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On $W^{1,p}$-solvability for special vectorial Hamilton–Jacobi systems

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Abstract

We study the solvability of special vectorial Hamilton–Jacobi systems of the form $F(Du(x)) = 0$ in a Sobolev space. In this paper we establish the general existence theorems for certain Dirichlet problems using suitable approximation schemes called $W^{1,p}$-reduction principles that generalize the similar reduction principle for Lipschitz solutions. Our approach, to a large extent, unifies the existing methods for the existence results of the special Hamilton–Jacobi systems under study. The method relies on a new Baire's category argument concerning the residual continuity of a Baire-one function. Some sufficient conditions for $W^{1,p}$-reduction are also given along with certain generalization of some known results and a specific application to the boundary value problem for special weakly quasiregular mappings.

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1. Introduction

In this paper we study the Dirichlet problem for a special class of Hamilton–Jacobi systems of the type:

$$\begin{cases} F(Du(x)) = 0, & x \in \Omega, \\ u(x) = \varphi(x), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded open set in $\mathbf{R}^n$ and $u:\Omega \to \mathbf{R}^m$ is a unknown vector field. Here $Du(x)$ is the Jacobi matrix of u defined as an $m \times n$ matrix function by $(Du)_{ij} = \partial u^i/\partial x_j$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, and F, φ are given in the problem.

When $m = 1$ the unknown u is a scalar function and problem (1.1) becomes a special case of the time-independent equations; in this case, the notion of viscosity solutions has been successfully introduced and quite extensively studied, cf., the monograph of P.-L. Lions [8] and also [2,3].

Recently, the Hamilton–Jacobi systems for vector-valued functions have attracted a great deal of attention in studying variational problems in the calculus of variations and nonlinear elasticity and in modeling phase transition problems in materials science; cf., Dacorogna and Marcellini [4,5], Müller [9], and Müller and Šverák [10,11]. Most of the existence results for such systems have been established for solutions that are Lipschitz continuous. Two most efficient approaches have been developed largely based on a Baire's category method (cf., [4,5,15]) and on a convex integration method of Gromov [6] as initiated by Müller and Šverák in [10] (see also [11–13]). Note that both methods rely essentially on the suitable approximation schemes.

In the present paper, we study a rather weak notion of almost everywhere solutions to the Dirichlet problem (1.1) in a Sobolev space. For such solutions, the zero set of the Hamiltonian F plays a descriptive role. Therefore, we let

$$K = F^{-1}(0) = \big\{\xi \in \mathbf{M}^{m\times n} \mid F(\xi) = 0\big\},$$

where $\mathbf{M}^{m\times n}$ is the space of $m \times n$ matrices. For a vector field $u:\Omega \to \mathbf{R}^m$, we write $u \in W^{1,p}(\Omega;\mathbf{R}^m)$ if each component of u belongs to the usual Sobolev space $W^{1,p}(\Omega)$. Similarly, define $W_0^{1,p}(\Omega;\mathbf{R}^m)$ to be the closure in $W^{1,p}(\Omega;\mathbf{R}^m)$ of the class $C_0^\infty(\Omega;\mathbf{R}^m)$ of vector fields from Ω to $\mathbf{R}^m$ that are smooth and have compact support in Ω. We say two functions u, v in $W^{1,p}(\Omega;\mathbf{R}^m)$ to have the same boundary values and write $u = v$ on $\partial\Omega$ or $u|_{\partial\Omega} = v|_{\partial\Omega}$ provided that $u - v \in W_0^{1,p}(\Omega;\mathbf{R}^m)$. Given $v \in W^{1,p}(\Omega;\mathbf{R}^m)$, we denote by $v + W_0^{1,p}(\Omega;\mathbf{R}^m)$ the *Dirichlet class* of all $u \in W^{1,p}(\Omega;\mathbf{R}^m)$ having the same boundary value as v. Note that for any $u \in W^{1,p}(\Omega;\mathbf{R}^m)$ the Jacobi matrix $Du(x)$ is defined for almost every $x \in \Omega$, each element being also an $L^p(\Omega)$ function.

Definition 1.1. Let $\varphi \in W^{1,p}(\Omega;\mathbf{R}^m)$. By a $W^{1,p}$-(*almost everywhere*) *solution* to the Dirichlet problem (1.1) we mean a function $u \in \varphi + W_0^{1,p}(\Omega;\mathbf{R}^m)$ that satisfies $Du(x) \in K$ for almost every $x \in \Omega$, where $K = F^{-1}(0)$. Moreover, define the solution set to be

$$S_\varphi^p(\Omega;K) = \big\{u \in \varphi + W_0^{1,p}(\Omega;\mathbf{R}^m) \mid Du(x) \in K \text{ a.e. } x \in \Omega\big\}. \tag{1.2}$$

Although it is an ultimate goal to characterize all the boundary data $\varphi \in W^{1,p}(\Omega;\mathbf{R}^m)$ for which the solution set $S_\varphi^p(\Omega;K)$ is nonempty and to establish a well-posed selection principle that renders a unique solution in $S_\varphi^p(\Omega;K)$, as the viscosity solution does in the scalar case, at this stage, only the existence problems have been studied and the selection principles for systems seem out of reach.

In this paper, we restrict ourselves only to the (countably) *piecewise affine* boundary data.

Definition 1.2. A function $\psi \in W^{1,p}(\Omega;\mathbf{R}^m)$ is said to be (countably) *piecewise affine* on Ω if there exists a family of at most countably many disjoint open subsets Ω_j, $j = 1,2,\dots,$ of Ω such that $|\Omega\setminus\bigcup_{j=1}^{\infty}\Omega_j| = 0$ and $\psi|_{\Omega_j} = \xi_j x + b_j$. In this case, we also write $\psi = \sum_{j=1}^{\infty}(\xi_j x + b_j)\chi_{\Omega_j}$, where χ_{Ω_j} is the characteristic function of Ω_j.

Often when dealing with piecewise affine functions or other piecewise-defined functions, we need to glue the piece functions together. The following elementary result turns out useful; the proof is left for the interested reader.

Lemma 1.3. *Let $\{\Omega_j\}$ be a set of at most countably many disjoint open subsets Ω. Let $u \in W^{1,p}(\Omega;\mathbf{R}^m)$ and $u_j \in W^{1,p}(\Omega_j;\mathbf{R}^m)$ satisfy that $u_j|_{\partial\Omega_j} = u|_{\partial\Omega_j}$ for each index j. Suppose $\sum_j \|u_j\|^p_{W^{1,p}(\Omega_j)} < \infty$ if $p<\infty$ or $\sup_j \|u_j\|_{W^{1,\infty}(\Omega_j)} < \infty$ if $p = \infty$. Then the map $\tilde{u} = u\chi_{\Omega\setminus\bigcup_j\Omega_j} + \sum_j u_j\chi_{\Omega_j}$ belongs to $u + W_0^{1,p}(\Omega;\mathbf{R}^m)$.*

The Dirichlet problem (1.1) with piecewise affine boundary data φ can be reduced to the similar problems with affine boundary data $\varphi = \xi x + b$, but on different open sets. For this reason, we denote by $\beta_p(K)$ the set of matrices ξ for which the problem (1.1) has a $W^{1,p}$-solution with boundary data $\varphi = \xi x$; that is,

$$\beta_p(K) = \left\{\xi \in \mathbf{M}^{m\times n} \mid S^p_{\xi x}(\Omega;K) \neq \emptyset\right\}. \tag{1.3}$$

Note that, for $\xi \in \beta_p(K)$, if the solution set $S^p_{\xi x}(\Omega;K)$ contains a nontrivial solution $u \not\equiv \xi x$, then a typical Vitali covering argument shows that the set $S^p_{\xi x}(\Omega;K)$ must contain infinitely many solutions; this is certainly the case when $\xi \in \beta_p(K)\setminus K$. The Vitali covering argument will play an important role throughout the whole theory developed in this paper; we refer to [5] for suitable and most commonly used versions in this regard. The existence results established below often indicate that in general when $\xi \in \beta_p(K)\setminus K$ the solution set $S^p_{\xi x}(\Omega;K)$ is *dense* in some complete metric space.

