Sequences and Series

2.1. The Limit of a Sequence

Definition 2.1. A **sequence** is a function whose domain is $\mathbf{N}$. If this function is denoted by f, then the values $f(n)$ $(n \in \mathbf{N})$ determine the sequence uniquely, and vise-versa. Therefore, a sequence is usually denoted by

$$(a_1, a_2, a_3, a_4, \cdots) \quad \text{or} \quad (a_n)_{n=1}^{\infty},$$

where $a_n = f(n)$ for $n \in \mathbf{N}$.

Throughout this course we only study sequences of real numbers; namely functions $f : \mathbf{N} \to \mathbf{R}$.

Example 2.1. Each of the following are common ways to describe a sequence.

(i) $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$,

(ii) $(\frac{n}{n+1})_{n=1}^{\infty}$,

(iii) (a_n), where $a_n = 2^n$ for all $n \in \mathbf{N}$,

(iv) (x_n), where $x_1 = 2$ and $x_{n+1} = \frac{x_n+1}{2}$. This is the induction way or recursion way to define a sequence.

Example 2.2. Notice the difference between a sequence (a_n) and a set $\{a_n \,:\, n \in \mathbf{N}\}$:

$((-1)^n)_{n=1}^{\infty} = (-1, 1, -1, 1, -1, 1, \cdots)$ is a sequence, having infinitely many terms (which can have repeated values);

$\{(-1)^n \,:\, n \in \mathbf{N}\} = \{1, -1\}$ is simply a set of two elements, not a countable set nor a sequence;

$(c) = (c, c, c, c, \cdots)$ is the constant sequence; $\{c\}$ is the set of single element c.

Definition 2.2 (Convergence of a Sequence)**.** A sequence (a_n) is said to **converge** to a real number a (called the **limit** of the sequence) if, for every number $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$. In this case we write either $\lim a_n = a$ or $a_n \to a$.
If a sequence (a_n) does not converge to any real number, we say (a_n) **diverges**.

Given a real number $a \in \mathbf{R}$ and positive number $\epsilon > 0$ the interval $V_\epsilon(a) = (a - \epsilon, a + \epsilon)$ is the ϵ-**neighborhood** of a.

A topological way to say $\lim a_n = a$ is the following: Given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a place in the sequence after which all of the terms are in $V_\epsilon(a)$.

Easy Fact: $\lim(c) = c$ for all constant sequences (c).

Quantifiers. The definition of $\lim a_n = a$ quantifies the closeness of a_n to a by an arbitrarily given $\epsilon > 0$ and the truth of this closeness for all terms after a term a_N. It is often the case the smaller ϵ is the larger N is needed to be. However the heart of the matter in this definition is that no matter how small $\epsilon > 0$ is there always exists such an integer N validating the requirement.

TEMPLATE FOR A PROOF OF $\lim a_n = a$

- Let $\epsilon > 0$ be arbitrary (not 1 or $\frac{1}{1000^5}$, but arbitrary; no numerical values are known).
- Try to solve the inequality $|a_n - a| < \epsilon$ to determine how to choose an $N \in \mathbf{N}$ so that this inequality holds for all $n \geq N$. This step usually requires the most work, almost of all of which is done prior to actually writing the formal proof.
- Now show that the N found actually works; namely for all $n \geq N$ the inequality $|a_n - a| < \epsilon$ indeed holds.

EXAMPLE 2.3. Show

$$\lim(\frac{n+1}{n}) = 1.$$

Proof. Let $a_n = \frac{n+1}{n}$ and $a = 1$. Then the inequality

$$|a_n - a| = \left|\frac{n+1}{n} - 1\right| = \frac{1}{n} < \epsilon$$

is the same as $n > \frac{1}{\epsilon}$. The existence of $N \in \mathbf{N}$ can be deduced by the AP(i): there always exists an $N \in \mathbf{N}$ such that $N > \frac{1}{\epsilon}$. The actual proof goes as follows.

Let $\epsilon > 0$ be arbitrary. By the AP(i), there exists an $N \in \mathbf{N}$ such that $N > \frac{1}{\epsilon}$. Then whenever $n \in N$ we have $1/n \leq 1/N < \epsilon$ and hence

$$|a_n - a| = \left|\frac{n+1}{n} - 1\right| = \frac{1}{n} < \epsilon.$$

Therefore, by definition, $\lim a_n = 1$. □

2.2. The Algebraic and Order Limit Theorems

Definition 2.3. A sequence (a_n) is **bounded above** (or **bounded below**) if there exists a number M such that $a_n \leq M$ (or $a_n \geq M$) for $n \in \mathbf{N}$. A sequence (a_n) is **bounded** if it is both bounded above and bounded above; namely, there exists a number $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbf{N}$.

