The Wave Equation

In this chapter we investigate the wave equation

$$u_{tt} - \Delta u = 0 \tag{5.1}$$

and the nonhomogeneous wave equation

$$u_{tt} - \Delta u = f(x,t) \tag{5.2}$$

subject to appropriate initial and boundary conditions. Here $x \in \Omega \subset \mathbb{R}^n$, $t > 0$; the unknown function $u = u(x,t) : \Omega \times [0,\infty) \to \mathbb{R}$.

We shall discover that solutions to the wave equation behave quite differently from solutions of Laplaces equation or the heat equation. For example, these solutions are generally not C^∞ and exhibit the **finite speed of propagation** of given disturbances.

5.1. Derivation of the wave equation

The wave equation is a simplified model for a vibrating string ($n = 1$), membrane ($n = 2$), or elastic solid ($n = 3$). In this physical interpretation $u(x,t)$ represents the displacement in some direction of the point at time $t \geq 0$.

Let V represent any smooth subregion of Ω. The acceleration within V is then

$$\frac{d^2}{dt^2}\int_V u dx = \int_V u_{tt} dx,$$

and the net force is

$$\int_{\partial V} \mathbf{F} \cdot \nu dS,$$

where $\mathbf{F}$ denoting the force acting on V through ∂V, ν is the unit outnormal on ∂V. Newton's law says (assume the mass is 1) that

$$\int_V u_{tt} = \int_{\partial V} \mathbf{F} \cdot \nu dS.$$

This identity is true for any region, hence the divergence theorem tells that

$$u_{tt} = \operatorname{div} \mathbf{F}.$$

For elastic bodies, F is a function of Du, i.e., $\mathbf{F} = F(Du)$. For small u and small Du, we use the linearization aDu to approximate $F(Du)$, and so

$$u_{tt} - a\Delta u = 0,$$

when $a = 1$, the resulting equation is the **wave equation**. The physical interpretation strongly suggests it will be mathematically appropriate to specify two initial conditions, $u(x,0)$ and $u_t(x,0)$.

5.2. One-dimensional wave equations and d'Alembert's formula

This section is devoted to solving the Cauchy problem for one-dimensional wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}, \end{cases} \tag{5.3}$$

where g, h are given functions.

Note that

$$u_{tt} - u_{xx} = (u_t - u_x)_t + (u_t - u_x)_x = v_t + v_x$$

where $v = u_t - u_x$. So $v(x,0) = u_t(x,0) - u_x(x,0) = h(x) - g'(x) := a(x)$. From $v_t + v_x = 0$, we have $v(x,t) = a(x-t)$. Then u solves the nonhomogeneous transport equation

$$u_t - u_x = v(x,t), \quad u(x,0) = g(x).$$

Solve this problem for u to obtain

$$\begin{aligned} u(x,t) &= g(x+t) + \int_0^t v(x+t-s,s)\,ds \\ &= g(x+t) + \int_0^t a(x+t-2s)\,ds = g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} a(y)\,dy \\ &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} (h(y) - g'(y))\,dy, \end{aligned}$$

from which we deduce **d'Alembert's formula**:

$$u(x,t) = \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy. \tag{5.4}$$

This formula defines a classical solution $u \in C^2$ if and only if $g \in C^2$ and $h \in C^1$.

Remark 5.1. (i) When the initial data g and h are not smooth, we still use formula (5.4) to define a (weak) solution to the Cauchy problem. If $g \in C^k$ and $h \in C^{k-1}$, then $u \in C^k$. Thus the wave equation does not have the smoothing effect like the heat equation has.

(ii) Any solution to the wave equation $u_{tt} = u_{xx}$ has the form

$$u(x,t) = F(x+t) + G(x-t)$$

for appropriate functions F and G. Usually, $F(x+t)$ is called a **traveling wave to the left** with speed 1; $G(x-t)$ is called a **traveling wave to the right** with speed 1.

(iii) The value $u(x_0,t_0)$ depends only on the values of initial data from $x_0 - t_0$ to $x_0 + t_0$; in this sense, the interval $[x_0 - t_0, x_0 + t_0]$ is called the **domain of dependence** for the point (x_0,t_0). The Cauchy data at $(x_0,0)$ influence the value of u in the set

$$I(x_0,t_0) = \{(x,t) \mid x_0 - t < x < x_0 + t, t > 0\},$$

which is called the **domain of influence** for $(x_0, 0)$. The **domain of determinacy** of $[x_1, x_2]$; that is, the set of points (x, t) at which the solution is completely determined by the values of initial data on $[x_1, x_2]$, is given by

$$D(x_1, x_2) = \left\{ (x,t) \;\middle|\; x_1 + t \le x \le x_2 - t,\ t \in \left[0, \frac{x_2 - x_1}{2}\right] \right\}.$$

Lemma 5.1 (Parallelogram property)**.** *Let Ω be an open set in $\mathbb{R} \times \bar{\mathbb{R}}^+$. Then any solution u of the one-dimensional wave equation in Ω satisfies*

$$u(A) + u(C) = u(B) + u(D), \tag{5.5}$$

where $ABCD$ is any parallelogram contained in Ω with the slope 1 or -1, with A and C being two opposite points as shown in Figure 5.1.

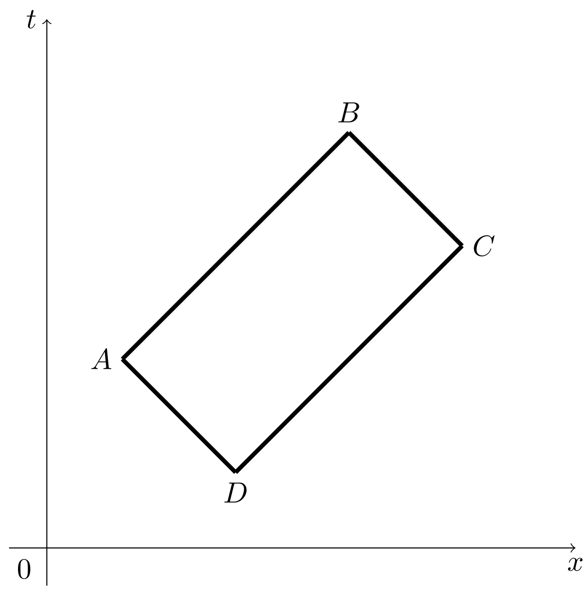

Figure 5.1. Parallelogram $ABCD$ in Lemma 5.1.

We may use this property to solve certain initial and boundary problems.

Example 5.2. Solve the initial boundary value problem:

$$\begin{cases} u_{tt} - u_{xx} = 0, & x > 0,\ t > 0, \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x > 0, \\ u(0,t) = k(t), & t > 0, \end{cases}$$

where g, h, k are given functions satisfying certain smoothness and compatibility conditions.

Solution. (See Figure 5.2.) If point $E = (x, t)$ is in the region (I); that is, $x \ge t > 0$, then $u(x, t)$ is given by (5.4):

$$u(x,t) = \frac{g(x+t) + g(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} h(y)\, dy.$$

In particular,

$$u(x,x) = \frac{g(2x) + g(0)}{2} + \frac{1}{2} \int_0^{2x} h(y) dy.$$

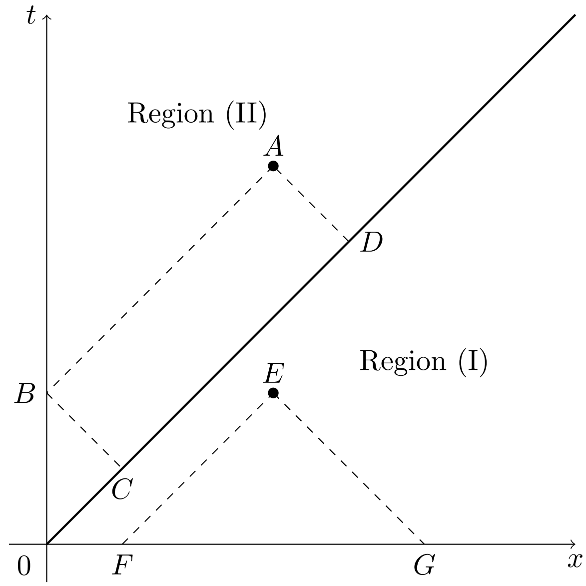

Figure 5.2. Mixed-value problem in $x > 0,\ t > 0$.

