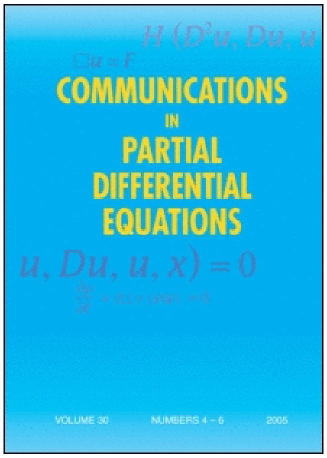

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Rellich Type Decay Theorem for Equation $P(D)u = f$ with f Supported in Infinite Cylinders

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Abstract

In this paper we prove that for a certain class of linear differential operators $P(\partial/i\partial x)$ if $P(\partial/i\partial x)\,u(x)$ has support inside a convex infinite cylinder and decays exponentially to zero in one direction of the cylinder, then $u(x)$ must have support inside the same cylinder provided that $u(x)$ satisfies a certain Rellich type decay condition at infinity. Some examples are given, including the reduced wave equation and the reduced system of crystal optics.

1 Introduction

For a large class of higher order partial differential equations

$$P((1/i)\partial/\partial x)\,u(x) = f(x) \tag{1.1}$$

on the whole space $\mathbf{R}^N$, it has been proved in Littman [10] that if $f(x)$ has compact support then $u(x)$ must have compact support if it satisfies the following Rellich type decay condition:

$$\lim_{R\to+\infty} R^{-1}\int_{R/2<|x|<R} |u(x)|^2\,dx = 0. \tag{1.2}$$

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This is an extension of a classical result of Rellich [17] about decay at infinity of solutions to the reduced wave equation

$$\Delta u(x) + u(x) = 0$$

in an exterior domain of $\mathbf{R}^N$. See also Trèves [18] for an earlier result in this direction. Further studies on the optimal decay rate at infinity for solutions to the more general equations like (1.1) have been also made in Littman [11], Hörmander [7], and Agmon and Hörmander [2].

Later on, similar results have been given for large classes of equations (1.1) in the case when $f(x)$ has support inside a closed proper cone by Littman [12], [13] and Murata and Shibata [16]. The methods in both [12] and [16] rely on analysis of the analytic Fourier–Laplace transforms in certain complex domains of $\mathbf{C}^N$ and the real Fourier transforms supported by lower dimensional algebraic varieties. An important link between the two transforms which assures the divisibility of $\hat{f}(\zeta)$ by $P(\zeta)$ is the connectedness relationship between certain complex zeros and real zeros of $P(\zeta)$. We shall explain these ideas in greater detail after we briefly discuss the problem we shall be concerned with in the present paper.

In this paper, we consider the equation

$$P((1/i)\partial/\partial x, (1/i)\partial/\partial y)\, u(x, y) = f(x, y) \tag{1.3}$$

in the whole space $\mathbf{R}^N = \mathbf{R}^n_x \times \mathbf{R}^1_y$. We assume $f(x, y)$ has support inside the cylinder $B_a \times \mathbf{R}^1$ and decays exponentially to zero in the positive direction of the cylinder (i.e., as $y \to +\infty$). Here B_a is a ball of radius $a > 0$ in $\mathbf{R}^n_x$. Our main result is that under certain conditions a solution u to (1.3) must have its support in the same cylinder if the following Rellich type decay condition holds:

$$\lim_{R\to\infty} \frac{1}{R} \int_{\mathbf{R}^1} \int_{|x|<R} |u(x, y)|^2\, dx\, dy = 0. \tag{1.4}$$

The precise statement of the result is presented in section 2 as our main theorem of the paper: Theorem 2.1. Some remarks about the assumptions of the theorem will also be given.

Note that if a tempered distribution $f = P((1/i)\partial/\partial x)u$ has support inside a proper cone K with vertex 0, its Fourier–Laplace transform $\hat{f}(\zeta)$ is analytic on the complex domain $\mathbf{R}^N + i\Gamma$, where $\Gamma \subset \mathbf{R}^N$ is the dual cone of $-K$. In this case, one has only to consider the complex zeros of $P(\zeta)$ in the domain $\mathbf{R}^N + i\Gamma$ and the properties of $\hat{f}(\zeta)$ in $\mathbf{R}^N + i\Gamma$. The divisibility of $\hat{f}(\zeta)$ by $P(\zeta)$ in $\mathbf{R}^N + i\Gamma$ is generally obtained by assuming some connectedness conditions concerning the complex zeros and real zeros of $P(\zeta)$. Then the classical Paley–Wiener theory would imply the inverse Fourier–Laplace transform of the quotient $\hat{f}(\zeta)/P(\zeta)$ gives rise to a solution of (1.1) with support inside the cone K (see [8], [12] and [16]). Consequently, a uniqueness result (see e.g. [7] and [12]) would then imply that solution u to (1.1) must have support inside K.

In the case we shall consider in the present paper, the one-sided exponential decay condition (2.4) on $f = P(D)u$ assures the analyticity of the Fourier–Laplace transform $\hat{f}(\zeta)$ in some complex "slab domain" $\mathcal{C}_\rho$ and also some estimates about the Fourier–Laplace transform $\hat{f}(\zeta)$ (see Proposition 3.1). Under the connectedness condition (C) stated in next section, the divisibility of $\hat{f}(\zeta)$ by $P(\zeta)$ in the domain $\mathcal{C}_\rho$ is obtained by using the arguments of the stationary phase method for surface-carried Fourier transforms (see [3], [8], and [10]) and the analytic continuation theory of several complex variables (see Lemmas 3.3 and 3.4). This is Theorem 4.1. It is noted that the connectedness condition (C) plays an important role in both aspects, as seen in the proof of Theorem 4.1.

Using the quotient of $\hat{f}(\zeta)$ by $P(\zeta)$ in the domain $\mathcal{C}_\rho$ a solution $v(x, y)$ to (1.3) having support inside the cylinder $B_a \times \mathbf{R}^1$ is constructed in section 5, using the general idea of a limiting absorption principle. Such a principle describes the limiting behavior of the solution operator (considered as a linear operator defined on certain function spaces) of $P(D_x, \lambda)$ as the complex "spectral" parameter λ approaches the boundary of the "resolvent" sets. We refer to [1], [2], [15], [19] and [20] for further references on limiting absorption principles. In Littman and Yan [15], we use this idea to study an elliptic boundary value problem in the complement of an infinite cylinder with general boundary conditions.

The rest of the paper is organized as follows. In section 2, we introduce some notation and state the main theorems: Theorems 2.1 and 2.2. In that section, we also give some examples as the corollaries of these theorems. Some preliminary results that are needed for proving the theorem are given in section 3. In section 4, we prove the divisibility of $\hat{f}(\zeta)$ by $P(\zeta)$ in the domain $\mathcal{C}_\rho$, i.e., Theorem 4.1. The proof of main results, Theorems 2.1 and 2.2, is given in section 5. And finally, in section 6, we make some remarks about the similar results concerning certain equations with variable coefficients.

2 Condition (C) and statement of main results

We first introduce some notation. We shall denote by $\mathbf{R}^d$ the d-dimensional real space of variables $x = (x_1, ..., x_d)$. A point in the dual space is denoted by $\xi = (\xi_1, ..., \xi_d) \in \mathbf{R}^d$. Let $D_x = (D_{x_1}, ..., D_{x_d})$, with D_{x_j} denoting $(1/i)\partial/\partial x_j$ or $(1/i)\partial_{x_j}$, where $j = 1, 2, ..., d$.