In this paper we are mainly interested in the nontrivial structures of the set $\beta_p(K)$ and we shall prove certain self-enlarging properties of $\beta_p(K)$. For example, given a set $U \subset \mathbf{M}^{m\times n}$, we would like to know whether and when one can have $U \subset \beta_p(K)$. For *compact sets K*, a nearly optimal condition, known as the *reduction principle*, has been given in Müller and Sychev [12]:

Definition 1.4 [12, Definition 1.1]. Let U, K be subsets of $\mathbf{M}^{m\times n}$. We say U is *reducible* to K if for every $\xi \in U$, $\varepsilon > 0$, there exists a piecewise affine function $v \in \xi x + W_0^{1,p}(\Omega;\mathbf{R}^m)$ such that

$$Dv(x) \in U \quad \text{a.e. } x \in \Omega; \qquad \int_\Omega \operatorname{dist}\big(Dv(x);\ K\big)\,dx < \varepsilon|\Omega|, \tag{1.4}$$

where dist $(\eta; K)$ is the distance function to K defined by

$$\operatorname{dist}(\eta;K) = \inf_{\xi\in K}|\eta-\xi|.$$

The reduction principle is an approximation scheme, which gives the existence of only approximate solutions that are piecewise affine. However, such an approximation scheme turns out to be sufficient for the existence of exact solutions; the following existence theorem has been established by Müller and Sychev in [12] using the reduction principle.

Theorem 1.5 [12, Theorem 1.2]. *Let U be bounded, K compact. If U is reducible to K, then $U \subset \beta_\infty(K)$. More generally, for any piecewise affine function $\varphi \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ with $D\varphi(x) \in U \cup K$ a.e. $x \in \Omega$ and for any $\varepsilon > 0$, there exists a solution $u \in S^\infty_\varphi(\Omega; K)$ to problem* (1.1) *satisfying $\|u - \varphi\|_{L^\infty(\Omega)} < \varepsilon$.*

The proof of Theorem 1.5 given in [12] relies on constructing $W^{1,1}$-Cauchy sequences with only control of L^∞-norms. A similar idea has been also exploited in Yan [13] to deal with certain unbounded sets K.

The main purpose of this paper is to generalize this reduction principle to the case where the set K can be unbounded and solutions u can belong to the Sobolev space $W^{1,p}(\Omega; \mathbf{R}^m)$. Our new approximation scheme allows for unbounded sets K and non-affine pieces that are exact solutions and it recovers Müller and Sychev's result quoted above. Furthermore, our approach is completely different from the one used in [12], even for compact sets K; our methods rely more on a new Baire's category argument motivated by a recent work of Kirchheim [7], which is also different from the Baire category method used in [4,5,15].

We now introduce our approximation scheme, called the *$W^{1,p}$-reduction principles*.

Definition 1.6. Let $U, K \subset \mathbf{M}^{m\times n}$, and let $1 \leqslant p < \infty$.

(a) *The $W^{1,p}$-reduction principle*: Let U be bounded. We say that U is *$W^{1,p}$-reducible* to K if there exists a constant $c(p, U, K) > 0$ such that, for some bounded open set $\Omega \subset \mathbf{R}^n$ with $|\partial\Omega| = 0$ and for every $\xi \in U$, $\varepsilon > 0$, there exists a function $v \in \xi x + W^{1,p}_0(\Omega; \mathbf{R}^m)$ satisfying the conditions (i) and (ii) given below:
  (i) $v = \sum_{i\in\mathbb{N}} v_i \chi_{\Omega_i}$, where $\{\Omega_i\}$ is a family of disjoint open subsets of Ω with $|\Omega \setminus \bigcup_{i\in\mathbb{N}} \Omega| = 0$ such that

$$\int_{\Omega_i} \left|Dv_i(x)\right|^p dx \leqslant c(p, U, K)|\Omega_i|, \quad \forall i \in \mathbb{N}, \tag{1.5}$$

  and, either $v_i|_{\Omega_i} = \xi_i x + b_i$ with $\xi_i \in U$ or $Dv_i(x) \in K$ a.e. $x \in \Omega_i$;
  (ii) $\int_\Omega \operatorname{dist}(Dv(x); K)\,dx < \varepsilon|\Omega|$.

(b) *The uniform local $W^{1,p}$-reduction principle*: For any set U, we say that U is *uniformly locally $W^{1,p}$-reducible* to set K if for each $\xi \in U$ there exists a bounded set $U_\xi \subset U$, containing ξ, such that U_ξ is $W^{1,p}$-reducible to K with constant

$$c(p, U_\xi, K) \leqslant C\big(1 + |\xi|^p\big), \tag{1.6}$$

where $C = C(p, U, K) \geqslant 1$ is a uniform constant independent of ξ.

Remarks. (1) It follows from (1.5) and Jensen's inequality that the function $v \in \xi x + W_0^{1,p}(\Omega; \mathbf{R}^m)$ satisfies:

$$|\xi|^p \leqslant \frac{1}{|\Omega|} \int\limits_{\Omega} |Dv|^p \, dx \leqslant c(p, U, K). \tag{1.7}$$

Therefore, if U is $W^{1,p}$-reducible to K then $\sup_{\xi \in U} |\xi|^p \leqslant c(p, U, K)$, and hence U must be bounded.

(2) A Vitali covering argument shows that the constant $c = c(p, U, K)$ is independent of bounded sets Ω with $|\partial\Omega| = 0$. Also, if U is $W^{1,p}$-reducible to K, then the requirements in the definition hold for arbitrary bounded open sets Ω. Moreover such a function $v \in \xi x + W_0^{1,p}(\Omega; \mathbf{R}^m)$ can also be chosen to satisfy

$$\|v - \xi x\|_{L^p(\Omega)} < \delta \tag{1.8}$$

for any given $\delta > 0$.

(3) If U is *bounded* and reducible to K in the sense of Müller and Sychev (cf., Definition 1.4), then, for any $1 \leqslant p < \infty$, U is $W^{1,p}$-reducible to K, with constant $c(p, U, K) = \sup\{|\xi|^p \mid \xi \in U\}$. Moreover, for bounded sets U, uniform local $W^{1,p}$-reduction principle is equivalent to $W^{1,p}$-reduction principle.