If point $A = (x, t)$ is in the region (II); that is, $0 \le x < t$, then we use the parallelogram $ABCD$ as shown in the figure to obtain $u(x,t) = u(B) + u(D) - u(C)$, where

$$u(B) = u(0, t-x) = k(t-x),$$

$$u(D) = u(\frac{x+t}{2}, \frac{x+t}{2}) = \frac{g(x+t)+g(0)}{2} + \frac{1}{2}\int_0^{x+t} h(y)\,dy,$$

$$u(C) = u(\frac{t-x}{2}, \frac{t-x}{2}) = \frac{g(t-x)+g(0)}{2} + \frac{1}{2}\int_0^{t-x} h(y)\,dy.$$

Hence, for $0 \le x < t$,

$$u(x,t) = k(t-x) + \frac{g(x+t)-g(t-x)}{2} + \frac{1}{2}\int_{t-x}^{x+t} h(y)\,dy.$$

Therefore the solution to this problem is given by

$$u(x,t) = \begin{cases} \dfrac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & (0 \le t \le x), \\ k(t-x) + \dfrac{g(x+t)-g(t-x)}{2} + \frac{1}{2}\int_{t-x}^{x+t} h(y)\,dy & (0 \le x < t). \end{cases} \tag{5.6}$$

Of course, some smoothness and compatibility conditions on g, h, k are needed in order for $u(x,t)$ to be a true solution of the problem. Derive such conditions as an exercise. □

Example 5.3. Solve the initial-boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in (0,\pi),\ t > 0, \\ u(x,0) = g(x),\ u_t(x,0) = h(x), & x \in (0,\pi), \\ u(0,t) = u(\pi,t) = 0, & t > 0. \end{cases}$$

Solution. (See Figure 5.3.) We divide the strip $(0,\pi)\times(0,\infty)$ by line segments of slope ± 1 starting first at $(0,0)$ and $(\pi,0)$ and then at all the intersection points with the boundaries.

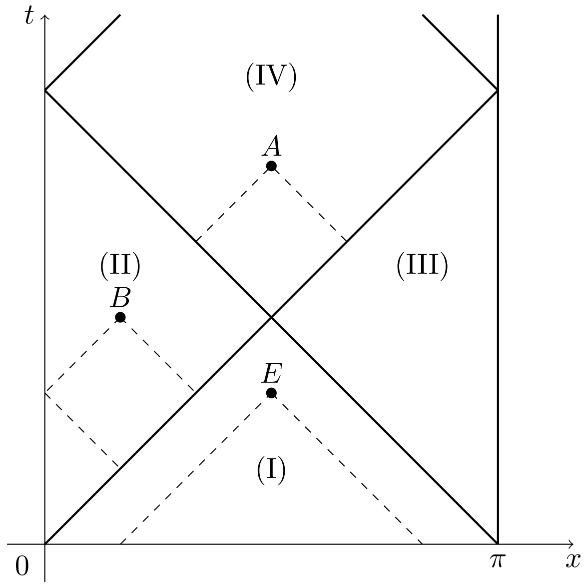

Figure 5.3. Mixed-value problem in $0 < x < \pi,\ t > 0$.

We can solve u in the region (I) by formula (5.4). In all other regions we use the parallelogram formula (5.5).

Another way to solve this problem is the method of **separation of variables**. First try $u(x,t) = X(x)T(t)$, then we should have

$$X''(x)T(t) = T''(t)X(x), \quad X(0) = X(\pi) = 0,$$

which implies that

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda,$$

where λ is a constant. From

$$X''(x) = \lambda X(x), \quad X(0) = X(\pi) = 0,$$

we find that $\lambda = -j^2$ with all $j = 1, 2, \ldots$, and

$$X_j(x) = \sin(jx), \quad T_j(t) = a_j \cos(jt) + b_j \sin(jt).$$

To make sure that u satisfies the initial condition, we consider

$$u(x,t) = \sum_{j=1}^{\infty} [a_j \cos(jt) + b_j \sin(jt)] \sin(jx). \tag{5.7}$$

To determine coefficients a_j and b_j, we use

$$u(x,0) = g(x) = \sum_{j=1}^{\infty} a_j \sin(jx), \quad u_t(x,0) = h(x) = \sum_{j=1}^{\infty} j b_j \sin(jx).$$

i.e., the a_j and jb_j are Fourier coefficients of functions $g(x)$ and $h(x)$. That is,

$$a_j = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(jx) dx, \quad b_j = \frac{2}{j\pi} \int_0^{\pi} h(x) \sin(jx) dx.$$

Substitute these coefficients into (5.7) and we obtain a formal solution u in terms of trigonometric series; the issue of convergence will not be discussed here. □

5.3. The method of spherical means

Suppose u solves the Cauchy problem of n-dimensional wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = g(x),\ u_t(x,0) = h(x), & x \in \mathbb{R}^n. \end{cases} \tag{5.8}$$

The idea is to reduce the problem to a problem of one-dimensional wave equation. This reduction requires the **method of spherical means**.

For $x \in \mathbb{R}$, $r > 0$, $t > 0$, define

$$U(x;r,t) = ⨍_{\partial B(x,r)} u(y,t)\, dS_y = M_{u(\cdot,t)}(x,r),$$

$$G(x;r) = ⨍_{\partial B(x,r)} g(y)\, dS_y = M_g(x,r), \quad H(x;r) = ⨍_{\partial B(x,r)} h(y)\, dS_y = M_h(x,r).$$

Note that

$$U(x;r,t) = \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(x,r)} u(y,t) dS_y = ⨍_{\partial B(0,1)} u(x + r\xi, t) dS_\xi.$$

So, for fixed x, the function $U(x;r,t)$ extends as a function of $r \in \mathbb{R}$ and $t \in \mathbb{R}^+$. One can recover $u(x,t)$ from $U(x;r,t)$ in terms of

$$u(x,t) = \lim_{r \to 0^+} U(x;r,t).$$

5.3.1. The Euler-Poisson-Darboux equation.

Let $u(x,t)$ be a smooth solution to (5.8). We show that, for each $x \in \mathbb{R}^n$, the function $U(x;r,t)$ solves a PDE for $(r,t) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Theorem 5.4 (Euler-Poisson-Darboux equation). *Let $u \in C^m(\mathbb{R}^n \times [0,\infty))$ with $m \geq 2$ solve (5.8). Then, for each $x \in \mathbb{R}^n$, the function $U(x;r,t) \in C^m([0,\infty) \times [0,\infty))$ and solves the Cauchy problem of the* **Euler-Poisson-Darboux equation**

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \text{in } (0,\infty) \times (0,\infty), \\ U = G,\ U_t = H & \text{on } (0,\infty) \times \{t = 0\}. \end{cases} \tag{5.9}$$

Proof. 1. Note that, for each $x \in \mathbb{R}^n$, the regularity of U on (r,t) follows easily as

$$U(x;r,t) = ⨍_{\partial B(0,1)} u(x + r\xi, t) dS_\xi.$$

2. Using the formula above, we have

$$U_r = \frac{r}{n} ⨍_{B(x,r)} \Delta u(y,t)\, dy = \frac{1}{n\alpha_n r^{n-1}} \int_{B(x,r)} u_{tt}(y,t)\, dy.$$

Hence

$$r^{n-1} U_r = \frac{1}{n\alpha_n} \int_0^r \left(\int_{\partial B(x,\rho)} u_{tt}(y,t) dS_y \right) d\rho$$

and so, differentiating with respect to r yields

$$(r^{n-1} U_r)_r = \frac{1}{n\alpha_n} \int_{\partial B(x,r)} u_{tt}(y,t) dS_y = r^{n-1} U_{tt},$$

which expands into the Euler-Poisson-Darboux equation. The initial condition is satisfied easily; this completes the proof of the theorem. $\square$