In this paper, we assume that $n \geq 2$ is an integer. We shall consider linear differential operators on $\mathbf{R}^N$. The variable in $\mathbf{R}^{n+1}$ will be denoted by $(x, y) \in \mathbf{R}^n \times \mathbf{R}^1$ and $(D_{x_1}, ..., D_{x_n}, D_y)$ by (D_x, D_y). The dual variable in $\mathbf{R}^{n+1}$ will be denoted by (ξ, σ) and its complexification will be denoted by $(\zeta, \lambda) \in \mathbf{C}^n \times \mathbf{C}^1$. Denote by $\mathcal{C}_\epsilon$ the "slab domain" of all (ζ, λ) in $\mathbf{C}^n \times \mathbf{C}^1$ with $0 < \operatorname{Im} \lambda < \epsilon$.

In what follows, we shall denote by $L^2_a(\mathbf{R}^n)$ the subspace of $L^2(\mathbf{R}^n)$ consisting of functions supported in the closed ball $\mathbf{B}^n_a = \mathbf{B}_a \equiv \{x \in \mathbf{R}^n \mid |x| \leq a\}$, where $a > 0$ is a finite number.

Let $P(\zeta, \lambda)$ be a polynomial of $n+1$ variables. We denote by $\Sigma(P)$ (resp. $S(P)$) the zero set of P in $\mathbf{C}^n \times \mathbf{C}^1$ (resp. in $\mathbf{R}^n \times \mathbf{R}^1$). Define $\Sigma_\epsilon(P) = \mathcal{C}_\epsilon \cap \Sigma(P)$, where $\mathcal{C}_\epsilon$ is the "slab domain" defined before.

Let $\nabla P = (\nabla_\zeta P, \partial_\lambda P)$ be the gradient of P, where $\nabla_\zeta P = (\partial_{\zeta_1} P, ..., \partial_{\zeta_n} P)$. Finally, define the real set $\mathbf{A}(P) \subset S(P)$ as follows

$$\mathbf{A}(P) = \{ (\xi, \sigma) \in \mathbf{R}^n \times \mathbf{R}^1 \mid P(\xi, \sigma) = 0,\ \nabla_\xi P(\xi, \sigma) \neq 0 \}. \tag{2.1}$$

We next introduce a condition on P which, roughly speaking, requires that each "nice" complex zero of $P(\zeta, \lambda)$ in the "slab" $\mathcal{C}_\epsilon$ be connected to these simple real zeros. More precisely, we introduce the following condition on polynomial $P(\zeta, \lambda)$.

Connectedness Condition (C): *There exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$, the closure of each connected component of the set*

$$\mathcal{A}_\epsilon(P) = \Sigma_\epsilon(P) \cap \{ (\zeta, \lambda) \in \mathbf{C}^N \mid \nabla P(\zeta, \lambda) \neq 0 \} \tag{2.2}$$

intersects the set $\mathbf{A}(P)$ defined by (2.1).

Remarks. In many cases, this condition (C) can be verified by finding a continuous path lying in the closure of set $\mathcal{A}_\epsilon(P)$ that connects any given point there to a point in $\mathbf{A}(P)$. For example, if $(\xi^0 + i\eta^0, \sigma^0 + i\tau^0) \in \mathcal{A}_\epsilon(P)$, one can try to find a path in the form of $(\xi(t) + it\eta^0, \sigma(t) + it\tau^0)$ which lies on $\mathcal{A}_\epsilon(P)$ for $0 < t \leq 1$ and satisfies $\xi(1) = \xi^0$, $\sigma(1) = \sigma^0$ and $(\xi(0), \sigma(0)) \in \mathbf{A}(P)$.

In this paper, we consider the following differential equation

$$P(D_x, D_y)\, u(x, y) = f(x, y) \tag{2.3}$$

defined in the whole space $\mathbf{R}^n \times \mathbf{R}^1$. We shall prove the following main theorem.

Theorem 2.1 *Let $P(\zeta, \lambda) = P_1(\zeta, \lambda)^{m_1} P_2(\zeta, \lambda)^{m_2} \cdots P_r(\zeta, \lambda)^{m_r}$ and $P_j(\zeta, \lambda)$ be distinct irreducible polynomials of real coefficients and satisfy the connectedness condition (C) given above.*

Let $u(x, y) \in L^2_{loc}(\mathbf{R}^n_x; L^2(\mathbf{R}^1_y))$ and $f(x, y) \in L^2(\mathbf{R}^N)$ satisfy equation (2.3). *Suppose $f(x, y) \equiv 0$ in $|x| > a$ and satisfies for some constant $\rho > 0$ the condition*

$$\int_{\mathbf{R}^1} \int_{\mathbf{R}^n} |f(x, y)|^2\, e^{2\rho y}\, dx\, dy < \infty, \tag{2.4}$$

and $u(x,y)$ satisfies the decay condition

$$\lim_{R\to\infty} \frac{1}{R} \int_{\mathbf{R}^1} \int_{|x|<R} |u(x,y)|^2 \, dx \, dy = 0. \tag{2.5}$$

Then $u(x,y) \equiv 0$ in $|x| > a$ and $y \in \mathbf{R}^1$.

We now discuss the system of differential equations with constant coefficients. We consider the system

$$\sum_{j=1}^{m} P_{kj}(D_x, D_y) \, u_j(x,y) = f_k(x,y) \tag{2.6}$$

in $\mathbf{R}^n_x \times \mathbf{R}^1_y$, where $P_{kj}(D_x, D_y)$ with $k,\ j = 1, ..., m$ are differential operators with constant coefficients.

For simplicity, we shall rewrite (2.6) in terms of matrix. To this end, let us introduce $\vec{u} = (u_1, u_2, ..., u_m)^T$ and $\vec{f} = (f_1, f_2, ..., f_m)^T$ to be column vectors and define the matrix of the system (2.6) by $\mathcal{M}(\zeta, \lambda) = (P_{kj}(\zeta, \lambda))$. Then (2.6) can be rewritten as

$$\mathcal{M}(D_x, D_y) \, \vec{u}(x,y) = \vec{f}(x,y) \tag{2.7}$$

in $\mathbf{R}^n_x \times \mathbf{R}^1_y$. We now present a result similar to Theorem 2.1 for systems.

Theorem 2.2 *Let $Q(\zeta, \lambda) = \det \mathcal{M}(\zeta, \lambda)$ be the determinant of matrix $\mathcal{M}$. Suppose that each irreducible factor of $Q(\zeta, \lambda)$ has real coefficients up to a complex constant factor and satisfies the connectedness condition (C). Let $\vec{u}$ and $\vec{f}$ satisfy the system (2.7), and let each component u_j and f_j satisfy (2.5) and (2.4) given in Theorem 2.1, respectively. Then $\vec{u}(x,y) \equiv 0$ in $|x| > a$ and $y \in \mathbf{R}^1$.*

The proof of these theorems will be given later in section 5. We first make several remarks about the conditions in Theorem 2.1.

Remarks. (i) The one-sided exponential decay condition (2.4) on the function f is necessary for the conclusion of Theorem 2.1 to hold. To see this, we take the inhomogeneous Helmholtz equation

$$\Delta_x u(x,y) + \partial_y^2 u(x,y) + u(x,y) = f(x,y) = g(x) \, h(y) \tag{2.8}$$

with $g \in C_0^\infty(\mathbf{R}^n)$. Now if $h(y)$ is chosen so that its Fourier transform $\hat{h}(\sigma)$ belongs to C_0^∞, vanishing for $\sigma \notin [2,3]$, then by using the Fourier transformation we can construct a classical solution $u \in L^2(\mathbf{R}^{n+1})$ which does not vanish outside any finite cylinder in the y-direction. Clearly this $f(x,y)$ does not satisfy condition (2.4).