The main result of this paper is the following existence theorem.

Theorem 1.7 (Main Theorem). *Let $1 < p < \infty$ and let U be uniformly locally $W^{1,p}$-reducible to a closed set K. Then $U \subset \beta_p(K)$. More generally, for any piecewise affine function $\varphi \in W^{1,p}(\Omega; \mathbf{R}^m)$ with $D\varphi(x) \in U \cup K$ a.e. $x \in \Omega$ and for any $\varepsilon > 0$, there exists a solution $u \in S_\varphi^p(\Omega; K)$ to problem* (1.1) *satisfying*

$$\|u - \varphi\|_{L^p(\Omega)} < \varepsilon; \qquad \int\limits_{\Omega} |Du|^p \leqslant C\left(|\Omega| + \int\limits_{\Omega} |D\varphi|^p\right), \tag{1.9}$$

where $C = C(p, U, K) \geqslant 1$ is the uniform constant in (1.6).

Remark. For bounded sets U, K, from Remark (3) of Definition 1.6 and the Sobolev embedding theorem, using $W^{1,p}$-reduction with $p > n$, we can easily see that our main theorem, Theorem 1.7, implies Theorem 1.5.

We prove our main theorem using a new approach which is quite different from that of [12,13]; the proof will be given in Section 2. Sections 3 and 4 will be devoted to several applications of this theorem where $W^{1,p}$-reduction principles can be established, including some known results obtained by using different methods.

2. Proof of the main theorem

The proof of the main theorem, Theorem 1.7, will be based on the following special case of the theorem.

Theorem 2.1. *Let $1 < p < \infty$ and let U be a bounded set which is $W^{1,p}$-reducible to a closed set K with constant $c(p,U,K)$. Then $U \subset \beta_p(K)$. Moreover, for any bounded open set $\Omega \subset \mathbf{R}^n$, and for any $\xi \in U,\ b \in \mathbf{R}^m$ and $\varepsilon > 0$, there exists a solution $u \in S^p_{\xi x+b}(\Omega; K)$ satisfying*

$$\|u - (\xi x + b)\|_{L^p(\Omega)} < \varepsilon; \quad \int_\Omega |Du|^p\, dx \leqslant c(p,U,K)|\Omega|. \tag{2.1}$$

The proof of this theorem will be given at the end of this section, but we first show this special case in fact implies the main theorem.

Proof of Theorem 1.7. Let U be uniformly locally $W^{1,p}$-reducible to K. Let $\varphi \in W^{1,p}(\Omega; \mathbf{R}^m)$ be a piecewise affine function with

$$D\varphi(x) \in U \cup K \quad \text{a.e. } x \in \Omega.$$

We write $\Omega^U = \{x \in \Omega \mid D\varphi(x) \in U\}$ and $\Omega^K = \{x \in \Omega \mid D\varphi(x) \in K \setminus U\}$. By the definition of piecewise affine functions, we can assume Ω^K and Ω^U are disjoint open sets except for a measure zero set and $|\Omega \setminus (\Omega^K \cup \Omega^U)| = 0$. Let

$$\varphi = \varphi\chi_{\Omega^K} + \sum_{i\in\mathbb{N}} (\xi_i x + b_i)\chi_{\Omega_i}, \quad \xi_i \in U; \quad \Bigl|\Omega^U \Bigm\backslash \bigcup_{i\in\mathbb{N}} \Omega_i\Bigr| = 0.$$

The fact that $\varphi \in W^{1,p}(\Omega; \mathbf{R}^m)$ implies

$$\|D\varphi\|^p_{L^p(\Omega)} = \int_{\Omega^K} |D\varphi|^p\, dx + \sum_{i\in\mathbb{N}} |\xi_i|^p |\Omega_i| < \infty. \tag{2.2}$$

By the uniform local $W^{1,p}$-reduction assumption, for each $i \in \mathbb{N}$, there exists a bounded set $U_i \subset U$, containing ξ_i, such that U_i is $W^{1,p}$-reducible to K with constant

$$c(p, U_i, K) \leqslant C\bigl(1 + |\xi_i|^p\bigr),$$

where $C = C(p,U,K) \geqslant 1$ is a constant. We apply Theorem 2.1 to U_i and K with open bounded set Ω_i to obtain a function $u_i \in S^p_{\xi_i x+b_i}(\Omega_i; K)$ satisfying

$$\begin{aligned} &\|u_i - (\xi_i x + b_i)\|^p_{L^p(\Omega_i)} < \varepsilon^p/2^i; \\ &\int_{\Omega_i} |Du_i|^p\, dx \leqslant C\bigl(1 + |\xi_i|^p\bigr)|\Omega_i|. \end{aligned} \tag{2.3}$$

Let

$$u = \varphi\chi_{\Omega^K} + \sum_{i\in\mathbb{N}} u_i \chi_{\Omega_i}.$$

Then, by Lemma 1.3, we easily have $u \in S^p_\varphi(\Omega; K)$ and, by (2.3), we also have

$$\|u - \varphi\|^p_{L^p(\Omega)} = \sum_{i\in\mathbb{N}} \|u_i - (\xi_i x + b_i)\|^p_{L^p(\Omega)} < \varepsilon^p \sum_{i\in\mathbb{N}} 1/2^i = \varepsilon^p.$$

Moreover, by (2.2), (2.3), using $C \geqslant 1$, it follows that

$$\int_{\Omega} |Du|^p\,dx = \int_{\Omega^K} |D\varphi|^p\,dx + \sum_{i\in\mathbb{N}} \int_{\Omega_i} |Du_i|^p\,dx$$
$$\leqslant \int_{\Omega^K} |D\varphi|^p\,dx + C\sum_{i\in\mathbb{N}}\big(|\Omega_i| + |\xi_i|^p|\Omega_i|\big)$$
$$\leqslant C\bigg(|\Omega| + \int_{\Omega} \big|D\varphi(x)\big|^p\,dx\bigg).$$

This completes the proof of our main theorem, Theorem 1.7. □

The rest of this section is devoted to the proof of Theorem 2.1.

Given $\xi \in U,\ b \in \mathbf{R}^m$, let V be the set of functions $v \in \xi x + b + W_0^{1,p}(\Omega;\mathbf{R}^m)$ that satisfy the condition (i) of Definition 1.6. Then the set V is nonempty since, by (1.7), $\xi x + b \in V$.

Let $\mathcal{X}$ be the closure of V in the metric space $L^p(\Omega;\mathbf{R}^m)$ with the metric defined by

$$\rho_1(f,g) = \sum_{1\leqslant i\leqslant m} \big\|f^i - g^i\big\|_{L^p(\Omega)}.$$

Then $(\mathcal{X},\rho_1)$ is a complete metric space. Furthermore, by Remark (1) of Definition 1.6, one easily has

Lemma 2.2. *One has $\mathcal{X} \subset \xi x + b + W_0^{1,p}(\Omega;\mathbf{R}^m)$. Moreover, $\forall v \in \mathcal{X}$,*

$$\int_{\Omega} \big|Dv(x)\big|^p\,dx \leqslant c(p,U,K)|\Omega|.$$

To continue the proof, we prove the following result.