5.3.2. Solutions in $\mathbb{R}^3$ and Kirchhoff's formula. For the most important case of three-dimensional wave equations, we can easily see that the Euler-Poisson-Darboux equation implies

$$(rU)_{rr} = rU_{rr} + 2U_r = \frac{1}{r}(r^2U_r)_r = rU_{tt} = (rU)_{tt}.$$

That is, for fixed $x \in \mathbb{R}^3$, the function $\tilde{U}(r,t) = rU(x;r,t)$ solves the 1-D wave equation $\tilde{U}_{tt} = \tilde{U}_{rr}$ in $r > 0$, $t > 0$, with

$$\tilde{U}(0,t) = 0, \quad \tilde{U}(r,0) = rG := \tilde{G}, \quad \tilde{U}(r,0) = rH := \tilde{H}.$$

This is the mixed-value problem studied in Example 5.2; hence, by (5.6) we have

$$\tilde{U}(r,t) = \frac{1}{2}[\tilde{G}(r+t) - \tilde{G}(t-r)] + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(y)\,dy \quad (0 \le r \le t).$$

We recover $u(x,t)$ by

$$u(x,t) = \lim_{r\to 0^+} U(x;r,t) = \lim_{r\to 0^+} \frac{\tilde{U}(r,t)}{r}$$

$$= \lim_{r\to 0^+} \left[\frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{-r+t}^{r+t} \tilde{H}(y)\,dy\right] = \tilde{G}'(t) + \tilde{H}(t).$$

Therefore, we have obtained the so-called **Kirchhoff's formula for 3-D wave equation**:

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial t}\left(t ⨍_{\partial B(x,t)} g(y)\,dS_y\right) + t ⨍_{\partial B(x,t)} h(y)\,dS_y \\ &= ⨍_{\partial B(x,t)} (th(y) + g(y) + Dg(y)\cdot(y-x))\,dS_y \\ &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)} (th(y) + g(y) + Dg(y)\cdot(y-x))\,dS_y \quad (x \in \mathbb{R}^3,\ t > 0). \end{aligned} \tag{5.10}$$

5.3.3. Solutions in $\mathbb{R}^2$ by Hadamard's method of descent. Assume $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ solves problem (5.8) with $n = 2$. We would like to derive a formula of u in terms of g and h. The trick is to consider u as a solution to a 3-dimensional wave problem with one added dimension x_3 and then to use Kirchhoff's formula to find u. This is the well-known **Hadamard's method of descent**.

Define $\tilde{u}(\tilde{x},t) = u(x,t)$ for $\tilde{x} = (x,x_3) \in \mathbb{R}^3$, $t > 0$, where $x = (x_1,x_2) \in \mathbb{R}^2$. Then $\tilde{u}$ solves

$$\begin{cases} \tilde{u}_{tt} - \tilde{\Delta}\tilde{u} = 0, & (\tilde{x},t) \in \mathbb{R}^3 \times (0,\infty), \\ \tilde{u}(\tilde{x},0) = \tilde{g}(\tilde{x}),\ \tilde{u}_t(\tilde{x},0) = \tilde{h}(\tilde{x}), & \tilde{x} \in \mathbb{R}^3, \end{cases}$$

where $\tilde{\Delta}$ is the Laplacian in $\mathbb{R}^3$, $\tilde{g}(\tilde{x}) = g(x)$ and $\tilde{h}(\tilde{x}) = h(x)$. Let $\bar{x} = (x,0) \in \mathbb{R}^3$. Then, by Kirchhoff's formula,

$$u(x,t) = \tilde{u}(\bar{x},t) = \frac{\partial}{\partial t}\left(t ⨍_{\partial\tilde{B}(\bar{x},t)} \tilde{g}(\tilde{y})\,dS_{\tilde{y}}\right) + t ⨍_{\partial\tilde{B}(x,t)} \tilde{h}(\tilde{y})\,dS_{\tilde{y}},$$

where $\tilde{B}(\bar{x},t)$ is the ball in $\mathbb{R}^3$ centered at $\bar{x}$ of radius t. To evaluate this formula, we parametrize $\partial\tilde{B}(\bar{x},t)$ by parameter $z = (z_1,z_2) \in B(x,t) \subset \mathbb{R}^2$ as

$$\tilde{y} = (z, \pm\gamma(z)), \quad \gamma(z) = \sqrt{t^2 - |z-x|^2} \quad (z \in B(x,t)).$$

Note that $D\gamma(z) = (x-z)/\sqrt{t^2 - |z-x|^2}$ and thus

$$dS_{\tilde{y}} = \sqrt{1 + |D\gamma(z)|^2}\, dz = \frac{t\, dz}{\sqrt{t^2 - |z-x|^2}};$$

hence, noting that $\partial B(\tilde{x}, t)$ has the top and bottom parts with $y_3 = \pm\gamma(z)$,

$$\begin{aligned}
⨍_{\partial \tilde{B}(\tilde{x},t)} \tilde{g}(\tilde{y})\, dS_{\tilde{y}} &= \frac{1}{4\pi t^2} \int_{\partial \tilde{B}(\tilde{x},t)} \tilde{g}(\tilde{y})\, dS_{\tilde{y}} \\
&= \frac{2}{4\pi t^2} \int_{B(x,t)} \frac{g(z) t\, dz}{\sqrt{t^2 - |z-x|^2}} \\
&= \frac{1}{2\pi t} \int_{B(x,t)} \frac{g(z)\, dz}{\sqrt{t^2 - |z-x|^2}} \\
&= \frac{t}{2} ⨍_{B(x,t)} \frac{g(z)\, dz}{\sqrt{t^2 - |z-x|^2}}.
\end{aligned}$$

Similarly, we obtain the formula for $⨍_{\partial \tilde{B}(\tilde{x},t)} \tilde{h}(\tilde{y})\, dS_{\tilde{y}}$; finally, we have obtained the so-called **Poisson's formula for 2-D wave equation**:

$$\begin{aligned}
u(x,t) &= \frac{\partial}{\partial t} \left(\frac{t^2}{2} ⨍_{B(x,t)} \frac{g(z)\, dz}{\sqrt{t^2 - |z-x|^2}} \right) + \frac{t^2}{2} ⨍_{B(x,t)} \frac{h(z)\, dz}{\sqrt{t^2 - |z-x|^2}} \\
&= \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t g(z) + t^2 h(z) + t Dg(z) \cdot (z-x)}{\sqrt{t^2 - |z-x|^2}}\, dz \quad (x \in \mathbb{R}^2,\ t > 0).
\end{aligned} \tag{5.11}$$

Remark 5.2. (i) There are some fundamental differences for the wave equation between the one dimension and the dimensions $n = 2, 3$. In both Kirchhoff's formula ($n = 3$) and Poisson's formula ($n = 2$), the solution u depends on the derivative Dg of the initial data $u(x,0) = g(x)$. For example, if $g \in C^m$, $h \in C^{m-1}$ for some $m \geq 1$ then u is only C^{m-1} and hence u_t is only C^{m-2} (but $u_t(x,0) = h(x) \in C^{m-1}$); therefore, there is **loss of regularity** for the wave equation when $n = 2, 3$ (in fact for all $n \geq 2$). However, this does not happen when $n = 1$, in which u, u_t are at least as smooth as g, h.

(ii) There are also some fundamental differences between the 3-D wave equation and the 2-D wave equation. In $\mathbb{R}^2$, we need the information of initial data g, h in the whole disc $B(x,t)$ to compute the value $u(x,t)$; that is, the **domain of dependence** for (x,t) is the whole disc $B(x,t)$, while in $\mathbb{R}^3$ we only need the information of g, h on the sphere $\partial B(x,t)$ to compute the value $u(x,t)$; that is, the **domain of dependence** for (x,t) is the sphere $\partial B(x,t)$, not the solid ball $B(x,t)$. In $\mathbb{R}^3$, a "disturbance" initiated at x_0 propagates along the sharp wavefront $\partial B(x_0, t)$ and does not affect the value of u elsewhere; this is known as the **strong Huygens's principle**. In $\mathbb{R}^2$, a "disturbance" initiated at x_0 will affect the values $u(x,t)$ in the whole region $|x - x_0| \leq t$. In both cases $n = 2, 3$ (in fact all cases $n \geq 1$), the **domain of influence** of the initial data grows (with time t) at speed 1; therefore, the wave equation has the **finite speed of propagation**.