Theorem 2.1. *Every convergent sequence is bounded.*

Proof. Let $(a_n) \to a$. Then there exists an $N \in \mathbf{N}$ such that

$$|a_n - a| < 1 \quad \forall\, n \geq N.$$

Hence, by the triangle inequality, $|a_n| = |(a_n - a) + a| \leq |a_n - a| + |a| \leq |a| + l$ for all $n \geq N$. Now let

$$M = \max\{|a_1|, |a_2|, \cdots, |a_{N-1}|, |a| + 1\}.$$

Then $|a_n| \le M$ for all $n \in N$. This proves (a_n) is bounded. □

Theorem 2.2 (**Algebraic Limit Theorem**). *Let* $\lim a_n = a$ *and* $\lim b_n = b$. *Then*

(i) $\lim(ca_n) = ca$, *for all* $c \in \mathbf{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim(a_n b_n) = ab$;

(iv) $\lim(a_n/b_n) = a/b$, *provided* $b_n \neq 0$ *and* $b \neq 0$.

Warning: *We can use these formulas only when the limits* $\lim a_n$ *and* $\lim b_n$ *both exist.*

Proof. Include only the product and quotient formula.
Proof of (iii): Note that

$$a_n b_n - ab = a_n b_n - a_n b + a_n b - ab = a_n(b_n - b) + (a_n - a)b.$$

Therefore, by the triangle inequality,

$$|a_n b_n - ab| \le |a_n(b_n - b)| + |(a_n - a)b| = |a_n||b_n - b| + |a_n - a||b|.$$

In order to make $|a_n b_n - ab| < \epsilon$, it suffices to make each of the two terms on the right of above inequalities $< \epsilon/2$. Since (a_n) converges, it is bounded and so $|a_n| \le M$ $(\forall\ n \in \mathbf{N})$ for some number $M > 0$. Hence the two terms are bounded as follows:

$$|a_n||b_n - b| \le M|b_n - b|, \quad |a_n - a||b| \le |a_n - a|(|b| + 1)$$

(here changing $|b|$ to $|b| + 1$ to make it positive).
Now, given arbitrary $\epsilon > 0$, since $(a_n) \to a$, we have $N_1 \in \mathbf{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2(|b| + 1)} \quad \forall \quad n \ge N_1.$$

Using $(b_n) \to b$, we have $N_2 \in \mathbf{N}$ such that

$$|b_n - b| < \frac{\epsilon}{2M} \quad \forall \quad n \ge N_2.$$

Let $N = \max\{N_1, N_2\}$ (or $N = N_1 + N_2$). Then, for this N, whenever $n \ge N$, it follows that

$$|a_n - a| < \frac{\epsilon}{2(|b| + 1)}, \quad |b_n - b| < \frac{\epsilon}{2M}$$

and hence

$$|a_n - a||b| \le \frac{\epsilon|b|}{2(|b| + 1)} < \frac{\epsilon}{2},$$

$$|a_n||b_n - b| \le M|b_n - b| \le \frac{\epsilon M}{2M} = \frac{\epsilon}{2}.$$

Finally, it follows that whenever $n \ge N$

$$|a_n b_n - ab| \le |a_n||b_n - b| + |a_n - a||b| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $(a_n b_n) \to ab$.

Proof of (iv): Note that

$$\frac{a_n}{b_n} - \frac{a}{b} = \frac{ba_n - ab_n}{b_n b} = \frac{b(a_n - a) + a(b - b_n)}{b_n b}.$$

First, using $(b_n) \to b \neq 0$, with $\epsilon = |b|/2 > 0$, there exists an $N_1 \in N$ such that $|b_n - b| < |b|/2$ for all $n \geq N_1$. Hence, by a form of the triangle inequality, $|b_n| \geq |b| - |b_n - b| \geq |b|/2$ for all $n \geq N_1$. For all such n, $|b_n b| \geq |b|^2/2$ and

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \leq \frac{|b||a_n - a|}{|b_n b|} + \frac{|a||b - b_n|}{|b_n b|}$$

$$\leq \frac{2}{|b|}|a_n - a| + \frac{2|a|}{|b|^2}|b_n - b| \leq \frac{2}{|b|}|a_n - a| + \frac{2|a|+1}{|b|^2}|b_n - b|.$$

Then we proceed as above to select N_2 and N_3 in $\mathbf{N}$ such that

$$\frac{2}{|b|}|a_n - a| < \epsilon/2 \quad \text{whenever } n \geq N_2$$

and

$$\frac{2|a|+1}{|b|^2}|b_n - b| < \epsilon/2 \quad \text{whenever } n \geq N_3.$$

Finally, let $N = \max\{N_1, N_2, N_3\}$. Then, whenever $n \geq N$, it follows that

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Theorem 2.3 (Order Limit Theorem). *Assume both* $\lim a_n = a$ *and* $\lim b_n = b$ *exist. If* $a_n \leq b_n$ *for all* $n \geq N$, *where* $N \in \mathbf{N}$ *is some number, then* $a \leq b$.