Proposition 2.3. *For any $f \in \mathcal{X}$, there exists a sequence $\{f_j\}$ in V such that*

$$\|f_j - f\|_{L^p(\Omega)} \to 0, \qquad \int_{\Omega} \operatorname{dist}\big(Df_j(x);K\big)\,dx \to 0. \tag{2.4}$$

Proof. Given any $\varepsilon > 0$, since $f \in \mathcal{X}$, there exists a $v \in V$ such that

$$\|f - v\|_{L^p(\Omega)} < \varepsilon. \tag{2.5}$$

By the definition of set V, we can write $v = \sum_{i\in\mathbb{N}} v_i\chi_{\Omega_i}$ as the condition (i) of Definition 1.6. Let A be the set of indices $i \in \mathbb{N}$ for which $v_i|_{\Omega_i} = \xi_i x + b_i$ with $\xi_i \in U$ and let B be the set of indices $i \in \mathbb{N}$ for which $Dv_i(x) \in K$ a.e. $x \in \Omega_i$. For each $i \in A$, since $\xi_i \in U$ and U is $W^{1,p}$-reducible to K, by virtue of Remark (2) of Definition 1.6, there exists $w_i \in v_i + W_0^{1,p}(\Omega;\mathbf{R}^m)$ satisfying

(a) $w_i = \sum_{j\in\mathbb{N}} w_i^j \chi_{\Omega_i^j}$, where $\{\Omega_i^j\}_{j\in\mathbb{N}}$ is a family of disjoint open subsets of Ω_i with $|\Omega_i \setminus \cup_{j\in\mathbb{N}} \Omega_i^j| = 0$ such that

$$\int_{\Omega_i^j} \left|Dw_i^j(x)\right|^p dx \leqslant c(p, U, K)|\Omega_i^j|, \quad \forall j \in \mathbb{N},$$

and, either $w_i^j|_{\Omega_i^j} = \xi_j x + b_j,\ \xi_j \in U$, or $Dw_i^j(x) \in K$ a.e. $x \in \Omega_i^j$;

(b) $\int_{\Omega_i} \operatorname{dist}(Dw_i(x); K)\, dx < \varepsilon|\Omega_i|$;

(c) $\|w_i - v_i\|^p_{L^p(\Omega_i)} < \varepsilon^p/2^i$.

Let

$$u = \sum_{i\in A} w_i \chi_{\Omega_i} + \sum_{i\in B} v_i \chi_{\Omega_i}.$$

Then, from the definition of V, it follows that $u \in V$ and, by property (c) above,

$$\|u - v\|^p_{L^p(\Omega)} = \sum_{i\in A} |w_i - v_i\|^p_{L^p(\Omega_i)} < \varepsilon^p.$$

Moreover, from (b) above,

$$\int_{\Omega} \operatorname{dist}(Du; K)\, dx = \sum_{i\in A} \int_{\Omega_i} \operatorname{dist}(Dw_i; K)\, dx < \varepsilon \sum_{i\in A} |\Omega_i| < \varepsilon|\Omega|.$$

Finally, choosing $\varepsilon = 1/j$ and $f_j = u \in V$ proves the result. □

We now follow some idea in a recent work of Kirchheim [7] of using a Baire's category theorem. We refer to [1, Chapter 10] for details on the Baire's category theory for sets and functions in metric spaces.

Let $\{e_j\}$ be the standard basis of $\mathbf{R}^n$. For $h > 0$, define

$$\Omega_{j,h} = \{x \in \Omega \mid x + te_j \in \Omega,\ \forall 0 \leqslant t \leqslant h\}.$$

Then $\Omega_{j,h}$ is an open subset of Ω and for any compact set $F \Subset \Omega$, there exists $h_0 > 0$ such that $F \subset \Omega_{j,h}$ for all $0 < h < h_0$ and hence $|\Omega\setminus\Omega_{j,h}| \to 0$ as $h \to 0^+$.

Let $\mathcal{Y} \equiv L^p(\Omega; \mathbf{M}^{m\times n})$ be the metric space endowed with the L^p-metric defined by

$$\rho_2(A, B) = \sum_{1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n} \|A_{ij} - B_{ij}\|_{L^p(\Omega)}.$$

Define $T_h : (\mathcal{X}, \rho_1) \to (\mathcal{Y}, \rho_2)$ by letting

$$(T_h f)_{ij} = \begin{cases} \frac{f^i(x+he_j)-f^i(x)}{h}, & x \in \Omega_{j,h}, \\ 0, & x \notin \Omega_{j,h}. \end{cases} \tag{2.6}$$

Proposition 2.4. *For $1 < p < \infty$, $h > 0$, map $T_h : (\mathcal{X}, \rho_1) \to (\mathcal{Y}, \rho_2)$ is continuous between the two metric spaces. Moreover, $\forall f \in \mathcal{X}$, it follows $T_h f \to Df$ in $\mathcal{Y}$ as $h \to 0^+$.*

Proof. It is easy to see that for $f, g \in \mathcal{X}$ and for any $h > 0$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$,

$$\big\|(T_h f - T_h g)_{ij}\big\|_{L^p(\Omega)} \leqslant \frac{2}{h}\big\|f^i - g^i\big\|_{L^p(\Omega)}.$$

This proves $T_h : (\mathcal{X}, \rho_1) \to (\mathcal{Y}, \rho_2)$ is continuous for any $h > 0$. To show

$$\lim_{h\to 0^+} \rho_2(T_h f, Df) = 0$$

for all $f \in \mathcal{X}$, since $1 < p < \infty$, it suffices to show, for all $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$,

(a) $(T_h f)_{ij}$ converges weakly to $\partial f^i/\partial x_j$ in $L^p(\Omega)$ as $h \to 0^+$, and
(b) $\lim_{h\to 0^+} \|(T_h f)_{ij}\|_{L^p(\Omega)} = \|\partial f^i/\partial x_j\|_{L^p(\Omega)}$.

Note that, by Lemma 2.2, $\mathcal{X} \subset W^{1,p}(\Omega; \mathbf{R}^m)$. Therefore, it is easy to show that, for any $h > 0$,

$$\big\|(T_h f)_{ij}\big\|_{L^p(\Omega)} \leqslant \big\|\partial f^i/\partial x_j\big\|_{L^p(\Omega)} < \infty. \tag{2.7}$$

Using this inequality, to prove (a), it is sufficient to prove

$$\lim_{h\to 0^+} \int\limits_{\Omega_{j,h}} \frac{f^i(x + he_j) - f^i(x)}{h} \phi(x)\, dx = \int\limits_{\Omega} \frac{\partial f^i(x)}{\partial x_j} \phi(x)\, dx \tag{2.8}$$

for each $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ and any test function $\phi \in C_0^\infty(\Omega)$. Given any such ϕ, let $h > 0$ be small enough that the support of ϕ is contained in $\Omega_{j,h}$. The right-hand side of (2.8) equals $-\int_\Omega f^i \partial\phi/\partial x_j$, while the integral on the left-hand side equals $\int_\Omega f^i(x)(\phi(x - he_j) - \phi(x))/h\, dx$, which, by Lebesgue Dominated Convergence Theorem, tends to $-\int_\Omega f^i \partial\phi/\partial x_j$ as $h \to 0^+$. Hence (a) is proved. From (a) we have

$$\big\|\partial f^i/\partial x_j\big\|_{L^p(\Omega)} \leqslant \liminf_{h\to 0^+} \big\|(T_h f)_{ij}\big\|_{L^p(\Omega)},$$

which, together with (2.7), proves (b). This completes the proof. □

The following result is crucial for proving the theorem.