(iii) To illustrate the differences between $n = 2$ and $n = 3$ of the wave equation, imagine you are at position x in $\mathbb{R}^n$ and there is a sharp initial disturbance at position x_0 away from you at time $t = 0$. If $n = 3$ then you will only feel the disturbance (e.g., hear a screaming) *once*, exactly at time $t = |x - x_0|$; however, if $n = 2$, you will feel the disturbance (e.g., you are on a boat in a large lake and feel the wave) at *all* times $t \geq |x - x_0|$, although the effect on you will die out at $t \to \infty$.

5.4. Solutions of the wave equation for general dimensions

We now try to find the solution $u(x,t)$ of problem (5.8) in $\mathbb{R}^n \times \mathbb{R}^+$ by solving its spherical mean $U(x;r,t)$ from the Euler-Poisson-Darboux equation. We need the following useful result.

Lemma 5.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be C^{m+1}, where $m \geq 1$. Then for $k = 1, 2, \ldots, m$,*

$$(i) \quad (\frac{d^2}{dr^2})(\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}f(r)) = (\frac{1}{r}\frac{d}{dr})^k(r^{2k}f'(r));$$

$$(ii) \quad (\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}f(r)) = \sum_{j=0}^{k-1} \beta_j^k r^{j+1} \frac{d^j f}{dr^j}(r),$$

where $\beta_0^k = (2k-1)!! = (2k-1)(2k-3)\cdots 3 \cdot 1$ and β_j^k are independent of f.

Proof. Homework. $\square$

5.4.1. Solutions for the odd-dimensional wave equation. Assume that $n = 2k+1$ $(k \geq 1)$ and $u \in C^{k+1}(\mathbb{R}^n \times [0,\infty))$ is a solution to the problem (5.8) in $\mathbb{R}^n \times \mathbb{R}^+$.

As above, let $U(x;r,t)$ be the spherical mean of $u(x,t)$. Then for each $x \in \mathbb{R}^n$ the function $U(x;r,t)$ is C^{k+1} in $(r,t) \in [0,\infty) \times [0,\infty)$ and solves the Euler-Poisson-Darboux equation. Let

$$V(r,t) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U(x;r,t)) \quad (r > 0,\ t \geq 0).$$

Lemma 5.6. *We have that $V_{tt} = V_{rr}$ and $V(0^+,t) = 0$ for $r > 0,\ t > 0$.*

Proof. By part (i) of Lemma 5.5, we have

$$\begin{aligned} V_{rr} &= (\frac{\partial^2}{\partial r^2})(\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U) = (\frac{1}{r}\frac{\partial}{\partial r})^k(r^{2k}U_r) \\ &= (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}[\frac{1}{r}(r^{2k}U_r)_r] = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}[r^{2k-1}U_{tt}] = V_{tt}, \end{aligned}$$

where we have used the Euler-Poisson-Darboux equation $(r^{2k}U_r)_r = r^{2k}U_{tt}$. Finally, it follows from part (ii) of Lemma 5.5 that $V(0^+,t) = 0$. $\square$

Now that

$$V(r,0) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U(x;r,0)) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}[r^{2k-1}M_g(x,r)] := \tilde{G}(r)$$

and

$$V_t(r,0) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U_t(x;r,0)) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}[r^{2k-1}M_h(x,r)] := \tilde{H}(r).$$

Hence, by (5.6), we have

$$V(r,t) = \frac{1}{2}[\tilde{G}(r+t) - \tilde{G}(t-r)] + \frac{1}{2}\int_{t-r}^{r+t} \tilde{H}(y)dy \quad (0 \leq r \leq t). \tag{5.12}$$

By (ii) of Lemma 5.5,

$$V(r,t) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U(x;r,t)) = \sum_{j=0}^{k-1} \beta_j^k r^{j+1} \frac{\partial^j}{\partial r^j} U(x;r,t)$$

and hence

$$\begin{aligned} u(x,t) = U(x;0^+,t) &= \lim_{r\to 0^+} \frac{V(r,t)}{\beta_0^k\, r} \\ &= \frac{1}{\beta_0^k} \lim_{r\to 0^+} \left[\frac{\tilde{G}(r+t)-\tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{r+t} \tilde{H}(y)dy \right] \\ &= \frac{1}{\beta_0^k} [\tilde{G}'(t)+\tilde{H}(t)]. \end{aligned}$$

Therefore, we obtain the formula for C^{k+1}-solution of $(2k+1)$-dimensional wave equation:

$$u(x,t) = \frac{1}{\beta_0^k} \left\{ \frac{\partial}{\partial t} \left[(\frac{1}{t}\frac{\partial}{\partial t})^{k-1} \Big(t^{2k-1} M_g(x,t) \Big) \right] + (\frac{1}{t}\frac{\partial}{\partial t})^{k-1} \Big(t^{2k-1} M_h(x,t) \Big) \right\}. \tag{5.13}$$

Note that when $n=3$ (so $k=1$) this formula agrees with Kirchhoff's formula derived earlier.

In fact, the formula (5.13) defines indeed a classical solution to problem (5.8) under some smoothness assumption on initial data.

Theorem 5.7 (Solution of wave equation in odd-dimensions)**.** *If $n=2k+1\geq 3$, $g\in C^{k+2}(\mathbb{R}^n)$ and $h\in C^{k+1}(\mathbb{R}^n)$, then the function $u(x,t)$ defined by* (5.13) *belongs to $C^2(\mathbb{R}^n\times(0,\infty))$, solves the wave equation $u_{tt}=\Delta u$ in $\mathbb{R}^n\times(0,\infty)$, and satisfies the Cauchy condition in the sense that, for each $x_0\in\mathbb{R}^n$,*

$$\lim_{x\to x_0,\, t\to 0^+} u(x,t) = g(x_0), \qquad \lim_{x\to x_0,\, t\to 0^+} u_t(x,t) = h(x_0).$$

Proof. We may separate the proof in two cases: (a) $g\equiv 0$, and (b) $h\equiv 0$. The proof in case (a) is given in the text. Here we give a similar proof for case (b) by assuming $h\equiv 0$.

1. The function $u(x,t)$ defined by (5.13) becomes

$$u(x,t) = \frac{1}{\beta_0^k}\frac{\partial}{\partial t} \left[(\frac{1}{t}\frac{\partial}{\partial t})^{k-1} \Big(t^{2k-1} G(x;t) \Big) \right], \quad G(x;t) = M_g(x,t).$$

By Lemma 5.5(ii),

$$u(x,t) = \frac{1}{\beta_0^k} \sum_{j=0}^{k-1} \beta_j^k \left[(j+1)t^j \frac{\partial^j G}{\partial t^j} + t^{j+1} \frac{\partial^{j+1} G}{\partial t^{j+1}} \right] \to G(x_0,0^+) = g(x_0)$$

as $(x,t)\to(x_0,0^+)$. Also from this formula,

$$\lim_{x\to x_0, t\to 0^+} u_t(x,t) = \frac{2}{\beta_0^k} \lim_{x\to x_0, t\to 0^+} G_t(x,t).$$

Note that

$$G_t(x,t) = \frac{t}{n} ⨍_{B(x,t)} \Delta g(y)\, dy = \frac{1}{n\alpha_n t^{2k}} \int_{B(x,t)} \Delta g(y)\, dy.$$

Hence $u_t(x,t)\to 0$ as $(x,t)\to(x_0,0^+)$.