Proof. We use the proof by contradiction. Suppose $a > b$. Then $\lim(a_n - b_n) = a - b > 0$. (The following argument was used above in the proof of (iv) of the theorem.) Using $\epsilon = \frac{a-b}{2} > 0$, we have an $N \in \mathbf{N}$ such that

$$|(a_n - b_n) - (a - b)| < \epsilon = \frac{a-b}{2} \quad \forall \quad n \geq N.$$

Hence $a - b - \epsilon < a_n - b_n < a - b + \epsilon$ for all $n \geq N$. But $a - b - \epsilon = \frac{a-b}{2} > 0$; this implies that $a_n - b_n > a - b - \epsilon > 0$ for all $n \geq N$. So $a_n > b_n$ for all $n \geq N$. This is a contradiction to the assumption $a_n \leq b_n$ for all $n \in \mathbf{N}$. Hence we must have $a \leq b$. □

Example 2.4. (Exercise 2.3.2.) Let $x_n \geq 0$ for all $n \in \mathbf{N}$ and $\lim(x_n) = x$. Show $\lim(\sqrt{x_n}) = \sqrt{x}$.

Proof. We must have $x \geq 0$ by the order limit theorem. We prove the statement in two cases.

Case 1: $x = 0$. Note that $\sqrt{x_n} < \epsilon$ if and only if $x_n < \epsilon^2$.

Case 2: $x > 0$. In this case

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}.$$

□

2.3. The Monotone Convergence Theorem and a First Look at Infinite Series

Definition 2.4. A sequence (a_n) is called **increasing** if $a_n \le a_{n+1}$ for all $n \in \mathbf{N}$ and **decreasing** if $a_n \ge a_{n+1}$ for all $n \in \mathbf{N}$. A sequence is said to be **monotone** if it is either increasing or decreasing.

Theorem 2.4 (**Monotone Convergence Theorem** (**MCT**))**.** *If an increasing (or decreasing) sequence is bounded above (or below) then it converges. In fact, the limit equals the supremum (or the infimum) of the set consisting of the terms of the sequence.*

Proof. We only prove the theorem for the increasing sequence. Let (a_n) be an increasing sequence and, for some number M, $a_n \le M$ for all $n \in \mathbf{N}$. Consider the set $S = \{a_n : n \in \mathbf{N}\}$. Then S is nonempty and bounded above (with M being an upper-bound). So by the AoC, $a = \sup S$ exists. We now prove $\lim a_n = a$. Since a is an upper-bound for S,

$$a_n \le a \quad \forall\, n \in \mathbf{N}.$$

On the other hand, given arbitrary $\epsilon > 0$, since $a = \sup S$, by the Lemma before, there exists an element $a_N \in S$ such that $a - \epsilon < a_N$. Then, by the monotonicity of a_n,

$$a_n \ge a_N > a - \epsilon \quad \forall\, n \ge N.$$

Combining above inequalities, we have $a - \epsilon < a_n \le a < a_n + \epsilon$; that is, $|a_n - a| < \epsilon$ for all $n \ge N$. Hence $\lim a_n = a$. □

The MCT is useful for the study of infinite series because it asserts the convergence of a sequence without explicit mention of the actual limit; of course, without needing to checking the definition involving arbitrary $\epsilon > 0$.

Example 2.5. (Exercise 2.4.4.) Show that the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots$$

converges and find the limit.

Solution. Let a_n be the n-th term of this sequence; that is, $a_1 = \sqrt{2}, a_2 = \sqrt{2\sqrt{2}}, \cdots$. We have

$$a_{n+1} = \sqrt{2a_n}; \quad \text{hence} \quad a_{n+1}^2 = 2a_n, \quad \forall\, n = 1, 2, 3, \cdots.$$

Use induction and we can show that

$$\sqrt{2} \le a_n \le 2 \text{ and } a_n \le a_{n+1} \text{ for all } n \in \mathbf{N}.$$

Hence (a_n) is bounded and increasing. Therefore, by the MCT, $\lim a_n = a$ exists. Moreover. the order limit theorem says $\sqrt{2} \le a \le 2$. Since $\lim(a_{n+1}) = a$, taking the limit on both sides of $a_{n+1}^2 = 2a_n$, we have $a^2 = 2a$. Since $a \neq 0$, it follows that $a = 2$; that is, $\lim a_n = 2$. □

Limit Superior and Limit Inferior*. This is covered in Exercise 2.4.6.