Proposition 2.5. *There exists a dense subset $G \subset \mathcal{X}$ such that for any $f \in G$ and any sequence $\{f_j\}$ in $\mathcal{X}$ with $\|f_j - f\|_{L^p(\Omega)} \to 0$ one has*

$$\|Df_j - Df\|_{L^p(\Omega)} \to 0.$$

Proof. Recall that a *Baire-one function* is defined to be a pointwise limit of a sequence of continuous functions between two metric spaces; cf., [1]. Proposition 2.4 asserts that the gradient operator $D : \mathcal{X} \to \mathcal{Y}$ is a Baire-one function. By a Baire's category theorem [1, Theorem 10.13], there exists a *residual set* $G \subset \mathcal{X}$, i.e., a set whose complement is of *first category* and hence itself is *dense*, such that $D : \mathcal{X} \to \mathcal{Y}$ is continuous at every $f \in G$; this continuity is exactly the conclusion of the proposition. □

Remark. Proposition 2.5 is a *reverse* Sobolev type estimate and is exactly what Müller and Sychev needed in [12] for their existence theorems; but they established this using a totally different approach.

Completion of proof of Theorem 2.1. Since K is closed, Propositions 2.3 and 2.5 imply that any $f \in G$ is a solution of

$$Df(x) \in K, \quad f - (\xi x + b) \in W_0^{1,p}\big(\Omega; \mathbf{R}^m\big).$$

Hence $G \subset S^p_{\xi x+b}(\Omega; K)$. Since G is dense in $\mathcal{X}$ and $\xi x + b \in \mathcal{X}$, we easily fulfill the first requirement of (2.1), whereas the second follows easily from Lemma 2.2. The proof of Theorem 2.1 is completed. □

3. Reduction by open lamination convex hulls

We first recall the notion of *lamination convex hulls* of sets of matrices. Given any set $K \subset \mathbf{M}^{m\times n}$, let

$$\gamma(K) = \big\{t\eta_1 + (1-t)\eta_2 \mid t \in (0,1),\ \eta_i \in K, \operatorname{rank}(\eta_1 - \eta_2) = 1\big\}. \tag{3.1}$$

Note that $\gamma(K) = \emptyset$ if K does not contain any two matrices with rank-one difference. Define $L_0(K) = K$ and inductively

$$L_{j+1}(K) = L_j(K) \cup \gamma\big(L_j(K)\big), \quad j = 0, 1, \ldots. \tag{3.2}$$

Then, define the *lamination convex hull* of K to be the set

$$\mathcal{L}(K) = K^{\mathrm{lc}} = \bigcup_{j=0}^{\infty} L_j(K). \tag{3.3}$$

Remark. From definition, $\mathcal{L}(K)$ is contained in the convex hull of K, and $\mathcal{L}(K)$ is open if K is open; moreover,

$$K \subset \mathcal{L}(K) = \mathcal{L}\big(\mathcal{L}(K)\big) \tag{3.4}$$

for any set $K \subset M^{m\times n}$.

The following important result elucidates the close relationship of lamination convex hulls with the reduction principles (or relaxation properties); we refer to Yan [13] for a detailed proof of this result.

Lemma 3.1 [13, Lemma 3.4]. *Let U be an open set in $\mathbf{M}^{m\times n}$ and let $\eta \in U$ and $\eta = t\eta_1 + (1-t)\eta_2$ with $t \in (0,1)$ and $\operatorname{rank}(\eta_1 - \eta_2) = 1$. Then, for any $\varepsilon > 0$, there exist finitely many points $\eta_3, \ldots, \eta_s$ in U and a piece-wise affine map $u \in \eta x + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ such that*

$$\begin{cases} Du(x) \in \{\eta_1, \eta_2, \eta_3, \ldots, \eta_s\} \quad \text{a.e. } x \in \Omega; \\ |\{x \in \Omega \mid Du(x) \notin \{\eta_1, \eta_2\}\}| < \varepsilon|\Omega|. \end{cases}$$

Definition 3.2. Let $A \subset \mathbf{M}^{m\times n}$ be a bounded set with nonempty interior (i.e., $\operatorname{int} A \neq \emptyset$). We say a subset B of ∂A is a *rank-one boundary set* of A provided that for each $\xi \in \operatorname{int} A$ there exist rank-one matrix η and numbers $t^- < 0 < t^+$ such that $\xi + t^\pm \eta \in B$ and $\xi + t\eta \in \operatorname{int} A$ for all $t \in (t^-, t^+)$.

Remark. It is easy to see that ∂A is itself a rank-one boundary set of A. However, later on, we shall see that there may be other rank-one boundary sets smaller than ∂A.

The following theorem provides another proof and a generalization of the result of Yan [13, Corollary 3.3].

Theorem 3.3. *Let $A \subset M^{m\times n}$ be bounded and let B be a rank-one boundary set of A when* $\operatorname{int} A \neq \emptyset$ *and let $B = \emptyset$ when* $\operatorname{int} A = \emptyset$. *Let $K = (A \cap \partial A) \cup B \subset \partial A$. If $U = \mathcal{L}(A)$ is open, then U is reducible to K. In particular, $\mathcal{L}(A) \subset \beta_\infty(\overline{K})$ if $\mathcal{L}(A)$ is open and bounded.*

Proof. Since $U = \mathcal{L}(A)$ is bounded, there exists a constant $M > 0$ such that

$$|\eta| + \operatorname{dist}(\eta; K) \leqslant M, \quad \forall \eta \in U. \tag{3.5}$$

Let $\xi \in U$ and $\varepsilon > 0$. Since $U = \mathcal{L}(A)$ is bounded and open, a repeated use of Lemma 3.1 shows that there exist two finite sets $\{\xi_1, \xi_2, \ldots, \xi_r\} \subset A \subset U$ and $\{\eta_1, \eta_2, \ldots, \eta_q\} \subset U$ and a piece-wise affine map $u \in \xi x + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ such that

$$\begin{cases} Du(x) \in \{\xi_1, \ldots, \xi_r\} \cup \{\eta_1, \ldots, \eta_q\} & \text{a.e. } x \in \Omega; \\ |\{x \in \Omega \mid Du(x) \notin \{\xi_1, \xi_2, \ldots, \xi_r\}\}| < \frac{\varepsilon|\Omega|}{2M}. \end{cases} \tag{3.6}$$

Note that this already shows that U is reducible to A and thus to $\bar{A}$, which, by our main theorem (Theorem 1.7), gives another proof of the result of Yan [13, Corollary 3.3].