2. By Lemma 5.5(i),

$$u_t(x,t) = \frac{1}{\beta_0^k} \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^k (t^{2k} G_t), \quad u_{tt}(x,t) = \frac{1}{\beta_0^k} \frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^k (t^{2k} G_t). \tag{5.14}$$

Since

$$G_t(x,t) = \frac{1}{n\alpha_n t^{2k}} \int_{B(x,t)} \Delta g(y)\,dy = \frac{1}{n\alpha_n t^{2k}} \int_0^t \left(\int_{\partial B(x,\rho)} \Delta g(y)\,dS_y \right) d\rho,$$

we have

$$(t^{2k}G_t)_t = t^{2k} \fint_{\partial B(x,t)} \Delta g(y)\,dS_y.$$

Hence

$$u_{tt}(x,t) = \frac{1}{\beta_0^k} \frac{\partial}{\partial t} \Big(\frac{1}{t}\frac{\partial}{\partial t} \Big)^{k-1} \Big(\frac{1}{t}(t^{2k}G_t)_t \Big) = \frac{1}{\beta_0^k} \frac{\partial}{\partial t} \Big(\frac{1}{t}\frac{\partial}{\partial t} \Big)^{k-1} \Big(t^{2k-1} \fint_{\partial B(x,t)} \Delta g\,dS \Big).$$

On the other hand,

$$\Delta u(x,t) = \frac{1}{\beta_0^k} \frac{\partial}{\partial t} \Big(\frac{1}{t}\frac{\partial}{\partial t} \Big)^{k-1} \Big(t^{2k-1} \Delta G(x;t) \Big)$$

and

$$\Delta G(x;t) = \fint_{\partial B(x,t)} \Delta g(y)\,dS_y.$$

This proves $u_{tt} = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$. □

Remark 5.3. (i) In the above theorem, the solution $u(x,t)$ defined by (5.13) can be extended to $t \le 0$ in the same way as $M_g(x,t)$ and $M_h(x,t)$. Then the extended function $u \in C^2(\mathbb{R}^n \times [0,\infty))$ takes the initial data in the classical sense: $u(x,0) = g(x)$, $u_t(x,0) = h(x)$.

(ii) Let $n = 2k+1 \ge 3$. To compute $u(x,t)$ we need the information of $g, Dg, \cdots, D^k g$ and that of $h, Dh, \cdots, D^{k-1}h$ only on $\partial B(x,t)$, not on the whole ball $\bar{B}(x,t)$. Therefore, for the odd-dimensional wave equations, the **domain of dependence** for (x,t) is also the sharp wavefront $\partial B(x,t)$; so one still has the **strong Huygens' principle**.

(iii) If $n = 1$, in order for u to be C^2, we need $g \in C^2$ and $h \in C^1$. However, if $n = 2k+1 \ge 3$, in order for u to be C^2, we need $g \in C^{k+2}$ and $h \in C^{k+1}$. So, the solutions in general lose k-orders of smoothness from the initial data.

5.4.2. Solutions for the even-dimensional wave equation. Assume that $n = 2k$ is even. Suppose u is a C^{k+1} solution to the Cauchy problem (5.8). Again we use **Hadamard's method of descent** similarly as in the case $n = 2$.

Let $\tilde{x} = (x, x_{n+1}) \in \mathbb{R}^{n+1}$, where $x \in \mathbb{R}^n$. Set

$$\tilde{u}(\tilde{x},t) = u(x,t), \quad \tilde{g}(\tilde{x}) = g(x), \quad \tilde{h}(\tilde{x}) = h(x).$$

Then $\tilde{u}$ is a C^{k+1} solution to the wave equation in $\mathbb{R}^{n+1} \times (0,\infty)$ with initial data $\tilde{u}(\tilde{x},0) = \tilde{g}(\tilde{x})$, $\tilde{u}_t(\tilde{x},0) = \tilde{h}(\tilde{x})$. Since $n+1 = 2k+1$ is odd, we use (5.13) to obtain $\tilde{u}(\tilde{x},t)$ and then $u(x,t) = \tilde{u}(\bar{x},t)$, where $\bar{x} = (x,0) \in \mathbb{R}^{n+1}$. In this way, we obtain

$$u(x,t) = \frac{1}{(2k-1)!!} \left\{ \frac{\partial}{\partial t} \left[(\frac{1}{t}\frac{\partial}{\partial t})^{k-1} \Big(t^{2k-1} M_{\tilde{g}}(\bar{x};t) \Big) \right] + (\frac{1}{t}\frac{\partial}{\partial t})^{k-1} \Big(t^{2k-1} M_{\tilde{h}}(\bar{x};t) \Big) \right\}.$$

Note that

$$\begin{aligned} M_{\tilde{g}}(\bar{x},t) &= \frac{1}{t^n (n+1)\alpha_{n+1}} \int_{y_{n+1}^2 + |x-y|^2 = t^2} \tilde{g}(y, y_{n+1}) dS \\ &= \frac{2}{t^{n-1}(n+1)\alpha_{n+1}} \int_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |y-x|^2}} dy, \end{aligned}$$

since $dS = \frac{t}{\sqrt{t^2-|y-x|^2}}dy$ on the surface $y_{n+1}^2 + |y-x|^2 = t^2$. Similarly we have

$$M_{\tilde{h}}(\bar{x},t) = \frac{2}{t^{n-1}(n+1)\alpha_{n+1}} \int_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}dy.$$

Therefore, we have the following representation formula for even n:

$$\begin{aligned} u(x,t) = {} & \frac{2}{(n+1)!!\alpha_{n+1}} \Big\{ \frac{\partial}{\partial t}\Big[\big(\frac{1}{t}\frac{\partial}{\partial t}\big)^{\frac{n-2}{2}} \int_{B(x,t)} \frac{g(y)\,dy}{\sqrt{t^2-|y-x|^2}}\Big] \\ & + \big(\frac{1}{t}\frac{\partial}{\partial t}\big)^{\frac{n-2}{2}} \int_{B(x,t)} \frac{h(y)\,dy}{\sqrt{t^2-|y-x|^2}} \Big\}. \end{aligned} \tag{5.15}$$

When $n=2$, this reduces to **Poisson's formula for 2-D wave equation** obtained above.

Theorem 5.8 (Solution of wave equation in even-dimensions)**.** *If $n = 2k \ge 2$, $g \in C^{k+2}(\mathbb{R}^n)$ and $h \in C^{k+1}(\mathbb{R}^n)$, then the function $u(x,t)$ defined by (5.15) belongs to $C^2(\mathbb{R}^n \times (0,\infty))$, solves the wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$, and satisfies the Cauchy condition in the sense that, for each $x_0 \in \mathbb{R}^n$,*

$$\lim_{x\to x_0,\, t\to 0^+} u(x,t) = g(x_0), \qquad \lim_{x\to x_0,\, t\to 0^+} u_t(x,t) = h(x_0).$$

Proof. This follows from the theorem in odd dimensions. □

Remark 5.4. (i) In the theorem, note that

$$\int_{B(x,t)} \frac{g(y)\,dy}{\sqrt{t^2-|y-x|^2}} = t^{n-1} \int_{B(0,1)} \frac{g(x+tz)\,dz}{\sqrt{1-|z|^2}}$$

and that the function $G(x,t) = \int_{B(0,1)} \frac{g(x+tz)\,dz}{\sqrt{1-|z|^2}}$ is in $C^{k+2}(\mathbb{R}^n \times \mathbb{R})$. Hence, the solution $u(x,t)$ defined by (5.15) can be extended to $t \le 0$ and the extended function $u \in C^2(\mathbb{R}^n \times [0,\infty))$ takes the initial data in the classical sense: $u(x,0) = g(x)$, $u_t(x,0) = h(x)$.

(ii) Let $n = 2k \ge 2$. To compute $u(x,t)$ we need the information of g, $Dg, \cdots, D^k g$ and that of $h, Dh, \cdots, D^{k-1}h$ in the solid ball $B(x,t)$. Therefore, for the even-dimensional wave equations, the **domain of dependence** for (x,t) is the whole solid ball $\bar{B}(x,t)$.

