of an arbitrary collection of closed sets is closed.*

Proof. Use **De Morgan's Laws** and Theorem 3.1. □

3.2. Compact Sets

Definition 3.4. A set $K \subseteq \mathbf{R}$ is called **compact** if every sequence in K has a subsequence that converges to a limit that is also in K.

Theorem 3.8 (Heine-Borel Theorem (HBT)). *A set $K \subseteq \mathbf{R}$ is compact if and only if K is bounded and closed.*

Proof. First let K be compact and we show that K is bounded and closed. Assume first, for contradiction, K is not bounded. This means that, for every number $n \in \mathbf{N}$, there exists a $x_n \in K$ such that $|x_n| > n$. Now, since K is compact, the sequence (x_n) in K has a subsequence, say (x_{n_k}), converging to a limit $x \in K$. However, since $|x_{n_k}| > n_k \geq k$, this convergent subsequence is not bounded, contradicting the result that every convergent sequence be bounded. So K must be bounded. Now we show K is closed; that is, K contains all its limit points. Assume x is a limit point of K. Then, there exists a sequence (x_n), with $x_n \in K$ and $x_n \neq x$, such that $(x_n) \to x$. Since K is compact, (x_n) has a convergent subsequence whose limit is in K; however, since (x_n) converges, any convergent subsequence must have the same limit as (x_n), which is x. So $x \in K$. Hence K is closed.

The proof of the converse statement is easier. For example, assume K is closed and bounded. Let (x_n) be a sequence in K. We show that (x_n) has a subsequence converging to some number in K. Since (x_n) is bounded, by the BW, there exists a subsequence (x_{n_k}) converging to some number $x \in \mathbf{R}$. Then (x_{n_k}) is a Cauchy sequence in K. Since K is closed, by Theorem 3.3 above, every Cauchy sequence in K converges to some number in K; hence $x \in K$. By the definition of compact sets, K is compact. □

Example 3.2. Let $K \subseteq \mathbf{R}$ be compact. Show that both $\sup K$ and $\inf K$ are in K; that is, $\max K$ and $\min K$ both exist.

Proof. Exercises! □

Theorem 3.9. *If* $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ *is a nested sequence of nonempty compact sets, then the intersection* $\cap_{n=1}^{\infty} K_n$ *is nonempty.*

Proof. For each $n \in \mathbf{N}$, since K_n is nonempty, select an element $x_n \in K_n$. Since $x_n \in K_1$ and K_1 is compact, it follows that (x_n) has a subsequence (x_{n_k}) converging to some $x \in K_1$. We show that this x in fact belongs to every K_n for $n \in \mathbf{N}$. Given a particular $n_0 \in \mathbf{N}$, since $n_k \geq k$, we have $n_k \geq n_0$ for all $k \geq n_0$. We select a subsequence of (x_{n_k}) consisting of terms with $k \geq n_0$; then this subsequence also converges to x and each of its terms is also in the compact set K_{n_0}. Hence the limit $x \in K_{n_0}$. But n_0 is arbitrary; so $x \in \cap_{n=1}^{\infty} K_n$ and hence $\cap_{n=1}^{\infty} K_n \neq \emptyset$. □