The following is devoted to proving U is in fact reducible to the set $K = (A \cap \partial A) \cup B$.

If $\operatorname{int} A = \emptyset$, then we have $A \subset \partial A$ and $B = \emptyset$ and thus $K = A$; the theorem is already proved from (3.6). Therefore, we assume $\operatorname{int} A \neq \emptyset$. Let $\Omega' = \{x \in \Omega \mid Du(x) \notin \{\xi_1, \xi_2, \ldots, \xi_r\}\}$. Then $|\Omega'| < \varepsilon|\Omega|/2M$. Let I be the set of indices $i \in \{1, 2, \ldots, r\}$ for which $\xi_i \in \operatorname{int} A$ and J the set of remaining indices for which $\xi_i \in A \setminus \operatorname{int} A = A \cap \partial A$, a subset of K. We now fix $i \in I$ and let $\Omega_i = \{x \in \Omega \mid Du(x) = \xi_i\} = \bigcup_{j\in\mathbb{N}} \Omega_{ij}$, where $u = \xi_i x + b_j$ on Ω_{ij} for each $j \in \mathbb{N}$ and $\xi_i \in \operatorname{int} A$. Since B is a rank-one boundary set of A, there exist a rank-one matrix η with $|\eta| = 1$ and numbers $t^- < 0 < t^+$ such that $\xi_i + t^\pm \eta \in B \subset K$ and $\xi_i + t\eta \in \operatorname{int} A$ for all $t \in (t^-, t^+)$. Choose $0 < \delta < \min\{-t^-, t^+, \varepsilon/4\}$ and let

$$\xi_\delta^- = \xi_i + (t^- + \delta)\eta, \quad \xi_\delta^+ = \xi_i + (t^+ - \delta)\eta; \qquad t_\delta = \frac{t^+ - \delta}{t^+ - t^- - 2\delta}.$$

Then, $\xi_\delta^\pm \in \operatorname{int} A$, $\operatorname{dist}(\xi_\delta^\pm; K) \leqslant \delta$ and $\xi_i = t_\delta \xi_\delta^- + (1 - t_\delta)\xi_\delta^+ \in \operatorname{int} A$. Using Lemma 3.1, we obtain a piecewise affine map $w_j \in (\xi_i x + b_j) + W_0^{1,\infty}(\Omega_{ij}; \mathbf{R}^m)$ such that

$$\begin{cases} Dw_j(x) \in \operatorname{int} A \subset A \subset U & \text{a.e. } x \in \Omega_{ij}; \\ |\{x \in \Omega_{ij} \mid Dw_j(x) \notin \{\xi_\delta^-, \xi_\delta^+\}\}| < \frac{\varepsilon|\Omega_{ij}|}{4M}. \end{cases} \tag{3.7}$$

Define $v_i = \sum_{j\in\mathbb{N}} w_j \chi_{\Omega_{ij}} \in u + W_0^{1,p}(\Omega_i; \mathbf{R}^m)$ and let

$$v = u\chi_{\Omega'} + \sum_{i\in I} v_i \chi_{\Omega_i} + \sum_{j\in J} u\chi_{\Omega_j}.$$

Then $v \in u + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ is piecewise affine and satisfies $Dv(x) \in U = \mathcal{L}(A)$ a.e. $x \in \Omega$. Moreover, by (3.7),

$$\begin{aligned}
\int_{\Omega_i} \operatorname{dist}(Dv_i; K) &= \sum_{j\in\mathbb{N}} \int_{\Omega_{ij}} \operatorname{dist}(Dw_j; K) \\
&= \sum_{j\in\mathbb{N}} \Bigg[\int_{\{Dw_j = \xi_\delta^\pm\}} + \int_{\{Dw_j \neq \xi_\delta^\pm\}} \operatorname{dist}(Dw_j; K) \Bigg] \\
&< \sum_{j\in\mathbb{N}} \delta|\Omega_{ij}| + \sum_{j\in\mathbb{N}} M \frac{\varepsilon|\Omega_{ij}|}{4M} \\
&< \varepsilon|\Omega_i|/4 + \varepsilon|\Omega_i|/4 = \varepsilon|\Omega_i|/2.
\end{aligned}$$

Hence $\sum_{i\in I} \int_{\Omega_i} \operatorname{dist}(Dv_i; K)\,dx < \varepsilon|\Omega|/2$. On the other hand,

$$\int_{\Omega'} \operatorname{dist}(Du; K)\,dx \leqslant M|\Omega'| < \varepsilon|\Omega|/2.$$

Finally, we have

$$\begin{aligned}
\int_{\Omega} \operatorname{dist}(Dv; K)\,dx &= \int_{\Omega'} \operatorname{dist}(Du; K) + \sum_{i\in I} \int_{\Omega_i} \operatorname{dist}(Dv_i; K) \\
&< \varepsilon|\Omega|/2 + \varepsilon|\Omega|/2 = \varepsilon|\Omega|,
\end{aligned}$$

as required by (ii) of Definition 1.6. This proves U is reducible to K; the proof is completed. □

Recall that in Müller and Šverák [10] (following [6]) a sequence of sets $\{U_j\}$ is called an *in-approximation* of a set K provided the following conditions hold:

(a) $U_j \subset \mathcal{L}(U_{j+1})$, $\forall j = 1, 2, \ldots$;
(b) $\eta \in K$ whenever $\eta_j \to \eta$ and $\operatorname{dist}(\eta_j; U_j) \to 0$ for all $j \in \mathbb{N}$.

Remark. Condition (a) implies $\mathcal{L}(U_j) \subset \mathcal{L}(U_{j+1})$ for all $j \in \mathbb{N}$.

Lemma 3.4. *Let $\{U_j\}$ be an in-approximation of K and let*

$$d(\eta) = \operatorname{dist}(\eta; K), \qquad d_j(\eta) = \operatorname{dist}(\eta; U_j).$$

Then, for any $\delta > 0$ and $j \in \mathbb{N}$, there exist constants $C > 0$ and $J \in \mathbb{N}$ depending on δ, j with $J \geqslant j$ such that

$$d(\eta) \leqslant \delta\big(|\eta|^2 + 1\big) + C d_J(\eta), \quad \forall \eta \in \mathbf{M}^{m\times n}. \tag{3.8}$$

Proof. Suppose not. Then, there exist $\delta_0 > 0$ and $j_0 \in \mathbb{N}$ such that for each $j \geqslant j_0$ there exists an $\eta_j \in \mathbf{M}^{m\times n}$ verifying

$$d(\eta_j) > \delta_0\big(|\eta_j|^2 + 1\big) + j d_j(\eta_j).$$

Since $d(\eta)$ grows linearly, this inequality implies $\{\eta_j\}$ is bounded; hence we assume $\eta_j \to \eta$. The same inequality also implies $d_j(\eta_j) \to 0$. The in-approximation property thus implies $\eta \in K$ and hence $d(\eta_j) \to 0$. This contradicts with $d(\eta_j) > \delta_0$. The result is proved. □

The following theorem has been proved by Müller and Šverák [10]. We provide a different proof using mainly the reduction principle.