(ii) Condition (2.4) is satisfied if $f(x,y)$ is supported in a semi-infinite cylinder in the negative y-direction, i.e., $f(x,y) = 0$ when $|x| > a$ or $y > b$ for some constant

b. Our result only asserts that (under all other assumptions in the theorem) the solution $u(x, y)$ is supported in the whole cylinder $B_a \times \mathbf{R}^1$.

(iii) Other type connectedness conditions similar to those given as in [12] or [16] would imply the support of $u(x, y)$ is contained in some cones. But our connectedness condition here is different and in some sense weaker. However, in this special case when f is supported in the convex semi-infinite cylinder, by our theorem, the solution u would be supported in the same semi-infinite cylinder if the operator $P(D_x, D_y)$ has a weak unique continuation property.

We now give some examples where our main theorems can yield some new results. We state these examples as the corollaries of the main theorems.

Corollary 2.3 *Let $u(x, y)$ and $f(x, y)$ satisfy the conditions in the previous theorem and also satisfy one of the following classical equations on $\mathbf{R}^n_x \times \mathbf{R}^1_y$ with $n \geq 2$:*

(1) The Helmholtz equation: $\Delta_x u + \partial_y^2 u + k^2 u = f, (k > 0)$;

(2) The Klein–Gordon equation: $\Delta_x u - \partial_y^2 u + ku = f$, (k real);

(3) The Schrödinger equation: $\Delta_x u - (1/i)\partial_y u + ku = f$, (k real).

Then the conclusion of Theorem 2.1 holds.

Proof. Note that the polynomials corresponding to the given equations are all irreducible and have real coefficients. Thus the corollary will follow if one can verify the Connectedness Condition (C) for these polynomials. This can be done by using the method mentioned in the above remark. We only consider the Helmholtz equation, the other two cases can be done similarly.

In the case of Helmholtz equation, the corresponding polynomial is

$$P(\zeta, \lambda) = -\zeta_1^2 - \cdots - \zeta_n^2 - \lambda^2 + k^2.$$

Note that $P(\zeta, \lambda)$ is irreducible and $\nabla P(\zeta, \lambda) \neq 0$ as long as $\lambda \neq 0$. Also, the set $\mathbf{A}(P)$ defined by (2.1) is now given as follows

$$\mathbf{A}(P) = \{ (\xi, \sigma) \in \mathbf{R}^n \times \mathbf{R}^1 \mid |\xi|^2 + |\sigma|^2 = k^2, \ |\sigma| \neq k \}.$$

Let $\epsilon > 0$ and $(\xi^0 + i\eta^0, \sigma^0 + i\tau^0) \in \mathcal{A}_\epsilon(P)$ be given. Define

$$\begin{cases} \xi(t) = \left(\frac{t^2(|\eta^0|^2 + (\tau^0)^2) + k^2}{|\eta^0|^2 + (\tau^0)^2 + k^2} \right)^{1/2} \xi^0, \quad \eta(t) = t\eta^0, \\ \sigma(t) = -\frac{1}{\tau^0} \xi(t) \cdot \eta^0, \quad \tau(t) = t\tau^0. \end{cases} \tag{2.9}$$

Then $(\xi(t) + i\eta(t), \sigma(t) + i\tau(t))$ belongs to the set $\mathcal{A}_\epsilon(P)$ for all $0 < t \leq 1$ and $(\xi(1) + i\eta(1), \sigma(1) + i\tau(1)) = (\xi^0 + i\eta^0, \sigma^0 + i\tau^0)$ and $(\xi(0), \sigma(0)) \in \mathbf{A}(P)$. Thus the condition (C) is verified, and proof is completed.

Our second example is the reduced system of crystal optics. We start with the system of crystal optics:

$$\begin{pmatrix} -\epsilon\partial_t & \mathrm{curl} \\ -\mathrm{curl} & -\mu\partial_t \end{pmatrix} \vec{U} = \vec{F}, \tag{2.10}$$

where $\vec{U} = \vec{U}(x,t) \in \mathbf{R}^6$ is the electric and magnetic field and $\vec{F} = \vec{F}(x,t)$ is the source field. Here ϵ and μ are constant coefficient 3×3 matrices which we shall assume to be of the form

$$\epsilon = \begin{pmatrix} \epsilon_1 & & 0 \\ & \epsilon_2 & \\ 0 & & \epsilon_3 \end{pmatrix}, \quad \mu = \begin{pmatrix} \nu & & 0 \\ & \nu & \\ 0 & & \nu \end{pmatrix} \tag{2.11}$$

where ϵ_j and ν are positive constants representing the dielectric constants and the magnetic permeability constants, respectively. Here $x = (x_1, x_2, x_3)$ and the $y = x_3$-direction will be considered as the axis of the cylinder. We refer to Courant and Hilbert [5] and Liess [9] for more discussions.

First of all, we need to get the reduced system of crystal optics. To do so, suppose the source term $\vec{F}(x,t)$ is time-harmonic, i.e., $\vec{F}(x,t) = \vec{f}(x)\, e^{-ikt}$ for some real $k \neq 0$, and we are looking for the time-harmonic solutions $\vec{U}(x,t) = \vec{u}(x)\, e^{-ikt}$ of the system (2.10). Then we obtain the reduced system of crystal optics for $\vec{u}(x)$:

$$\begin{pmatrix} ik\epsilon & \mathrm{curl} \\ -\mathrm{curl} & ik\mu \end{pmatrix} \vec{u} = \vec{f},$$

which can be written as (recall $D_{x_j} = (1/i)\partial_{x_j}$)

$$\mathcal{M}_k(D_x, D_y)\, \vec{u}(x,y) = \vec{f}(x,y), \; x = (x_1, x_2), y = x_3, \tag{2.12}$$

where $\mathcal{M}_k(\zeta_1, \zeta_2, \zeta_3)$ is the 6×6 matrix given by

$$\mathcal{M}_k(\zeta) = i \begin{pmatrix} k\epsilon & C(\zeta) \\ -C(\zeta) & k\mu \end{pmatrix}, \quad C(\zeta_1, \zeta_2, \zeta_3) = \begin{pmatrix} 0 & \zeta_3 & -\zeta_2 \\ -\zeta_3 & 0 & \zeta_1 \\ \zeta_2 & -\zeta_1 & 0 \end{pmatrix}.$$

We can prove the following result.

Corollary 2.4 *Let $\vec{u}$ and $\vec{f}$ satisfy the reduced system of crystal optics* (2.12) *for a real $k \neq 0$, and let each component u_j and f_j satisfy* (2.5) *and* (2.4) *given in Theorem* 2.1, *respectively. Then $\vec{u}(x,y) \equiv 0$ in $|x| > a$ and $y \in \mathbf{R}^1$.*

Proof. The proof of this result will be given elsewhere with some applications since we think all calculations leading to the verification of conditions of Theorem 2.2 are also useful in understanding the nature of this system. It certainly deserves more attention from the point of view of applications.

In next two sections, we shall present some preliminary results that are needed for the proof of the main results: Theorems 2.1 and 2.2.

3 Notation and preliminaries

Let $f(x,y) \in L^2(\mathbf{R}^1_y;\, L^2_a(\mathbf{R}^n_x))$ be a given function which satisfies the exponential decay condition (2.4).