(iii) If $n = 2k \ge 2$, in order for u to be C^2, we need $g \in C^{k+2}$ and $h \in C^{k+1}$. So, again, the solutions in general lose k-orders of smoothness from the initial data.

5.4.3. Solution of the wave equation from the heat equation*. We study another method of solving the odd-dimensional wave equations by the heat equation.

Suppose u is a bounded, smooth solution to the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x), \quad u_t(x,0) = 0 & \text{on } x \in \mathbb{R}^n, \end{cases} \tag{5.16}$$

where n is odd and g is smooth with nice decay at ∞. We extend u to negative times by *even extension* of t and then define

$$v(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} u(x,s) e^{-\frac{s^2}{4t}}\,ds \quad (x \in \mathbb{R}^n,\ t > 0).$$

Then v is bounded,

$$\begin{aligned}\Delta v(x,t) &= \frac{1}{(4\pi t)^{1/2}}\int_{-\infty}^{\infty} \Delta u(x,s)e^{-\frac{s^2}{4t}}\,ds\\ &= \frac{1}{(4\pi t)^{1/2}}\int_{-\infty}^{\infty} u_{ss}(x,s)e^{-\frac{s^2}{4t}}\,ds\\ &= \frac{1}{(4\pi t)^{1/2}}\int_{-\infty}^{\infty} u_{s}(x,s)\frac{s}{2t}e^{-\frac{s^2}{4t}}\,ds\\ &= \frac{1}{(4\pi t)^{1/2}}\int_{-\infty}^{\infty} u(x,s)\Big(\frac{s^2}{4t^2}-\frac{1}{2t}\Big)e^{-\frac{s^2}{4t}}\,ds\end{aligned}$$

and

$$v_t(x,t) = \frac{1}{(4\pi t)^{1/2}}\int_{-\infty}^{\infty} u(x,s)\Big(\frac{s^2}{4t^2}-\frac{1}{2t}\Big)e^{-\frac{s^2}{4t}}\,ds.$$

Moreover

$$\lim_{t\to 0^+} v(x,t) = g(x) \quad (x\in\mathbb{R}^n).$$

Therefore, v solves the Cauchy problem for the heat equation:

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \mathbb{R}^n\times(0,\infty)\\ v(x,0)=g(x) & \text{on } x\in\mathbb{R}^n.\end{cases}$$

As v is bounded, by uniqueness, we have

$$v(x,t) = \frac{1}{(4\pi t)^{1/2}}\int_{\mathbb{R}^n} g(y)e^{-\frac{|y-x|^2}{4t}}\,dy \quad (x\in\mathbb{R}^n,\ t>0).$$

We have two formulas for $v(x,t)$ and take $4t = 1/\lambda$ in the two formulas to obtain

$$\begin{aligned}\int_0^\infty u(x,s)e^{-\lambda s^2}\,ds &= \frac{1}{2}\Big(\frac{\lambda}{\pi}\Big)^{\frac{n-1}{2}}\int_{\mathbb{R}^n} e^{-\lambda|y-x|^2}g(y)\,dy\\ &= \frac{n\alpha_n}{2}\Big(\frac{\lambda}{\pi}\Big)^{\frac{n-1}{2}}\int_0^\infty e^{-\lambda r^2}r^{n-1}G(x;r)\,dr\end{aligned}\tag{5.17}$$

for all $\lambda>0$, where

$$G(x;r) = M_g(x,r) = ⨍_{\partial B(x,r)} g(y)\,dS_y.$$

So far, we have not used the odd dimension assumption. We will solve for u from (5.17) when $n=2k+1\ge 3$ is odd. Noticing that $-\frac{1}{2r}\frac{d}{dr}(e^{-\lambda r^2}) = \lambda e^{-\lambda r^2}$, we have

$$\begin{aligned}\lambda^{\frac{n-1}{2}}\int_0^\infty e^{-\lambda r^2}r^{n-1}G(x;r)\,dr &= \int_0^\infty \lambda^k e^{-\lambda r^2}r^{2k}G(x;r)\,dr\\ &= \frac{(-1)^k}{2^k}\int_0^\infty \Big[\Big(\frac{1}{r}\frac{d}{dr}\Big)^k(e^{-\lambda r^2})\Big]r^{2k}G(x;r)\,dr\\ &= \frac{1}{2^k}\int_0^\infty r\Big[\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^k(r^{2k-1}G(x;r))\Big]e^{-\lambda r^2}\,dr,\end{aligned}$$

where we integrated by parts k times (be careful with the operator $(\frac{1}{r}\frac{d}{dr})^k$). We can then write (5.17) as

$$\int_0^\infty u(x,r)e^{-\lambda r^2}\,dr = \frac{n\alpha_n}{\pi^k 2^{k+1}}\int_0^\infty r\Big[\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^k(r^{2k-1}G(x;r))\Big]e^{-\lambda r^2}\,dr$$

for all $\lambda > 0$. If we think of r^2 as τ, then this equation says that the Laplace transforms of two functions of τ are the same; therefore, the two functions of τ must be the same, which also implies the two functions of r are also the sam. So we obtain

$$u(x,t) = \frac{n\alpha_n}{\pi^k 2^{k+1}} t\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^k (t^{2k-1}G(x;t)) = \frac{n\alpha_n}{\pi^k 2^{k+1}} \frac{\partial}{\partial t}\left[(\frac{1}{t}\frac{\partial}{\partial t})^{k-1} \Big(t^{2k-1}G(x;t)\Big)\right], \tag{5.18}$$

which, except for the constant, agrees with the formula (5.13) with $h \equiv 0$. In fact the constant here, using $\alpha_n = \frac{\pi^{n/2}}{\Gamma(\frac{n+2}{2})}$, $\Gamma(\frac{1}{2}) = \pi^{1/2}$ and $\Gamma(s+1) = s\Gamma(s)$ for all $s > 0$,

$$\frac{n\alpha_n}{\pi^k 2^{k+1}} = \frac{(2k+1)\pi^{1/2}}{2^{k+1}\Gamma(k+1+\frac{1}{2})} = \frac{1}{(2k-1)!!} = \frac{1}{\beta_0^k}$$

is also in agreement with the constant in (5.13).

5.5. Nonhomogeneous wave equations and Duhamel's principle

We now turn to the initial value problem for nonhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = 0, \;\; u_t(x,0) = 0 & \text{on } x \in \mathbb{R}^n, \end{cases} \tag{5.19}$$

where $f(x,t)$ is a given function.

Motivated by **Duhamel's principle** used to solve the nonhomogeneous heat equations, for each $s \geq 0$, let $U(x,t;s)$ be the solution to the homogeneous Cauchy problem

$$\begin{cases} U_{tt}(x,t;s) - \Delta U(x,t;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty), \\ U(x,s;s) = 0, \;\; U_t(x,s;s) = f(x,s) & \text{on } x \in \mathbb{R}^n. \end{cases} \tag{5.20}$$

Define

$$u(x,t) = \int_0^t U(x,t;s)ds \tag{5.21}$$

Note that if $v = v(x,t;s)$ is the solution to the Cauchy problem

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = 0, \;\; v_t(x,0) = f(x,s) & \text{on } x \in \mathbb{R}^n, \end{cases} \tag{5.22}$$

then $U(x,t;s) = v(x,t-s;s)$ for all $x \in \mathbb{R}^n, \; t \geq s$. Therefore,

$$u(x,t) = \int_0^t U(x,t;s)\,ds = \int_0^t v(x,t-s;s)\,ds \quad (x \in \mathbb{R}^n, \; t > 0).$$

Theorem 5.9. *Assume $n \geq 2$ and $f \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \times [0,\infty))$. Let $U(x,t;s)$ be the solution of (5.20). Then the function u defined by (5.21) is in $C^2(\mathbb{R}^n \times [0,\infty))$ and a solution to (5.19).*

Proof. 1. The regularity of f guarantees a solution $U(x,t;s)$ is given by (5.13) if n is odd or (5.15) if n is even. In either case, $u \in C^2(\mathbb{R}^n \times [0,\infty))$.