Theorem 3.5. *Let $\{U_j\}$ be a family of uniformly bounded open sets, which forms an in-approximation of a compact set K. Let $U = \bigcup_{j\in\mathbb{N}} \mathcal{L}(U_j)$. Then U is reducible to K. Therefore $U \subset \beta_\infty(K)$.*

Proof. The uniform boundedness of $\{U_j\}$ implies $U = \bigcup_{j\in\mathbb{N}} \mathcal{L}(U_j)$ is bounded, so we assume $|\eta| \leqslant M$ for all $\eta \in U$. Let $\xi \in U$ and $\varepsilon > 0$ be given. Assume $\xi \in \mathcal{L}(U_{j_0})$ for some $j_0 \in \mathbb{N}$. Let $\delta = \varepsilon/2(M^2+1)$ and $j = j_0$ in the lemma above. We obtain constants $C > 0$ and $J \geqslant j_0$ such that

$$d(\eta) \leqslant \delta\big(|\eta|^2 + 1\big) + C d_J(\eta), \quad \forall \eta \in \mathbf{M}^{m\times n}. \tag{3.9}$$

Since $\xi \in \mathcal{L}(U_{j_0}) \subset \mathcal{L}(U_J)$, by a similar argument as in the proof of Theorem 3.3, we have a piecewise affine map $u \in \xi x + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ with the property that

$$\begin{cases} Du(x) \in \mathcal{L}(U_J) \subset U & \text{a.e. } x \in \Omega; \\ |\{x \in \Omega \mid Du(x) \notin U_j\}| < \frac{\varepsilon|\Omega|}{4CM}. \end{cases} \tag{3.10}$$

Using (3.9) it follows that

$$\begin{aligned} \int_\Omega d\big(Du(x)\big)\,dx &\leqslant \delta \int_\Omega \big(|Du|^2 + 1\big)\,dx + C \int_\Omega d_J(Du)\,dx \\ &\leqslant \delta \int_\Omega \big(M^2 + 1\big)\,dx + C \int_{\{Du(x)\notin U_J\}} 2M\,dx \\ &\leqslant \frac{\varepsilon}{2}|\Omega| + \frac{\varepsilon}{2}|\Omega| = \varepsilon|\Omega|. \end{aligned}$$

Hence, by definition, U is reducible to K. This proves the theorem. □

A modification of proof of Theorem 3.3 also yields a sufficient condition for $W^{1,p}$-reduction; the following is some kind of self-enlarging property of the set $\beta_p(K)$. See also Yan [13, Theorem 3.2].

Theorem 3.6. *Let $K \subset M^{m\times n}$ be a closed set and let $A \subset \beta_p(K)$ be a set satisfying*

$$c_0 = \sup_{\xi\in A} \frac{1}{|\Omega|} \int_\Omega |Du_\xi|^p\,dx < \infty, \tag{3.11}$$

where $u_\xi \in S^p_{\xi x}(\Omega; K)$ *is some solution for given* $\xi \in A$. *Suppose* $U = \mathcal{L}(A)$ *is open and bounded. Then* U *is* $W^{1,p}$*-reducible to* K *with constant* $c = c(p, U, K) = \max\{c_0, \sup_{\eta \in U} |\eta|^p\}$. *Therefore,* $U = \mathcal{L}(A) \subset \beta_p(K)$.

Proof. This result has been proved in [13] and here we provide a different proof using the $W^{1,p}$-reduction principle. We adopt the proof of Theorem 3.3 up to (3.6). We then modify the piecewise affine u on the set $\{x \in \Omega \mid Du(x) \in A\}$. On each piece, say $\widetilde{\Omega}$, of this set where $u = \xi x + b$ with some $\xi \in A$ we replace u by the solution $\tilde{u} \in S^p_{\xi x+b}(\widetilde{\Omega}; K)$ obtained by a Vitali covering argument from the function $u_\xi \in S^p_{\xi x}(\Omega; K)$ given in (3.11). We keep u unchanged elsewhere. The new function so obtained satisfies the condition (i) of Definition 1.6 with constant

$$c(p, U, K) = \max\bigl\{c_0, \sup\{|\eta|^p \mid \eta \in U\}\bigr\}.$$

Clearly the new function also satisfies condition (ii) of Definition 1.6 in view of (3.6). This proves the $W^{1,p}$-reduction principle and hence the theorem follows by our main theorem, Theorem 1.7. □

4. Boundary value problem for special weakly quasiregular mappings

As a specific application of our $W^{1,p}$-reduction principle, we study the boundary value problem for certain special weakly quasiregular mappings in higher dimensions. In the following, we assume $n \geqslant 3$, $L > 1$. Let

$$K_L = \bigl\{\xi \in \mathbf{M}^{n\times n} \mid |\xi|^n = L \det \xi\bigr\}, \tag{4.1}$$

$$U_L = \bigl\{\xi \in \mathbf{M}^{n\times n} \mid |\xi|^n < L \det \xi\bigr\}, \tag{4.2}$$

where the matrix norm $|\xi|$ is defined to be the operator norm given by

$$|\xi| = \max_{h \in \mathbf{R}^n,\ |h| \leqslant 1} |\xi h|.$$

A map $u \in W^{1,p}(\Omega; \mathbf{R}^n)$ is called a *special weakly* L*-quasiregular* mapping if

$$\bigl|Du(x)\bigr|^n = L \det Du(x), \quad \text{i.e.,} \quad Du(x) \in K_L \quad \text{a.e. } x \in \Omega.$$

We are interested in the Dirichlet boundary value problem for special weakly quasiregular mappings:

$$Du(x) \in K_L \quad \text{a.e. } x \in \Omega; \qquad u \in \varphi + W^{1,p}_0\bigl(\Omega; \mathbf{R}^n\bigr). \tag{4.3}$$

If $p \geqslant n$ and $\varphi = \xi x + b$ is affine, then a necessary condition for (4.3) to have a solution is $|\xi|^n \leqslant L \det \xi$. It turns out this is also a sufficient condition.

Theorem 4.1. *Let* $p > 1$. *Then, for any* $\varepsilon > 0$ *and any piecewise affine map* $\varphi \in W^{1,p}(\Omega; \mathbf{R}^n)$ *with*

$$\bigl|D\varphi(x)\bigr|^n \leqslant L \det D\varphi(x) \quad \text{a.e. } x \in \Omega, \tag{4.4}$$

there exists a function $u \in \varphi + W_0^{1,p}(\Omega; \mathbf{R}^m)$ *such that*

$$Du(x) \in K_L \quad a.e.\ x \in \Omega; \qquad \|u - \varphi\|_{L^p(\Omega)} < \varepsilon.$$

However, condition (4.4) may not be needed for certain values of $p < n$. In fact, no such conditions are needed at all if p is not too large.

Theorem 4.2. *Let* $1 < p < \frac{nL}{L+1}$. *Then, for any* $\varepsilon > 0$ *and any piecewise affine map* $\varphi \in W^{1,p}(\Omega; \mathbf{R}^n)$, *there exists a function* $u \in \varphi + W_0^{1,p}(\Omega; \mathbf{R}^n)$ *such that*

$$Du(x) \in K_L \quad a.e.\ x \in \Omega; \qquad \|u - \varphi\|_{L^p(\Omega)} < \varepsilon.$$

Theorems 4.1 and 4.2 have been proved in Yan [14,15] using different methods. We show below that these results also follow from our main theorem by reduction principle.