Define the Fourier-Laplace transform $\hat{f}(\zeta,\lambda)$ of $f(x,y)$ on $\zeta \in \mathbf{C}^n$ and $\lambda = \sigma + i\tau$ with $0 < \tau < \rho$ as follows

$$\hat{f}(\zeta,\lambda) = (2\pi)^{-\frac{n+1}{2}} \iint\limits_{\mathbf{R}^n\times\mathbf{R}^1} f(x,y)\, e^{-i(x\cdot\zeta + y\sigma)}\, e^{y\tau}\, dx\, dy. \tag{3.1}$$

Let $\hat{f}(\xi,\sigma)$ be the L^2-Fourier transform of $f(x,y) \in L^2(\mathbf{R}^1_y;\, L^2_a(\mathbf{R}^n_x))$. Since $f(x,y) \in L^2(\mathbf{R}^1_y;\, L^2_a(\mathbf{R}^n_x))$, it follows that $\hat{f}(\xi,\sigma)$ can be analytically extended to the all complex $\xi \in \mathbf{C}^n$ for almost every $\sigma \in \mathbf{R}^1$. Define also $F(x,\sigma)$ to be the L^2-Fourier transform of $f(x,y)$ with respect to $y \in \mathbf{R}^1$. Then for almost every $\sigma \in \mathbf{R}^1$, we can consider $F(x,\sigma)$ as a function in $L^2_a(\mathbf{R}^n_x)$ and define $\hat{F}(\zeta,\sigma)$ to be the Fourier-Laplace transform of $F(x,\sigma)$ defined on $\zeta \in \mathbf{C}^n$. Note that our notation here implies that for a.e. $\sigma \in \mathbf{R}^1$,

$$\hat{F}(\zeta,\sigma) = \hat{f}(\zeta,\sigma) \quad \text{for } \zeta \in \mathbf{C}^n.$$

We now have the following result.

Proposition 3.1 *The function $\hat{f}(\zeta,\lambda)$ is analytic in the slab domain C_ρ, and*

$$\lim_{\tau\to 0^+} \int_{\mathbf{R}^1}\int_{\mathbf{R}^n} |\hat{f}(\xi,\sigma+i\tau) - \hat{f}(\xi,\sigma)|^2\, d\xi\, d\sigma = 0. \tag{3.2}$$

Furthermore, for $0 \le \operatorname{Im}\lambda < \rho$ we define

$$f^*(\lambda) = \Big(\int_{\mathbf{R}^n} |\hat{f}(\xi,\lambda)|^2 d\xi\Big)^{1/2}. \tag{3.3}$$

When σ is real, $f^*(\sigma)$ is understood as a function in $L^2(\mathbf{R}^1_\sigma)$, which is well-defined. Then it follows that for any complex λ with $0 < \operatorname{Im}\lambda < \rho$ or almost every $\lambda \in \mathbf{R}^1$, one has $|\hat{f}(\zeta,\lambda)| \le C\, f^*(\lambda)\, e^{a|\operatorname{Im}\zeta|}$ for all $\zeta \in \mathbf{C}^n$.

Proof. This result follows from the classical Paley–Wiener theorem and the Fourier inversion theorem, see e.g., [8, Theorem 7.3.1].

Our next result is a regularization or smoothing result using the certain separated mollifiers.

Proposition 3.2 *Let $\phi_\epsilon \in C_0^\infty(\mathbf{R}^n)$ with $\operatorname{supp}\phi_\epsilon \subset \mathbf{B}^n_\epsilon$ and $\psi_\delta \in C_0^\infty(\mathbf{R}^1)$ with*

$\operatorname{supp}\psi_\delta \subset [-\delta,\delta]$. *Let* $\rho_{\epsilon,\delta}(x,y) = \phi_\epsilon(x)\,\psi_\delta(y)$. *For any* $g \in L^2_{loc}(\mathbf{R}^{n+1})$, *define the convolution* $g_{\epsilon,\delta} = g * \rho_{\epsilon,\delta}$. *Then we have the following estimates:*

$$\int_{\mathbf{R}^1}\int_{|x|<R} |g_{\epsilon,\delta}(x,y)|^2\,dx\,dy \le \|\phi_\epsilon\|^2_{L^1}\,\|\psi_\delta\|^2_{L^1} \int_{\mathbf{R}^1}\int_{|x|<R+\epsilon} |g(x,y)|^2\,dx\,dy$$

and

$$\int_{\mathbf{R}^1}\int_{\mathbf{R}^n} |g_{\epsilon,\delta}(x,y)|^2\,e^{2\rho y}\,dx\,dy \le e^{2\rho\delta}\,\|\phi_\epsilon\|^2_{L^1}\,\|\psi_\delta\|^2_{L^1} \int_{\mathbf{R}^1}\int_{\mathbf{R}^n} |g(x,y)|^2\,e^{2\rho y}\,dx\,dy.$$

Furthermore if $g(x,y) \in L^2_{loc}(\mathbf{R}^1_y; L^2_a(\mathbf{R}^n_x))$ *then* $g_{\epsilon,\delta}(x,y) \in L^2_{loc}(\mathbf{R}^1_y; L^2_{a+\epsilon}(\mathbf{R}^n_x))$.

Proof. Both inequalities follow from the Young's inequality for convolutions. We omit the details.

The following two results are well known and standard to the experts of several complex variables. We include them here for the convenience of the reader.

Lemma 3.3 *Let* Ω *be a domain in* $\mathbf{C}^N$, $(N \ge 2)$, *and* Γ *a closed subset of* Ω *with the* $(2N-2)$*-Hausdorff measure* $\mathcal{H}_{2N-2}(\Gamma) = 0$. *If* $H(z)$ *is an analytic function on* $\Omega\backslash\Gamma$, *then there is an analytic continuation* $\tilde{H}(z)$ *of* $H(z)$ *into* Ω.

Lemma 3.4 *Let* D *be a domain in* $\mathbf{C}^N$, *and let* E *be a closed subset in* D *such that the* $(2N-1)$*-Hausdorff measure* $\mathcal{H}_{2N-1}(E\cap K) < +\infty$ *for any compact set* $K \subset D$. *Then every function* $H(z)$ *continuous in* D *and analytic in* $D\backslash E$ *is analytic in all of* D.

Proof. The proof of both results can be found in Chirka [4, pp.298–301].

4 Analysis of the Fourier-Laplace transforms

In this section, we prove the divisibility of $\hat{f}(\zeta,\lambda)$ by $P(\zeta,\lambda)$ in the domain $\mathcal{C}_\rho$ under the assumptions of Theorem 2.1. We shall use arguments closely related to the theory of the stationary phase method. See, e.g., [7], [8], [10], [11] and [12].

Theorem 4.1 *Under the assumptions of Theorem* 2.1, *one can find an analytic function* $H(\zeta,\lambda)$ *in the slab* $\mathcal{C}_\rho$ *such that*

$$P(\zeta,\lambda)\,H(\zeta,\lambda) = \hat{f}(\zeta,\lambda) \tag{4.1}$$

for all $(\zeta,\lambda) \in \mathcal{C}_\rho$.

To prove this theorem, we need the following result.

Theorem 4.2 *Under the assumptions of Theorem* 2.1, *it follows that* $\hat{f}(\zeta,\lambda)$ *along* *with all its derivatives up to the order* $(m_j - 1)$ *vanish on any connected component* U *of the set* $\mathcal{A}_\rho(P_j)$, *for all* $j = 1, 2, ..., r$.

The proof of Theorem 4.2 will be given in the end of this section. We first show that Theorem 4.2 implies Theorem 4.1.