Let (a_n) be a bounded sequence. Let

$$x_n = \inf\{a_k \,|\, k \ge n\} = \inf\{a_n, a_{n+1}, a_{n+2}, \cdots\},$$
$$y_n = \sup\{a_k \,|\, k \ge n\} = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\}.$$

Then (x_n) is increasing and bounded, and (y_n) is decreasing and bounded. Hence, by the **MCT,** both $\lim x_n = x$ and $\lim y_n = y$ exist.

Definition 2.5 (Limit superior and limit inferior)**.** Define

$$\liminf a_n = \lim x_n = x, \quad \limsup a_n = \lim y_n = y$$

to be the **limit inferior** and **limit superior** of sequence (a_n), respectively.

Theorem 2.5. *One has that* $\liminf a_n \le \limsup a_n$. *Furthermore,* $\lim a_n$ *exists if and only if* $\liminf a_n = \limsup a_n$.

Proof. 1. The first statement is easy from the **order limit theorem** and the **inequality**

$$x_n \le a_n \le y_n \quad \forall\, n \in \mathbf{N}, \tag{2.1}$$

where x_n, y_n are defined as above.

2. Assume $\liminf a_n = \limsup a_n = l$; hence $\lim x_n = \lim y_n = l$. Then, from (2.1) and the **Squeeze Theorem,** $\lim a_n = l$.

3. Now assume $\liminf a_n < \limsup a_n$. Take two numbers a, b such that $\liminf a_n < a < b < \limsup a_n$. Since $\lim x_n < a$ and x_n is increasing, it follows that $x_k < a\ \forall\, k \in \mathbf{N}$. So, as $x_k = \inf\{a_n \mid n \ge k\}$, $\forall\, k \in \mathbf{N}$, $\exists\, n_k \ge k$ such that $a_{n_k} < a$. Similarly, $\forall\, k \in \mathbf{N}$, $\exists\, m_k \ge k$ such that $a_{m_k} > b$. Hence

$$a_{m_k} - a_{n_k} > b - a \quad \forall\, k \in \mathbf{N}. \tag{2.2}$$

We show that (a_n) does not converge. For a contradiction, suppose that $(a_n) \to l$. Then, for $\epsilon = \frac{b-a}{4} > 0$, $\exists\, N \in \mathbf{N}$ such that

$$|a_n - l| < (b-a)/4 \quad \forall\, n \ge N.$$

Let $k \ge N$; then m_k, $n_k \ge k \ge N$. Hence $|a_{m_k} - l| < \frac{b-a}{4}$ and $|a_{n_k} - l| < \frac{b-a}{4}$. This implies

$$|a_{m_k} - a_{n_k}| \le |a_{m_k} - l| + |a_{n_k} - l| < (b-a)/2,$$

contradicting with (2.2) above. This completes the proof. □

A First Look at Infinite Series. Let (b_n) be a sequence. An **infinite series** of (b_n) is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \cdots.$$

The corresponding **sequence of partial sums** (s_n) is defined by

$$s_n = b_1 + b_2 + \cdots + b_n \quad \forall\, n \in \mathbf{N}.$$

We say the series $\sum_{n=1}^{\infty} b_n$ **converges** (to $B \in \mathbf{R}$) if $\lim s_n = B$. In this case, we write $\sum_{n=1}^{\infty} b_n = B$ and B is called the value or the sum of the infinite series. If a series does not converge then we say it **diverges**.

Note that if $b_n \ge 0$ then its partial sum sequence (s_n) is increasing. Therefore, in this case, to show the series to converge, by the MCT, it suffices to show that (s_n) is *bounded above.*

Example 2.6. (i) Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Note that

$$b_n = \frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} \quad \forall\, n \ge 2.$$

We have

$$s_n = b_1 + b_2 + \cdots + b_n < 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 2 - \frac{1}{n} < 2.$$

That is, (s_n) is bounded above; hence (s_n) converges, so does the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. But we do not know the value of the series.

(ii) Consider the **harmonic series**:

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

We show this series diverges by showing its partial sum sequence (s_n) is not bounded. Note

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2}.$$

We can look at s_8 to see $s_8 > 2\frac{1}{2}$. In general, look at s_{2^k} and we find that

$$\begin{aligned}
s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^k} + \cdots + \frac{1}{2^k}\right) \\
&= 1 + \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8} + \cdots + 2^{k-1} \times \frac{1}{2^k} \\
&= 1 + k \times \frac{1}{2} = \frac{k+2}{2}.
\end{aligned}$$

Hence (s_n) can not be bounded above by any number. This proves that the harmonic series diverges.