First of all, we have the following result.

Theorem 4.3. (i) *For any* $1 \leqslant p < \infty$, U_L *is uniformly locally* $W^{1,p}$*-reducible to* K_L.

(ii) *For any* $1 \leqslant p < \frac{nL}{L+1}$, *the whole set* $\mathbf{M}^{n\times n}$ *is uniformly locally* $W^{1,p}$*-reducible to* K_L.

Note then that Theorems 4.1 and 4.2 follow easily from this theorem and our main theorem, Theorem 1.7.

To prove Theorem 4.3, we define the following bounded sets in $\mathbf{M}^{n\times n}$ for any $\lambda > 0$.

$$B_\lambda = \{|\xi| < \lambda\},$$
$$U^\lambda = \left\{|\xi|^n < L \det \xi < \lambda^n\right\},$$
$$P^\lambda = \left\{|\xi|^n = L \det \xi < \lambda^n\right\},$$
$$Q_\lambda = \left\{|\xi|^n = |\det \xi| < \lambda^n\right\}.$$

Lemma 4.4. $\mathcal{L}(U^\lambda) = U^\lambda$, P^λ *is a rank-one boundary set of* U^λ, *and* U^λ *is reducible to* K_L.

Proof. It is easy to see that $\gamma(U^\lambda) = U^\lambda$ and hence $L_1(U^\lambda) = U^\lambda$; this shows that $\mathcal{L}(U^\lambda) = U^\lambda$. We next show that P^λ is a rank-one boundary set of U^λ. To this end, let $\xi \in U^\lambda$; that is, $|\xi|^n < L \det \xi < \lambda^n$. By matrix polar decompositions, we find rotations $R, Q \in SO(n)$ such that

$$\xi = R \begin{pmatrix} \varepsilon_n & & & 0 \\ & \varepsilon_{n-1} & & \\ & & \ddots & \\ 0 & & & \varepsilon_1 \end{pmatrix} Q \equiv R\tilde{\xi} Q,$$

where $0 < \varepsilon_1 \leqslant \varepsilon_2 \leqslant \cdots \leqslant \varepsilon_{n-1} \leqslant \varepsilon_n$ satisfy $\varepsilon_n^n < L\varepsilon_1\varepsilon_2 \cdots \varepsilon_n < \lambda^n$. Let $\eta(t) = \tilde{\xi} + t\tilde{\eta}$, where $\tilde{\eta} = e_1 \otimes e_2$ is the rank-one matrix with the only nonzero element at $(1,2)$-position. Then it is easy to show (cf., [15]) that there exists a unique $t_0 > 0$ such that

$$\eta(\pm t_0) = \tilde{\xi} \pm t_0\tilde{\eta} \in P^\lambda, \qquad \eta(t) = \tilde{\xi} + t\tilde{\eta} \in U^\lambda, \quad \forall t \in (-t_0, t_0).$$

Now let $\eta = R\tilde{\eta}Q$ and $t^{\pm} = \pm t_0$. Then we have rank $\eta = 1$, $\xi + t^{\pm}\eta \in P^{\lambda}$ and $\xi + t\eta \in U^{\lambda}$ for all $t \in (t^-, t^+)$. This proves that P^{λ} is a rank-one boundary set of U^{λ}. Finally, using Theorem 3.3 with $A = U^{\lambda}$ and $B = P^{\lambda}$, since U^{λ} is open, and hence $K = (A \cap \partial A) \cup B = P^{\lambda}$, we have $U^{\lambda} = \mathcal{L}(U^{\lambda})$ is reducible to $K = P^{\lambda}$. Since $P^{\lambda} \subset K_L$, we have thus proved that U^{λ} is reducible to K_L. $\square$

Lemma 4.5. *$B_\lambda = \mathcal{L}(Q_\lambda)$ and, for $1 \leqslant p < \frac{nL}{L+1}$, $Q_\lambda \subset \beta_p(K_L)$; moreover, for all $\xi \in Q_\lambda$, there exists $u = u_\xi \in S^p_{\xi x}(\mathbf{B}; K_L)$ such that $\int_{\mathbf{B}} |Du_\xi|^p\,dx \leqslant C_1|\xi|^p|\mathbf{B}|$, where $\mathbf{B}$ is the unit ball in $\mathbf{R}^n$ and $C_1 = C_1(n, p, L) \geqslant 1$ is a constant. In particular, for $1 \leqslant p < \frac{nL}{L+1}$, B_λ is $W^{1,p}$-reducible to K_L with constant $c(p, B_\lambda, K_L) = C_1\lambda^p$.*

Proof. $B_\lambda = \mathcal{L}(Q_\lambda)$ follows from direct calculation (cf., [13,14]). For any $\xi \in Q_\lambda$, consider $u = u_\xi = \xi x/|x|^r$. One can select r so that, for $1 \leqslant p < \frac{nL}{L+1}$, $u_\xi \in S^p_{\xi x}(\mathbf{B}; K_L)$ and $\int_{\mathbf{B}} |Du_\xi|^p\,dx \leqslant C_1|\xi|^p|\mathbf{B}|$ for a constant $C_1 = C_1(n, p, L) \geqslant 1$ (cf., [14]). Finally, Theorem 3.6 implies B_λ is $W^{1,p}$-reducible to K_L with constant $c(p, B_\lambda, K_L) = C_1\lambda^p$. $\square$

Proof of Theorem 4.3. (i) For any $\xi \in U_L$, let $\lambda = (2L)^{1/n}|\xi| > 0$ and $U_\xi = U^{\lambda}$ defined above. Then $\xi \in U_\xi$. By Lemma 4.4, $U_\xi = U^{\lambda}$ is reducible to K_L and is thus $W^{1,p}$-reducible to K_L with constant

$$c(p, U_\xi, K_L) = \sup\bigl\{|\eta|^p \mid \eta \in U_\xi\bigr\} \leqslant \lambda^p = (2L)^{p/n}|\xi|^p.$$

Therefore, by Definition 1.6, U_L is uniformly locally $W^{1,p}$-reducible to K_L for all $p \geqslant 1$.

(ii) For any $\xi \in \mathbf{M}^{n\times n}$, let $\lambda = |\xi| + 1 > 0$ and $U_\xi = B_\lambda$ as above. Then $\xi \in U_\xi$. Let $1 \leqslant p < \frac{nL}{L+1}$. Then, by Lemma 4.5, $U_\xi = B_\lambda$ is $W^{1,p}$-reducible to K_L with constant

$$c(p, U_\xi, K_L) = C_1\lambda^p = C_1\bigl(|\xi| + 1\bigr)^p \leqslant C\bigl(|\xi|^p + 1\bigr),$$

which proves the uniform local $W^{1,p}$-reduction of $\mathbf{M}^{n\times n}$ to K_L for $1 \leqslant p < \frac{nL}{L+1}$. $\square$

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