2. A direct computation shows that

$$u_t(x,t) = U(x,t;t) + \int_0^t U_t(x,t;s)ds = \int_0^t U_t(x,t;s)ds,$$

$$u_{tt}(x,t) = U_t(x,t;t) + \int_0^t U_{tt}(x,t;s)ds = f(x,t) + \int_0^t U_{tt}(x,t;s)ds,$$

$$\Delta u(x,t) = \int_0^t \Delta U(x,t;s)ds.$$

Hence $u_{tt} - \Delta u = f(x,t)$. Clearly $u(x,0) = u_t(x,0) = 0$. □

EXAMPLE 5.10. Find a solution of the following problem

$$\begin{cases} u_{tt} - u_{xx} = te^x, \;\; (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = 0, \; u_t(x,0) = 0. \end{cases}$$

Solution. First, for each $s \geq 0$ solve

$$\begin{cases} v_{tt} = v_{xx} \;\text{ in } \mathbb{R} \times (0,\infty), \\ v(x,0) = 0, \;\; v_t(x,0) = se^x. \end{cases}$$

From d'Alembert's formula, we have

$$v = v(x,t;s) = \frac{1}{2}\int_{x-t}^{x+t} se^y dy = \frac{1}{2}s(e^{x+t} - e^{x-t}).$$

Hence $U(x,t;s) = v(x,t-s;s) = \frac{1}{2}s(e^{x+t-s} - e^{x+s-t})$ and so

$$\begin{aligned} u(x,t) &= \int_0^t U(x,t;s)ds = \frac{1}{2}\int_0^t s(e^{x+t-s} - e^{x+s-t})\, ds \\ &= \frac{1}{2}[e^{x+t}(-te^{-t} - e^{-t} + 1) - e^{x-t}(te^t - e^t + 1)] \\ &= \frac{1}{2}(-2te^x + e^{x+t} - e^{x-t}). \end{aligned}$$

□

EXAMPLE 5.11. Find a solution of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = f(x,t), & x \in \mathbb{R}^3, \; t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = 0, & x \in \mathbb{R}^3. \end{cases}$$

Solution. By Kirchhoff's formula, the solution v of the Cauchy problem

$$\begin{cases} v_{tt} - \Delta v = 0, & x \in \mathbb{R}^3, \; t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = f(x,s), & x \in \mathbb{R}^3 \end{cases}$$

is given by

$$v = v(x,t;s) = \frac{1}{4\pi t}\int_{\partial B(x,t)} f(y,s)\, dS_y \quad (x \in \mathbb{R}^3, \; t > 0).$$

Hence

$$\begin{aligned} u(x,t) &= \int_0^t v(x,t-s;s)\, ds = \frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{t-s}\, dS_y ds \\ &= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\, dS_y dr \\ &= \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\, dy. \end{aligned}$$

Note that the domain of dependence (on f) is the finite set $\{(y, t-|y-x|) \mid y \in \bar{B}(x,t)\}$, which is the boundary of a solid cone in $\mathbb{R}^3 \times \mathbb{R}^+$; the integrand on the right is called a *retarded potential.* □

5.6. Energy method and the uniqueness

There are some subtle issues about the uniqueness of the Cauchy problem of wave equations. The formulas (5.13) for odd n or (5.15) for even n hold under more and more smoothness conditions of the initial data g, h as the dimension n gets larger and larger. For initial data not too smooth we cannot use such formulas to claim the uniqueness. Instead, we use certain quantities that behave nicely for the wave equation. One such quantity is the **energy**.

5.6.1. Domain of dependence. Let $u \in C^2$ be a solution to the wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^n \times (0, \infty)$. Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$, and consider the **backward wave cone** with apex (x_0, t_0):

$$K(x_0, t_0) = \{(x, t) \mid 0 \le t \le t_0,\ |x - x_0| \le t_0 - t\}.$$

Theorem 5.12 (Domain of dependence)**.** *Let $u \in C^2(\mathbb{R}^n \times [0, \infty))$ solve the wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^n \times (0, \infty)$. If $u = u_t = 0$ on $B(x_0, t_0) \times \{t = 0\}$, then $u \equiv 0$ within $K(x_0, t_0)$.*

Proof. Define the local energy

$$e(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} \Big(u_t^2(x, t) + |Du(x, t)|^2\Big) dx \quad (0 \le t \le t_0).$$

Then

$$e(t) = \frac{1}{2} \int_0^{t_0 - t} \int_{\partial B(x_0, \rho)} \Big(u_t^2(x, t) + |Du(x, t)|^2\Big) dS_x \, d\rho$$

and so, by the divergence theorem,

$$\begin{aligned}
e'(t) &= \int_0^{t_0 - t} \int_{\partial B(x_0, \rho)} \Big(u_t u_{tt}(x, t) + Du \cdot Du_t(x, t)\Big) dS_x d\rho \\
&\quad - \frac{1}{2} \int_{\partial B(x_0, t_0 - t)} \Big(u_t^2(x, t) + |Du(x, t)|^2\Big) dS_x \\
&= \int_{B(x_0, t_0 - t)} (u_t u_{tt} + Du \cdot Du_t) dx - \frac{1}{2} \int_{\partial B(x_0, t_0 - t)} \Big(u_t^2(x, t) + |Du(x, t)|^2\Big) dS \\
&= \int_{B(x_0, t_0 - t)} u_t(u_{tt} - \Delta u) dx + \int_{\partial B(x_0, t_0 - t)} u_t \frac{\partial u}{\partial \nu} dS \\
&\quad - \frac{1}{2} \int_{\partial B(x_0, t_0 - t)} \Big(u_t^2(x, t) + |Du(x, t)|^2\Big) dS \\
&= \int_{\partial B(x_0, t_0 - t)} \Big(u_t \frac{\partial u}{\partial \nu} - \frac{1}{2} u_t^2(x, t) - \frac{1}{2} |Du(x, t)|^2\Big) dS \le 0
\end{aligned} \tag{5.23}$$

because the last integrand is less than zero; in fact,

$$u_t \frac{\partial u}{\partial \nu} - \frac{1}{2} u_t^2 - \frac{1}{2} |Du|^2 \le |u_t| |\frac{\partial u}{\partial \nu}| - \frac{1}{2} u_t^2 - \frac{1}{2} |Du|^2 \le -\frac{1}{2} (|u_t| - |Du|)^2 \le 0.$$

Now that $e'(t) \le 0$ implies that $e(t) \le e(0) = 0$ for all $0 \le t \le t_0$. Thus $u_t = Du = 0$, and consequently $u \equiv 0$ in $K(x_0, t_0)$. $\square$

Theorem 5.13 (Uniqueness of Cauchy problem for wave equation)**.** *Given any f, g, h the Cauchy problem*

$$\begin{cases} u_{tt} - \Delta u = f(x, t), & x \in \mathbb{R}^n,\ t > 0, \\ u(x, 0) = g(x), \quad u_t(x, 0) = h(x), & x \in \mathbb{R}^n \end{cases}$$

can have at most one solution u in $C^2(\mathbb{R}^n \times [0,\infty))$.

EXAMPLE 5.14. (a) Show that there exists a constant K such that

$$|u(x,t)| \le \frac{K}{t} U(0) \quad \forall\, 0 < t < T$$

whenever $T > 0$ and u is a smooth solution to the Cauchy problem of 3-D wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^3,\ 0 < t < T, \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}^3, \end{cases}$$

where $U(0) = \int_{\mathbb{R}^3} (|g| + |h| + |Dg| + |Dh| + |D^2 g|)\, dy$.

(b) Let u be a smooth solution to the wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^3 \times (0,\infty)$ satisfying

$$\lim_{t\to\infty} \frac{1}{t} \int_{\mathbb{R}^3} (|u(x,t)| + |Du(x,t)| + |u_t(x,t)| + |Du_t(x,t)| + |D^2 u(x,t)|)\, dx = 0.$$

Show that $u \equiv 0$ on $\mathbb{R}^3 \times (0,\infty)$.