2.4. Subsequences and the Bolzano-Weierstrass Theorem

We know that every convergent sequence is bounded; however, every bounded sequence may not be convergent, e.g., $((-1)^n)$ is a bounded sequence, but diverges. But part of the sequence consists of only number 1 and, as a sequence itself, does converge. This is in fact valid for all bounded sequences. But we need to use the *subsequences*.

Definition 2.6. Let (a_n) be a sequence, and let $n_1 < n_2 < n_3 < \cdots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_k})_{k=1}^{\infty} = (a_{n_1}, a_{n_2}, a_{n_3}, \cdots)$$

is called a **subsequence** of (a_n).

Note that order of the terms in a subsequence is the same as in the original sequence.

Theorem 2.6. *Any subsequence of a convergent sequence converges to the same limit as the original convergent sequence.*

Therefore if a sequence has two subsequences converging to two distinct limits then the sequence must diverge. On the other hand, if we know a sequence converges, then we may use some of its subsequence to find the limit.

EXAMPLE 2.7. Let $0 < b < 1$. Show that $(b^n) \to 0$.

Proof. Note that
$$b > b^2 > b^3 > b^4 > \cdots > 0.$$
Hence the sequence $(a_n) = (b^n)$ is a decreasing sequence and bounded below by 0. By the MCT, $s = \lim(b^n)$ exists. By the order theorem $0 \le s \le b < 1$. We consider its subsequence $(a_{2n}) = (b^{2n})$, by the theorem above, $\lim(a_{2n}) = s$. However, $a_{2n} = b^{2n} = b_n b_n$ and hence, by the product rule,
$$s = \lim(a_{2n}) = \lim(b_n b_n) = \lim(b_n)\lim(b_n) = s \times s = s^2.$$
Hence $s(1-s) = 0$; but $s < 1$ so we have $s = 0$. Therefore $(b^n) \to 0$. □

Theorem 2.7 (**Bolzano-Weierstrass Theorem** (**BW**))**.** *Every bounded sequence contains a convergent subsequence.*

Proof. Let (a_n) be a bounded sequence; namely there exists a number $M > 0$ such that $|a_n| \le M$ for all $n \in \mathbf{N}$. Bisect the closed interval $[-M, M]$ into two closed intervals $[-M, 0]$ and $[0, M]$. Now it must be that at least one of these two intervals contains an infinite number of terms of (a_n). Select one interval for which this is the case and call it I_1. Let a_{n_1} be a term such that $a_{n_1} \in I_1$. Next, bisect I_1 into two closed intervals meeting at the mid-point of I_1. Let I_2 be one half that again contains an infinite number of terms of the original sequence. There must be a term $a_{n_2} \in I_2$ with $n_2 > n_1$ (if not, I_2 would contain only finite number of terms I_k with $k \le n_1$). Continue in this way, and we construct the closed interval I_k by taking a half of I_{k-1} that contains an infinite number of terms of (a_n) and then select a term $a_{n_k} \in I_k$, where $n_k > n_{k-1} > \cdots > n_1$. Therefore we obtain a subsequence (a_{n_k}) of (a_n). We want to show (a_{n_k}) converges. First, using the NIP, we have a point $a \in I_k$ for all $k \in \mathbf{N}$. Since both a and a_{n_k} are in I_k, it follows that $|a_{n_k} - a| \le$ the length of the interval I_k. But note that the length of I_k is $M(1/2)^{k-1}$. Hence, given any $\epsilon > 0$, there exists a $N \in \mathbf{N}$ such that for all $k \ge N$ we have $|a_{n_k} - a| < \epsilon$. This proves $\lim a_{n_k} = a$. □

Homework II. Exercises: 1.5.4, 2.2.1(b), 2.3.2, 2.4.4, 2.5.5

2.5. The Cauchy Criterion

Definition 2.7. A sequence (a_n) of real numbers is called a **Cauchy sequence** if, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $m, n \ge N$ in $\mathbf{N}$ it follows that $|a_n - a_m| < \epsilon$.

Theorem 2.8. *Every convergent sequence is a Cauchy sequence.*

Proof. Assume (a_n) is a convergent sequence with limit a. Then, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n \ge N$ in $\mathbf{N}$ it follows that $|a_n - a| < \epsilon/2$. (Notice here we use $\epsilon/2$ in place of ϵ in the definition.) Hence, whenever $n, m \ge N$ in $\mathbf{N}$, it follows by the triangle inequality that
$$|a_n - a_m| = |(a_n - a) + (a - a_m)| \le |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
Hence, by the definition, (a_n) is a Cauchy sequence. □