Proof of Theorem 4.1. Using the notation given before, let $\Omega = \mathcal{C}_\rho$ and $D = \Omega \backslash \Gamma$ with

$$\Gamma = \left(\cup_{j=1}^{r} K_\rho(P_j)\right) \cup \Delta$$

where for each $j = 1, ..., r$

$$K_\rho(P_j) = \{(\zeta, \lambda) \in \Sigma_\rho(P_j) \,|\, \nabla P_j(\zeta, \lambda) = 0\}, \quad \Delta = \cup_{j \neq j'} \left(\Sigma(P_j) \cap \Sigma(P_{j'})\right).$$

Note that since $P_j \neq P_{j'}$ $(j \neq j')$ are irreducible and have real coefficients, the complex dimension of Γ is not greater than $N-2$, and thus the $(2N-2)$-Hausdorff measure $\mathcal{H}_{2N-2}(\Gamma) = 0$ (see [4] or [10] and the references therein). Define also

$$E = \cup_{j=1}^{r} \mathcal{A}_\rho(P_j).$$

Since $D \backslash E$ is contained in $\mathcal{C}_\rho \backslash \Sigma_\rho(P)$, it follows from Theorem 4.2 that the function $H(\zeta, \lambda)$ defined by $\hat{f}(\zeta, \lambda)/P(\zeta, \lambda)$ on $D \backslash E$ can be extended as a continuous function in D, still denoted by $H(\zeta, \lambda)$. On the other hand, it is easy to see that $\mathcal{H}_{2N-1}(E \cap K) < +\infty$ for any compact set $K \subset D$. Thus by Lemma 3.4 the continuous function $H(\zeta, \lambda)$ is analytic in $D = \Omega \backslash \Gamma$. Therefore, by Lemma 3.3, the analytic continuation of this function $H(\zeta, \lambda)$ will be the function required in Theorem 4.1.

Before proving Theorem 4.2, we make some observations and prove several lemmas. In what follows, we assume functions $u(x, y)$ and $f(x, y)$ both satisfy all the conditions stated in Theorem 2.1. First of all, by Proposition 3.2, without loss of generality, we assume $u(x, y)$ and $f(x, y)$ are both of C^∞.

Now, let $1 \leq j \leq r$ be fixed. Let U be a connected component of the set $\mathcal{A}_\rho(P_j)$ and $\bar{U}$ denote the closure of the set U in $\mathbf{C}^{n+1}$. By the connectedness condition (C) it follows that $\bar{U} \cap \mathbf{A}(P_j) \neq \emptyset$. Let (ξ^0, σ^0) be a point in $\bar{U} \cap \mathbf{A}(P_j)$. Without loss of generality, we assume that $\partial_{\xi_1} P_j(\xi^0, \sigma^0) \neq 0$. Then locally one can parameterize the surface $\Sigma(P_j)$ by its projection onto the ζ_1-plane. To be more precise, one can find a complex neighborhood $\mathcal{N}$ of (ξ^0, σ^0) and an analytic function $s(\zeta_*, \lambda)$ defined there satisfying

$$\mathcal{N} \cap \Sigma(P_j) = \{(s(\zeta_*, \lambda), \, \zeta_*, \, \lambda) \,|\, \zeta_* \in G + iT, \ \lambda \in I + iJ\}, \tag{4.2}$$

where T and J are neighborhoods of the origin in $\mathbf{R}^{n-1}$ and $\mathbf{R}^1$, respectively; and G and I are neighborhoods of ξ_*^0 and σ^0, respectively. Note that the set

$$\{(s(\zeta_*, \lambda), \zeta_*, \lambda) \,|\, \zeta_* \in G + iT, \ \lambda \in I + iJ_\epsilon\}$$

is contained in $\mathcal{N} \cap U$ for some $J_\epsilon = (0, \epsilon)$.

For a fixed $0 \le \nu \le (m_j - 1)$, define an analytic function $W(\zeta_*, \lambda)$ as follows,

$$W(\zeta_*, \lambda) \equiv (\partial/\partial\zeta_1)^\nu \hat{f}(\zeta_1, \zeta_*, \lambda)\Big|_{\zeta_1 = s(\zeta_*, \lambda)} \tag{4.3}$$

for all $(\zeta_*, \lambda) \in (G + iT) \times (I + iJ_\epsilon)$.

We recall that $F(x, \sigma)$ is the partial Fourier transform of $f(x, y)$ with respect to y and for a.e. $\sigma \in \mathbf{R}^1$, $\hat{F}(\zeta, \sigma)$ is the Fourier–Laplacian transform of $F(x, \sigma)$ in $\mathbf{R}^n_x$. Define

$$W_0(\xi_*, \sigma) \equiv (\partial/\partial\xi_1)^\nu \hat{F}(\xi_1, \xi_*, \sigma)\Big|_{\xi_1 = s(\xi_*, \sigma)} \tag{4.4}$$

for all $(\xi_*, \sigma) \in G \times I$.

We now proceed with several lemmas.

Lemma 4.3 *For all $\phi \in C_0^\infty(G \times I)$, it follows that*

$$\lim_{\substack{\eta_* \to 0 \\ \tau \to 0^+}} \iint_{G \times I} W(\xi_* + i\eta_*, \sigma + i\tau)\, \phi(\xi_*, \sigma) d\xi_*\, d\sigma = \iint_{G \times I} W_0(\xi_*, \sigma)\, \phi(\xi_*, \sigma) d\xi_*\, d\sigma. \tag{4.5}$$

Proof. By (4.3) and Fubini's theorem it follows that

$$\iint_{G \times I} W(\xi_* + i\eta_*, \sigma + i\tau)\, \phi(\xi_*, \sigma) d\xi_*\, d\sigma = \iint_{\mathbf{R}^n \times \mathbf{R}^1} g_{\eta_*, \tau}(x, y)\, k_{\eta_*, \tau}(x, y)\, dx\, dy, \tag{4.6}$$

where $g_{\eta_*, \tau}(x, y) = (-ix_1)^\nu f(x, y)\, e^{x_* \cdot \eta_* + y\tau}$ with $x = (x_1, x_*)$, and

$$k_{\eta_*, \tau}(x, y) = (2\pi)^{-n/2} \iint_{G \times I} \phi(\xi_*, \sigma)\, e^{-i\left(s(\xi_* + i\eta_*, \sigma + i\tau)x_1 + \xi_* \cdot x_* + \sigma y\right)} d\xi_*\, d\sigma \tag{4.7}$$

for all sufficiently small η_* and $\tau > 0$. Note that $g_{0,0}$ and $k_{0,0}$ are also well-defined accordingly.

By the Fourier inversion formula, it is seen that (4.6) also holds for $\eta_* = 0$ and $\tau = 0$ if $W(\zeta_*, \lambda)$ on the left-hand side is replaced by $W_0(\xi_*, \sigma)$ defined by (4.4). Now, integration by parts in (4.7) implies that

$$|k_{\eta_*, \tau}(x, y)| \le C_s\, (1 + |y|)^{-s}$$

for all $s = 0, 1, 2, \ldots$ with sufficiently small τ and η_* and $x \in \mathbf{B}_a$ and $y \in \mathbf{R}^1$. Therefore, for all sufficiently small η_* and $\tau \ge 0$, it follows that

$$|g_{\eta_*, \tau}(x, y)\, k_{\eta_*, \tau}(x, y)| \le C_s \cdot m_s(x, y), \tag{4.8}$$

where

$$m_s(x,y) = \begin{cases} |f(x,y)|\, e^{\rho y/2} & \text{if } y \geq 0, \\ |f(x,y)|\,(1+|y|)^{-s} & \text{if } y < 0. \end{cases}$$

Note that $m_s(x,y)$ belongs to $L^1(\mathbf{B}_a \times \mathbf{R}^1)$ if $s \geq 1$. Furthermore, it is easily seen that

$$\lim_{\substack{\eta_* \to 0 \\ \tau \to 0^+}} g_{\eta_*,\tau}(x,y)\, k_{\eta_*,\tau}(x,y) = g_{0,0}(x,y)\, k_{0,0}(x,y), \quad (x,y) \in \mathbf{B}_a \times \mathbf{R}^1.$$

Therefore, by the Lebesgue dominated convergence theorem and (4.6), it follows that

$$\lim_{\substack{\eta_* \to 0 \\ \tau \to 0^+}} \iint_{G\times I} W(\xi_* + i\eta_*, \sigma + i\tau)\, \phi(\xi_*, \sigma) d\xi_*\, d\sigma = \iint_{G\times I} W_0(\xi_*, \sigma)\, \phi(\xi_*, \sigma) d\xi_*\, d\sigma.$$

The lemma is thus proved.