Theorem 2.9. *Every Cauchy sequence is bounded.*

Proof. Let (a_n) be Cauchy. Then there exists an $N \in \mathbf{N}$ such that $|a_n - a_m| < 1$ for all $n, m \in \mathbf{N}$ and $n, m \geq N$. With $m = N$ we have $|a_n - a_N| < 1$ for all $n \geq N$. Hence, by the triangle inequality, $|a_n| \leq |a_n - a_N| + |a_N| \leq |a_N| + 1$ for all $n \geq N$. Let

$$M = \max\{|a_1|, |a_2|, \cdots, |a_{N-1}|, |a_N| + 1\}.$$

Then it follows that $|a_n| \leq M$ for all $n \in \mathbf{N}$. Hence (a_n) is bounded. □

Theorem 2.10 (**Cauchy's Criterion** (**CC**))**.** *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

Proof. We have already shown that a convergent sequence is Cauchy. We need to prove that every Cauchy sequence converges. So let (a_n) be a Cauchy sequence. The previous theorem asserts that (a_n) is bounded. Hence, by the BW, (a_n) has a convergent subsequence (a_{n_k}); let $\lim a_{n_k} = a$.

We now show that the whole sequence (a_n) converges to a, using the definition of convergence. Given any $\epsilon > 0$, first since (a_n) is Cauchy, there exists an $N_1 \in \mathbf{N}$ such that

$$|a_m - a_n| < \epsilon/2 \quad \forall \ n, m \geq N_1.$$

Secondly, since $\lim a_{n_k} = a$, there exists an $N_2 \in \mathbf{N}$ such that

$$|a_{n_k} - a| < \epsilon/2 \quad \forall \quad k \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. We claim that whenever $m \geq N$ it follows that $|a_m - a| < \epsilon$; this proves that $\lim a_m = a$ and hence completes the proof. To prove this claim, assume $m \geq N$. Then $m \geq N_2$ and hence

$$|a_{n_m} - a| < \epsilon/2.$$

Also note that since $1 \leq n_1 < n_2 < n_3 < \cdots$ are natural numbers, it follows that $n_k \geq k$ for all $k \in \mathbf{N}$. Hence $n_m \geq m \geq N_1$. So

$$|a_m - a_{n_m}| < \epsilon/2.$$

Combining the previous two inequalities and using the triangle inequality again, we have

$$|a_m - a| \leq |a_m - a_{n_m}| + |a_{n_m} - a| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This completes the proof. □

Completeness Revisited. We have so far proved the NIP and the MCT from the AoC and proved the BW from NIP (hence from the AoC) and proved the CC from the BW. Each statement has its own way to characterize the property of $\mathbf{R}$ which we called the completeness. However it seems that the AoC is the center step iin deriving other statements. In fact, each of these statements (AoC, NIP, MCT, BW, CC) implies all other statements. You can try to prove these implications in Exercise 2.6.6.

2.6. Properties of Infinite Series

As above, an infinite series $\sum_{k=1}^{\infty} a_k$ is said to converge (to the sum A) if $\lim s_n = A$, where (s_n) is the sequence of the partial sums of the series defined by

$$s_n = \sum_{k=1}^{n} a_k \quad \forall \quad n \in \mathbf{N}.$$

Theorem 2.11 (**Algebraic Limit Theorems for Series**). *If* $\sum_{k=1}^{\infty} a_k = A$ *and* $\sum_{k=1}^{\infty} b_k = B$, *then*

$$\sum_{k=1}^{\infty} (ca_k + db_k) = cA + dB \quad \textit{for all } c, d \in \mathbf{R}.$$

Note that there is no similar rule for the series of product $\sum_{k=1}^{\infty}(a_k b_k)$ or quotient $\sum_{k=1}^{\infty}(a_k/b_k)$.

Theorem 2.12 (**Cauchy Criterion for Series**). *The series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if, given any* $\epsilon > 0$, *there exists an* $N \in \mathbf{N}$ *such that whenever* $n > m \geq N$ *it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Proof. Note that $s_n - s_m = a_{m+1} + a_{m+2} + \cdots + a_n$. □

Theorem 2.13. *If* $\sum_{k=1}^{\infty} a_k$ *converges, then* $(a_k) \to 0$.

Proof. $a_k = s_k - s_{k-1}$. □

This easy result is often used to show a series *diverges* by showing the sequence of terms does not converge to 0. However, it can not be used to show the convergence simply from the limit $(a_k) \to 0$, as seen from the divergent harmonic series.