As before, let $F(x,\sigma)$ and $U(x,\sigma)$ be the Fourier transforms of $f(x,y)$ and $u(x,y)$ with respect to $y \in \mathbf{R}^1$, respectively. Then one can prove the following results.

Lemma 4.4 *For almost every $\sigma \in \mathbf{R}^1$ the function $U(x,\sigma)$ is a tempered distribution on $\mathbf{R}^n$ and satisfies*

$$P(D_x, \sigma)\, U(x,\sigma) = F(x,\sigma) \quad \text{on } \mathbf{R}^n, \tag{4.9}$$

and

$$\liminf_{R\to\infty} \frac{1}{R} \int_{|x|<R} |U(x,\sigma)|^2\, dx = 0. \tag{4.10}$$

Proof. Using the separated test functions, one can easily prove (4.9), see [14]. To prove (4.10), we observe that from (2.5) it follows that for some constant $C > 0$

$$\int_{\mathbf{R}^1} \int_{R/2 \leq |x| < R} |u(x,y)|^2\, dx\, dy \leq C \cdot R$$

for every $R \geq 1$. From this inequality and the fact $u(x,y) \in L^2_{loc}(\mathbf{R}^n_x; L^2(\mathbf{R}^1_y))$, it follows that

$$\int_{\mathbf{R}^1} \int_{\mathbf{R}^n} \frac{|U(x,\sigma)|^2}{1+|x|^2}\, dx\, d\sigma = \int_{\mathbf{R}^1} \int_{\mathbf{R}^n} \frac{|u(x,y)|^2}{1+|x|^2}\, dx\, dy < \infty.$$

From this it easily follows that for almost every $\sigma \in \mathbf{R}^1$ the function $U(\cdot,\sigma)$ is a tempered distribution on $\mathbf{R}^n$. Similarly, the decay property (4.10) follows from (2.5) by using Fatou's lemma.

Lemma 4.5 *For all $\phi \in C_0^\infty(G \times I)$ with sufficiently small support, it follows that*

$$\int_G W_0(\xi_*, \sigma)\, \phi(\xi_*, \sigma) d\xi_* = 0 \tag{4.11}$$

for a.e. $\sigma \in I$.

Proof. This follows from (4.9) and (4.10) by using the arguments of the stationary phase method for the surface-carried Fourier transforms as used in Littman [10] and Hörmander [7]. We give the details of proof below for the convenience of the reader.

In what follows, we fix a $\sigma \in I$ so that both (4.9) and (4.10) hold and write

$$F(x,\sigma),\ U(x,\sigma),\ \phi(\xi_*,\sigma),\ P(\xi,\sigma)$$

as $F_\sigma(x)$, $U_\sigma(x)$, $\phi_\sigma(\xi_*)$ and $P_\sigma(\xi)$, respectively. Let $\psi \in C_0^\infty(\mathbf{R}^1)$ such that

$$\psi(t) = \begin{cases} 1 & \text{if } |t| \le 1/2; \\ 0 & \text{if } |t| > 1, \end{cases}$$

and define $g_R(\xi) = \phi(\xi_*, \sigma)\, R^{\nu+1} \hat{\psi}^{(\nu)}(R(s(\xi_*, \sigma) - \xi_1))$ for $R > 0$. Using (4.4), by the dominated convergence theorem it follows that

$$\lim_{R\to\infty} \hat{F}_\sigma(g_R) = -(2\pi)^{1/2} \int_G W_0(\xi_*, \sigma)\, \phi(\xi_*, \sigma) d\xi_*.$$

By virtue of the previous lemma we have $\hat{F}_\sigma = P_\sigma \hat{U}_\sigma$ and $\hat{F}_\sigma(g_R) = \hat{U}_\sigma(h_R)$, where

$$h_R(\xi) = P_\sigma(\xi)\, g_R(\xi) = R^{\nu+1}\, \hat{\psi}^{(\nu)}(R(s(\xi_*, \sigma) - \xi_1)) \sum_{\mu=m_j}^{m} (\xi_1 - s(\xi_*, \sigma))^\mu\, a_\mu(\xi_*)$$

with $a_\mu \in C_0^\infty(G)$ and $\operatorname{supp} a_\mu \subseteq \operatorname{supp} \phi_\sigma$. The Fourier transform of $t^\mu\, \hat{\psi}^{(\nu)}(-t)$ is

$$\psi_\mu(\tau) = (i\partial/\tau)^\mu\, ((-i\tau)^\nu\, \psi(\tau)) \in C_0^\infty(\mathbf{R}^1).$$

Since $\mu \ge m_j > \nu$, we have $\operatorname{supp} \psi_\mu \subseteq \operatorname{supp} \psi'$ which does not contain the origin. Define

$$I_\mu(x) = (2\pi)^{-\frac{n-1}{2}} \int\limits_G a_\mu(\xi_*)\, e^{-i\left(s(\xi_*,\sigma)\, x_1 + \xi_* \cdot x_*\right)}\, d\xi_*, \tag{4.12}$$

then it follows that

$$\int_{|x|\le R} |I_\mu(x)|^2 dx \le C\, R, \quad R > 0. \tag{4.13}$$

Also by an easy calculation we have that

$$\hat{h_R}(x) = \sum_{\mu=m_j}^{m} R^{\nu-\mu}\, \psi_\mu(x_1/R)\, I_\mu(x)$$

and

$$\hat{U}_\sigma(h_R) = U_\sigma(\hat{h_R}) = \sum_{\mu=m_j}^{m} R^{\nu-\mu} \int_{\mathbf{R}^n} U_\sigma(x)\, \psi_\mu(x_1/R)\, I_\mu(x)\, dx. \tag{4.14}$$

The proof of the lemma will be complete if we show that each term in the sum of (4.14) approaches zero when $R \to \infty$ if $\operatorname{supp} \phi_\sigma$ is sufficiently close to ξ_*^0.

Using the technique of localization (see Lemma 2.3 and P. 110 of [7]), we may assume that $\hat{U}_\sigma(\xi)$ is compactly supported on the following surface of $\mathbf{R}_\xi^n$

$$\mathcal{S}_\sigma = \{(s(\xi_*, \sigma), \xi_*) \mid \xi_* \in G\}.$$

By a change of variables, we may assume the normal direction of $\mathcal{S}_\sigma$ at the point ξ^0 is the x_1-direction in the dual space $\mathbf{R}_x^n$. Let $\omega \subset G$ be a small neighborhood of ξ_*^0 such that all the normal directions of $\mathcal{S}_\sigma$ at points for $\xi_* \in \omega$ are contained in the open conic neighborhood

$$V = \{(x_1, x_*) \in \mathbf{R}^n \mid 0 < |x_*| < |x_1|\}$$

of the x_1-axis with 0 being removed. By the stationary phase method, it follows that $I_\mu(x)$ is rapidly decreasing at infinity outside V if $\operatorname{supp} a_\mu \subseteq \omega$.

Having all these at our disposal, by virtue of (4.13) we obtain using Cauchy-Schwarz' inequality that for each $\mu \geq m_j$,

$$R^{\nu-\mu}\Big|\int_{\mathbf{R}^n} U_\sigma(x)\, \psi_\mu(x_1/R)\, I_\mu(x)\, dx\Big| = R^{\nu-\mu}\Big|\int_{\frac{R}{2} \leq |x_1| \leq R} U_\sigma(x)\, \psi_\mu(x_1/R)\, I_\mu(x)\, dx\Big|$$

$$\leq C\left(R^{\nu-\mu} \int_{|x| \leq \sqrt{2}R} |U_\sigma(x)|^2\, dx\right)^{1/2} + C\, R^{\nu-\mu} \int_{\mathbf{R}^n \setminus V} |I_\mu(x)\, U_\sigma(x)|\, dx.$$

By the assumption that $\hat{U}_\sigma(\xi)$ is compactly supported the second term on the right-hand side of the above estimate goes to zero as $R \to \infty$ since $I_\mu(x)$ decreases rapidly outside V and $\nu - \mu \leq -1$. The first term also goes to zero as $R \to \infty$ in view of (4.10). We have thus completed the proof.

We now complete the proof of Theorem 4.2.

Proof of Theorem 4.2. By the previous lemma and (4.5), it follows that

$$\lim_{\substack{\eta_* \to 0 \\ \tau \to 0^+}} \iint_{G \times I} W(\xi_* + i\eta_*,\, \sigma + i\tau)\, \phi(\xi_*, \sigma) d\xi_*\, d\sigma = 0$$

for all $\phi \in C_0^\infty(G \times I)$ with sufficiently small support. This implies

$$W(\zeta_*, \lambda) = 0$$

for all $(\zeta_*, \lambda) \in (G + iT) \times (I + iJ_\epsilon)$. Consequently, $\hat{f}(\zeta, \lambda)$ along with all its derivatives up to the order $(m_j - 1)$ vanish on an open set of each connected component U of $\mathcal{A}_\rho(P_j)$, for all $j = 1, 2, ..., r$. This proves the theorem.

5 Proof of main results

In this section, we shall prove our main results: Theorems 2.1 and 2.2. Assume that $u(x,y)$ and $f(x,y)$ are functions satisfying all conditions described in Theorem 2.1, and we shall prove that $u(x,y) \equiv 0$ in $|x| > a$.

As before, let $F(x,\sigma)$ and $U(x,\sigma)$ be the Fourier transforms of $f(x,y)$ and $u(x,y)$ with respect to $y \in \mathbf{R}^1$, respectively. We prove $u(x,y) \equiv 0$ in $|x| > a$ by proving that $U(x,\sigma) \equiv 0$ in $|x| > a$.

First of all, we need the following result due to Hörmander [6].

Lemma 5.1 *Let $Q(\xi)$ be an arbitrary polynomial in n-variables with principal part $q(\xi)$. Then the following estimate is valid for all real ξ_0 with $|\xi_0| = 1$ and all $u \in C_0^\infty(\mathbf{B}_r)$:*

$$\|u\|_{L^2(\mathbf{B}_r)} \leq \frac{e^r}{|q(\xi_0)|} \cdot \|Q(D_x)\,u\|_{L^2(\mathbf{B}_r)}. \tag{5.1}$$

Of course, this result is useful only in the case $q(\xi_0) \neq 0$, which is always possible for some ξ_0.

Theorem 5.2 *Assume that all the conditions of Theorem 2.1 are satisfied. Then for every $\lambda \in \mathbf{C}^1$ with $0 < \operatorname{Im}\lambda < \epsilon$, there exists a function $V(\cdot,\lambda) \in L^2_a(\mathbf{R}^n)$ satisfying*

$$P(D_x,\lambda)\,V(x,\lambda) = F(x,\lambda) \tag{5.2}$$

for $x \in \mathbf{R}^n$, where $F(x,\lambda)$ is the Fourier-Laplace transform of $f(x,y)$ with respect to $y \in \mathbf{R}^1$.

Proof. Let $H(\zeta,\lambda)$ be the analytic function in the slab domain $\mathcal{C}_\epsilon$ determined in Theorem 4.1. By virtue of Proposition 3.1 and a Paley–Wiener theory (see e.g. Theorems 7.3.1 and 7.3.2 in Hörmander [8]), it follows that for each λ, the analytic function $H(\zeta,\lambda)$ is the Fourier–Laplace transform of a distribution $V(x,\lambda)$ which is compactly supported in the ball $\mathbf{B}_a$ and satisfies equation (5.2). The regularizations of $V(x,\lambda)$ will satisfy an equation similar to (5.2) with $F(x,\lambda)$ replaced by its regularizations. Therefore by using the estimate (5.1), one easily sees that $V(x,\lambda)$ actually belongs to $L^2_a(\mathbf{R}^n)$. This completes the proof of the theorem.

We next follow the general idea of a limiting absorption principle to examine the limiting behavior of functions $V(x,\lambda)$ when λ approaches the reals through the slab $\mathcal{C}_\rho$. More precisely, we prove the following result:

Theorem 5.3 *Let $V(x,\lambda)$ be the function determined by the previous theorem.*

Then for almost every $\sigma \in \mathbf{R}^1$, for a sequence of $\tau \to 0^+$, it follows that $\{V(x, \sigma+i\tau)\}$ has a weak limit in $L^2_a(\mathbf{R}^n_x)$, denoted by $V(x,\sigma) \in L^2_a(\mathbf{R}^n_x)$. Moreover, this limit function $V(x,\sigma)$ satisfies the following equation on $\mathbf{R}^n_x$ for almost every $\sigma \in \mathbf{R}^1$:

$$P(D_x, \sigma)\, V(x,\sigma) = F(x,\sigma). \tag{5.3}$$

Proof. Let $p^\lambda(\zeta)$ be the principal part of the polynomial $P(\zeta, \lambda)$ considered as a polynomial in ζ. It is easily seen that the set of $\lambda \in \mathbf{C}^1$ such that $p^\lambda(\xi) \equiv 0$ for all real $\xi \neq 0$ is a finite set, hence for all but finitely many $\sigma_0 \in \mathbf{R}^1$ one can choose a $\xi_0 \in \mathbf{R}^n$ with $|\xi_0| = 1$ such that $|p^\lambda(\xi_0)| \geq \gamma > 0$ for every complex λ in a neighborhood $\mathcal{N}_0$ of σ_0. Then by using estimate (5.1), it follows that

$$\|V(\cdot,\lambda)\|_{L^2} \leq \frac{e^a}{\gamma} f^*(\lambda) \tag{5.4}$$

for every $\lambda \in \mathcal{C}_\epsilon \cap \mathcal{N}_0$, where the function $f^*(\lambda)$ was defined early in Section 3.

By Proposition 3.1 it follows that $f^*(\sigma + i\tau)$ converges to $f^*(\sigma)$ in $L^2(\mathbf{R}^1_\sigma)$ as $\tau \to 0^+$. By (5.4), it follows that for almost every $\sigma \in \mathbf{R}^1 \cap \mathcal{N}_0$ the sequence $\{V(\cdot, \sigma + i\tau)\}$ converges weakly to a function $V(\cdot,\sigma)$ in $L^2_a(\mathbf{R}^n)$ through a sequence $\tau \to 0^+$. Finally equation (5.3) follows easily from (5.2) and the weak convergence of $V(\cdot,\lambda)$. The theorem is thus proved.