Theorem 2.14 (**Comparison Test**). *Assume* (a_k) *and* (b_k) *are sequences satisfying*

$$0 \leq a_k \leq b_k \quad \forall \quad k \geq N,$$

where $N \in \mathbf{N}$ *is some integer.*

(i) If $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

(ii) If $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

Proof. Note that (i), (ii) state the same thing. Both follow from the Cauch Criterion for Series or the MCT. However, note that no information on the convergence of (larger) series $\sum_{k=1}^{\infty} b_k$ if we know that the (smaller) series $\sum_{k=1}^{\infty} a_k$ converges. □

Example 2.8. (**Geometric Series.**) A series of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots \tag{2.3}$$

is called a geometric series of **ratio** r. If $r \neq 1$ then the partial sum

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r} \quad (r \neq 1), \quad n \in \mathbf{N}.$$

The sequence (s_n) converges if and only if $|r| < 1$. Therefore the geometric series (2.3) converges if and only if its ratio r satisfies $|r| < 1$. In this case, the sum of the convergent geometric series is given by

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1 - r} \quad \text{when } |r| < 1. \tag{2.4}$$

The well-known **ratio test** and **root test** are based on comparison with geometric series; they are given in the Exercises.

Theorem 2.15 (**Absolute Convergence Test**). *If the series* $\sum_{k=1}^{\infty} |a_k|$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges as well.*

Proof. Use the CC for series. □

Definition 2.8. We say that the series $\sum_{k=1}^{\infty} a_k$ **converges absolutely** if $\sum_{k=1}^{\infty} |a_k|$ converges. We say that the series $\sum_{k=1}^{\infty} a_k$ **converges conditionally** if $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ diverges.

Theorem 2.16 (**Alternating Series Test**). *Let* (a_n) *satisfy*

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0, \quad (a_n) \to 0.$$

Then the **alternating series** $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ *converges.*

Proof. This is Exercise 2.7.1. We study the sequence of partial sums of series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$:

$$s_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots + (-1)^{n+1} a_n.$$

If $n > m \geq 1$, then $s_n - s_m = (-1)^{m+1}(a_{m+1} - a_{m+2} + \cdots + (-1)^{n-m-1} a_n)$. We claim

$$\begin{aligned} |s_n - s_m| &= |a_{m+1} - a_{m+2} + \cdots + (-1)^{n-m-1} a_n| \\ &= a_{m+1} - a_{m+2} + \cdots + (-1)^{n-m-1} a_n \\ &\leq a_{m+1}. \end{aligned} \tag{2.5}$$

The sum inside the absolue value has $k = n - m$ terms with the last term $(-1)^{n-m-1} a_n$ being $-a_n$ if k is even and being $+a_n$ if k is odd. By grouping the terms inside the absolute value by consecutive pairs, like $a_{m+1} - a_{m+2} \geq 0$, $a_{m+3} - a_{m+4} \geq 0$, etc, we can see this sum is always ≥ 0. Then, if $k = n - m$ is even, we have

$$a_{m+1} - a_{m+2} + \cdots - a_n = a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} - a_{m+5}) - \cdots - (a_{n-2} - a_{n-1}) - a_n \leq a_{m+1};$$

if $k = n - m$ is odd, we have

$$a_{m+1} - a_{m+2} + \cdots + a_n = a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} - a_{m+5}) - \cdots - (a_{n-1} - a_n) \leq a_{m+1}.$$

Since $a_{m+1} \to 0$, using (2.5), we deduce that (s_k) is Cauchy and hence the proof is done. □

Other Tests. There are other useful tests (e.g., **Dirichlet's test** and **Abel's test**) that can be proved using the **summation by parts**; see Exercises 2.7.12-14. Later we will prove **Abel's test** in applications to power series.

Example 2.9. (i) The **alternating harmonic series:**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges by the **alternating series test.** But since the series of absolute values $\sum \frac{1}{n}$ diverges, this alternating series converges **conditionally**.

(ii) The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is also a convergent alternating series, but converges **absolutely** because the absolute series $\sum \frac{1}{n^2}$ converges. So there are two tests we can use to see the convergence of this series; however, the **alternating series test** only asserts the convergence but does not tell whether the convergence is conditional or absolute.

(iii) Often, you should first try to use the **absolute convergence test**; if it does not work, try to use other tests.

Rearrangements. Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, we say $\sum_{n=1}^{\infty} b_n$ is a **rearrangement** of $\sum_{n=1}^{\infty} a_n$ if there exists a *1-1 and onto* function $f\colon \mathbf{N} \to \mathbf{N}$ such that

$$b_n = a_{f(n)} \quad \forall\ n \in \mathbf{N}.$$

Since, using the inverse function f^{-1} of f, we also have $a_n = b_{f^{-1}(n)}$ for all $n \in \mathbf{N}$, we see every term of $\sum_{n=1}^{\infty} b_n$ appears *exactly once* in $\sum_{n=1}^{\infty} a_n$ and vice-versa, every term of $\sum_{n=1}^{\infty} a_n$ appears *exactly once* in $\sum_{n=1}^{\infty} b_n$. Now if $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ both converge, is $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$?