We are now ready to prove our main results: Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Let $V(x,\sigma)$ be the function determined in Theorem 5.3. From (4.10) and (5.2) it follows that for almost every $\sigma \in \mathbf{R}^1$ the function $Z(x,\sigma) \equiv U(x,\sigma) - V(x,\sigma)$ satisfies the equation

$$P(D_x, \sigma)\, Z(x,\sigma) = 0 \tag{5.5}$$

and also satisfies the following decay condition:

$$\liminf_{R\to\infty} \frac{1}{R} \int_{|x|<R} |Z(x,\sigma)|^2\, dx = 0. \tag{5.6}$$

If for a $\sigma \in \mathbf{R}^1$ the polynomial $P(\xi,\sigma)$ has a real zero, then by a uniqueness theorem of Hörmander [7, Corollary 2.5], it follows from (5.5) and (5.6) that $Z(x,\sigma) = 0$ on $\mathbf{R}^n_x$. If for a $\sigma \in \mathbf{R}^1$, the polynomial $P(\xi,\sigma)$ has no real zeros, then it follows easily from (5.5) and (5.6) that $Z(x,\sigma) = 0$. Therefore, $Z(x,\sigma) = 0$ for almost every $\sigma \in \mathbf{R}^1$. Thus $U(x,\sigma) = V(x,\sigma)$ has compact support in $x \in B_a$ for a.e. $\sigma \in \mathbf{R}^1$; thus by the inverse Fourier transform it follows that $u(x,y)$ has support inside $B_a \times \mathbf{R}^1_y$. Theorem 2.1 is thus proved.

Proof of Theorem 2.2. Let $\phi_\epsilon(x)$ and $\psi_\epsilon(y)$ be the standard mollifiers in $\mathbf{R}^n_x$ and $\mathbf{R}^1_y$, respectively; and let $\rho_\epsilon(x,y) = \phi_\epsilon(x)\,\psi_\epsilon(y)$. Define $\vec{u}_\epsilon = \vec{u} * \rho_\epsilon$ and $\vec{f}_\epsilon = \vec{f} * \rho_\epsilon$. Then by (6.2), it follows that

$$\mathcal{M}(D_x, D_y)\,\vec{u}_\epsilon(x,y) = \vec{f}_\epsilon(x,y).$$

Let u_j^ϵ and f_j^ϵ be the components of $\vec{u}_\epsilon$ and $\vec{f}_\epsilon$, respectively. Then one has the following scalar equations:

$$Q(D_x, D_y)\,u_j^\epsilon(x,y) = g_j^\epsilon(x,y), \quad j = 1, ..., m, \tag{5.7}$$

where

$$g_j^\epsilon(x,y) = \sum_{k=1}^{m} P_{jk}^*(D_x, D_y)\,f_k^\epsilon(x,y), \quad j = 1, 2, ..., m, \tag{5.8}$$

and $(P_{jk}^*(\zeta,\lambda))$ is the adjoint matrix of $\mathcal{M}(\zeta,\lambda)$.

By Proposition 3.2 and conditions (2.4) and (2.5), it follows that the functions g_j^ϵ and u_j^ϵ satisfy all the conditions of the functions f and u stated in Theorem 2.1, respectively. Therefore, by Theorem 2.1, it follows that the solution u_j^ϵ of (5.7) will satisfy $u_j^\epsilon(x,y) = 0$ for all $|x| \geq a + \epsilon$. Finally, observe that $u_j^\epsilon(x,y)$ approaches $u_j(x,y)$ in $L^2(\mathbf{R}_y^1; L_{loc}^2(\mathbf{R}_x^n))$ as $\epsilon \to 0^+$. From this it follows that $u_j(x,y) = 0$ for all $|x| \geq a$. We thus complete the proof.

6 Remarks about equations with variable coefficients

In this final section, we discuss the similar results for partial differential equations with the variable coefficients that are equal to constants outside a cylinder. Consider the following partial differential operator

$$L(x, y, D_x, D_y) = Q(D_x, D_y) + \sum_{\alpha\in I, \beta\in J} c_{\alpha\beta}(x,y)\,D_x^\alpha\,D_y^\beta, \tag{6.1}$$

where $Q(D_x, D_y)$ is a linear differential operator with real constant coefficients, I and J are finite sets of indices, and $c_{\alpha\beta}(x,y) \in C^\infty(\mathbf{R}_x^n \times \mathbf{R}_y^1)$ is assumed to be bounded and satisfy

$$c_{\alpha\beta}(x,y) \equiv 0 \quad \text{for } |x| \geq b, \text{ and} \tag{6.2}$$

$$\sup_{|x|\leq b, y\geq 0} \left(|\partial_x^\mu \partial_y^j\, c_{\alpha\beta}(x,y)|\, e^{\delta y}\right) + \sup_{|x|\leq b, y<0} \left(|\partial_x^\mu \partial_y^j\, c_{\alpha\beta}(x,y)|\right) < +\infty \tag{6.3}$$

for all $\alpha \in I$, $\beta \in J$, and all $0 \leq |\mu| \leq |\alpha|$ and $0 \leq j \leq \beta$ with some constants $b \geq a$, $\delta > 0$. The following result can be proved by using Theorem 2.1 and Proposition 3.2.

Theorem 6.1 *Let $u(x,y)$ and $f(x,y)$ satisfy all the conditions as described in Theorem 2.1. Suppose that $Lu = f$ and L is the operator defined by (6.1) with coefficients $c_{\alpha\beta}$ satisfying conditions (6.2) and (6.3) above. Suppose each irreducible factor of $Q(\zeta, \lambda)$ in (6.1) has real coefficients up to a complex constant factor and satisfies the connectedness condition (C). Then the conclusion of Theorem 2.1 still holds for u with constant $a > 0$ being replaced by $b > 0$.*

The proof of this theorem makes use of the following result.

Lemma 6.2 *Let $u(x,y) \in L^2(\mathbf{R}^1_y; L^2_{loc}(\mathbf{R}^n_x))$ and $\chi(x,y)$ a C^∞-function satisfying the following:*

(i) $\chi(x,y) \equiv 0$ *when* $|x| \geq b$; *and*

(ii) $\sup_{|x|\leq b, y\geq 0}(|\chi(x,y)|\, e^{\delta y}) + \sup_{|x|\leq b, y<0} |\chi(x,y)| < +\infty$.

Let $\rho(x,y) = \rho_{\epsilon,\epsilon}(x,y)$ be as defined in Proposition 3.2. Define

$$v(x,y) = ((\chi\, u) * \rho)(x,y).$$

Then it follows that

$$\int_{\mathbf{R}^1} \int_{\mathbf{R}^n} |v(x,y)|^2\, e^{2\delta y}\, dx\, dy < \infty. \tag{6.4}$$

This lemma follows easily from the second estimate of Proposition 3.2 and the assumptions on u and χ.

Proof of Theorem 6.1. Let u_ϵ and f_ϵ be defined similarly as in the proof of Theorem 2.2. Observe that

$$Q(D_x, D_y)\, u_\epsilon = f_\epsilon - \sum_{\alpha\in I, \beta\in J} ((c_{\alpha\beta}\, D_x^\alpha\, D_y^\beta u) * \rho_\epsilon) \equiv g_\epsilon \tag{6.5}$$

and each term inside the summation in (6.5) can be written as

$$\sum_{\mu,\nu;j,k} ((D_x^\mu\, D_y^j c_{\alpha\beta})\, u) * (D_x^\nu\, D_y^k \rho_\epsilon).$$

Therefore, by (6.4) and Proposition 3.2 it follows that the function $g_\epsilon(x,y)$ satisfies all the conditions that the function $f(x,y)$ satisfies. The conclusion on the support of $u_\epsilon(x,y)$ follows then from (6.5) in much the same way as in the proof of Theorem 2.2. The proof of Theorem 6.1 is thus completed.

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