EXAMPLE 2.10. Consider the alternating harmonic series

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

Then

$$\frac{1}{2} S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots + (-1)^{n+1} \frac{1}{2n} + \cdots .$$

So

$$S + \frac{1}{2} S = \frac{3}{2} S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

becomes a rearrangement of $S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$; but certainly they are not equal.

Theorem 2.17. *If $\sum_{n=1}^{\infty} |a_n|$ converges, then, for any rearrangement function $f\colon \mathbf{N} \to \mathbf{N}$, it follows that*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{f(n)}.$$

Proof. Let $b_k = a_{f(k)}$ for $k \in \mathbf{N}$. For $n, m \in \mathbf{N}$, define

$$s_n = a_1 + a_2 + \cdots + a_n; \quad t_m = b_1 + b_2 + \cdots + b_m.$$

Let $(s_n) \to A$. We show that $(t_m) \to A$. Given any $\epsilon > 0$, we find an $N \in \mathbf{N}$ such that

$$|s_N - A| < \epsilon/2, \qquad \sum_{k=m+1}^{n} |a_k| < \epsilon/2 \quad \forall\ n > m \geq N.$$

Since $f\colon \mathbf{N} \to \mathbf{N}$ is 1-1 and onto, let $\{i_1, i_2, \cdots, i_N\} \subseteq \mathbf{N}$ be such that $f(i_k) = k$ for each $k = 1, 2, \cdots, N$. Let

$$M = \max\{i_1, i_2, \cdots, i_N\}.$$

Then $M \geq N$. Let $m \in \mathbf{N}$ be such that $m \geq M$. Then, since $\{i_1, i_2, \cdots, i_N\} \subseteq \{1, 2, 3, \cdots, m\}$, it follows that

$$\begin{aligned} t_m = b_1 + b_2 + \cdots + b_m &= a_{f(1)} + a_{f(2)} + \cdots + a_{f(m)} \\ &= a_{f(i_1)} + a_{f(i_2)} + \cdots + a_{f(i_N)} + \sum_{j \in J} a_{f(j)} \\ &= a_1 + a_2 + \cdots + a_N + \sum_{j \in J} a_{f(j)} \\ &= S_N + \sum_{j \in J} a_{f(j)}, \end{aligned}$$

where $J = \{1, 2, 3, \cdots, m\} \setminus \{i_1, i_2, \cdots, i_N\}$. Since $J \cap \{i_1, i_2, \cdots, i_N\} = \emptyset$, we have $f(j) \geq N+1$ for all $j \in J$. Let $K = \max\{f(j) \,:\, j \in J\} \geq N+1$. Then $N+1 \leq f(j) \leq K$ for all $j \in J$ and hence

$$\left|\sum_{j\in J} a_{f(j)}\right| \leq \sum_{j\in J} |a_{f(j)}| \leq \sum_{k=N+1}^{K} |a_k| < \epsilon/2.$$

Finally, it follows that, for all $m \geq M$,

$$\begin{aligned} |t_m - A| &\leq |S_N - A| + \left|\sum_{j\in J} a_{f(j)}\right| \\ &< \epsilon/2 + \sum_{j\in J} |a_{f(j)}| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

This proves $(t_m) \to A$. □

EXAMPLE 2.11. (**Hints for some exercises**.)

#3 Assume $a_n = p_n + q_n$ and $|a_n| = p_n - q_n$. Then, at least one of the series $\sum p_n$ and $\sum q_n$ diverges if $\sum a_n$ diverges.

If $\sum a_n$ converges conditionally, then both $\sum p_n$ and $\sum q_n$ diverge. This is because $|a_n| = 2p_n - a_n$ and $|a_n| = 2q_n + a_n$; hence either convergence of $\sum p_n$ or $\sum q_n$ will imply the convergence of $\sum |a_n|$.

#5 (a) If $\sum |a_n| < \infty$, then $\sum a_n^2 < \infty$. This is because (a_n) is bounded and hence $a_n^2 \leq M|a_n|$ and then using the comparison test.

(b) If $a_n \leq 0$ and $\sum a_n < \infty$, then $\sum \sqrt{a_n}$ could diverge or converge. Take $a_n = 1/n^4$ or $a_n = 1/n^2$.

#10 If $a_n > 0$ and $\lim(na_n) = l \neq 0$, then $l > 0$ and hence $a_n > \frac{l}{2}\frac{1}{n}$ for $n \geq N$ for some $N \in \mathbf{N}$. Hence $\sum a_n$ diverges.

If $\lim(n^2 a_n) = l$, then $(n^2 a_n)$ is bounded and hence $|a_n| \leq \frac{M}{n^2}$; so $\sum a_n$ converges absolutely.