Proof. The details are left as Homework. Part (b) follows from (a) by considering $\tilde{u}(x,t) = u(x, T-t)$ on $\mathbb{R}^n \times (0,T)$. ☐

5.6.2. Energy method for mixed-value problem of wave equation. Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and let $\Omega_T = \Omega \times (0,T]$. Let

$$\Gamma_T = \partial' \Omega_T = \overline{\Omega_T} \setminus \Omega_T.$$

We are interested in the uniqueness of initial-boundary value problem (mixed-value problem)

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega_T, \\ u = g & \text{on } \Gamma_T, \\ u_t = h & \text{on } \Omega \times \{t = 0\}. \end{cases} \tag{5.24}$$

Theorem 5.15 (Uniqueness of mixed-value problem for wave equation)**.** *There exists at most one solution $u \in C^2(\overline{\Omega_T})$ of* **mixed-value problem** (5.24).

Proof. Let u_1, u_2 be any two solutions; then $v = u_1 - u_2$ is in $C^2(\overline{\Omega_T})$ and solves

$$\begin{cases} v_{tt} - \Delta v = 0 \quad \text{in } \Omega_T, \\ v = 0 \quad \text{on } \Gamma_T, \\ v_t = 0 \quad \text{on } \Omega \times \{t = 0\}. \end{cases}$$

Define the energy

$$e(t) = \frac{1}{2} \int_\Omega \left(v_t^2(x,t) + |Dv(x,t)|^2 \right) dx \quad (0 \le t \le T).$$

Then, by the divergence theorem

$$e'(t) = \int_\Omega (v_t v_{tt} + Dv \cdot Dv_t) dx = \int_\Omega v_t (v_{tt} - \Delta v) dx + \int_{\partial\Omega} v_t \frac{\partial v}{\partial \nu} dS = 0$$

since $v = 0$ on $\partial\Omega$ for all $0 < t < T$ implies that $v_t = 0$ on $\partial\Omega$ for all $0 < t < T$. Therefore $v \equiv 0$ from $v = 0$ on Γ_T. ☐

5.6.3. Other initial and boundary value problems. Uniqueness of wave equation can be used to find the solutions to some mixed-value problems. Since solution is unique, any solution found in special forms will be the unique solution.

Example 5.16. Solve the Cauchy problem of the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u(x,0) = 0, \;\; u_t(x,0) = h(|x|), \end{cases}$$

where $h(r)$ is a given function.

Solution. In theory, we could use Kirchhoff's formula to find the solution; however, the computation would be too complicated. Instead, we can try to find a solution in the form of $u(x,t) = v(|x|,t)$ by solving an equation for v, which becomes exactly the Euler-Poisson-Darboux equation that can be solved easily when $n = 3$; some condition on h is needed in order to have a classical solution. Details are left as an exercise. $\square$

Example 5.17. Let $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. Solve

$$\begin{cases} u_{tt} - \Delta u + \lambda u = 0 \;\text{ in } \mathbb{R}^n \times (0, \infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x). \end{cases}$$

Solution. We use the idea of Hadamard's descent method. We first make u a solution to the wave equation in $\mathbb{R}^{n+1} \times (0, \infty)$ and recover u by this solution.

If $\lambda = \mu^2 > 0$ (the equation is called the **Klein-Gordon equation**), let $v(\tilde{x},t) = u(x,t)\cos(\mu x_{n+1})$, where $\tilde{x} = (x, x_{n+1}) \in \mathbb{R}^{n+1}$ and $x \in \mathbb{R}^n$.

If $\lambda = -\mu^2 < 0$, let $v(\tilde{x},t) = u(x,t)e^{\mu x_{n+1}}$. Then, in both cases, $v(\tilde{x},t)$ solves the wave equation and can be solved by using the formula (5.13) or (5.15). Then we recover $u(x,t)$ in both cases by $u(x,t) = v(\bar{x},t)$, where $\bar{x} = (x,0) \in \mathbb{R}^{n+1}$. $\square$

Example 5.18. Solve

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0, & x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^+, \, t > 0, \\ u(x,0) = g(x), \;\; u_t(x,0) = h(x), & x_n > 0, \\ u(x',0,t) = 0, & x' \in \mathbb{R}^{n-1}. \end{cases}$$

Solution. We extend the functions g, h to odd functions $\tilde{g}$ and $\tilde{h}$ in x_n; e.g., $\tilde{g}(x', -x_n) = -\tilde{g}(x', x_n)$ for all $x_n \in \mathbb{R}$ and $\tilde{g}(x', x_n) = g(x', x_n)$ when $x_n > 0$. We then solve

$$\begin{cases} \tilde{u}_{tt} - \Delta \tilde{u} = 0, & x \in \mathbb{R}^n, \, t > 0, \\ \tilde{u}(x,0) = \tilde{g}(x), \;\; \tilde{u}_t(x,0) = \tilde{h}(x), & x \in \mathbb{R}^n. \end{cases}$$

Since $V(x,t) = \tilde{u}(x', x_n, t) + \tilde{u}(x', -x_n, t)$ solves

$$V_{tt} - \Delta V = 0, \;\; V(x,0) = 0, \;\; V_t(x,0) = 0,$$

the uniqueness result implies $V \equiv 0$, i.e., $\tilde{u}$ is an odd function in x_n. Hence $u = \tilde{u}|_{x_n > 0}$ is the solution to the original problem. $\square$

Example 5.19. Let Ω be a bounded domain. Solve

$$\begin{cases} u_{tt} - \Delta u = f(x,t) & \text{in } \Omega \times (0, \infty), \\ u(x,0) = g(x), \;\; u_t(x,0) = h(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, \; t \geq 0. \end{cases}$$

Solution. Use the method of separation variables and try to find the solution of the form

$$u = \sum_{j=1}^{\infty} u_j(x) T_j(t),$$

where

$$-\Delta u_j = \lambda_j u_j, \quad u_j|_{\partial\Omega} = 0,$$
$$T_j''(t) - \lambda_j T(t) = w_j(t), \quad T_j(0) = a_j, \quad T_j'(0) = b_j,$$

By the elliptic theory, eigenfunctions $\{u_j(x)\}_{j=1}^{\infty}$ form an orthonormal basis of $L^2(\Omega)$, and $w_j(t)$, a_j, b_j are the Fourier coefficients of $w(x,t)$, $g(x)$ and $h(x)$ with respect to $u_j(x)$, respectively.

The question is whether the series gives indeed a true solution; we do not study such questions in this course.

□

5.7. Finite speed of propagation for second-order linear hyperbolic equations

We study a class of special second-order linear partial differential equations of the form

$$u_{tt} + Lu = 0 \quad (x \in \mathbb{R}^n, \ t > 0),$$

where L has a special form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u,$$

with smooth symmetric coefficients $(a^{ij}(x))$ satisfying uniform ellipticity condition on $\mathbb{R}^n$. In this case, we say the operator $\partial_{tt} + L$ is **uniformly hyperbolic**.

Let $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. Assume $q(x)$ is a continuous function on $\mathbb{R}^n$, positive and smooth in $\mathbb{R}^n \setminus \{x_0\}$ and $q(x_0) = 0$. Consider a *curved backward cone*

$$C = \{(x,t) \in \mathbb{R}^n \times (0, t_0) \mid q(x) < t_0 - t\}$$

and for each $0 \le t \le t_0$ let

$$C_t = \{x \in \mathbb{R}^n \mid q(x) < t_0 - t\}.$$

Assume $\partial C_t = \{x \in \mathbb{R}^n \mid q(x) = t_0 - t\}$ is a smooth surface for each $t \in [0, t_0)$. In addition, we assume

$$\sum_{i,j=1}^{n} a^{ij}(x) D_i q(x) D_j q(x) \le 1 \quad (x \in \mathbb{R}^n \setminus \{x_0\}). \tag{5.25}$$

Lemma 5.20. *Let $\beta(x,t)$ be a smooth function and*

$$\alpha(t) = \int_{C_t} \beta(x,t)\, dx \quad (0 < t < t_0).$$

Then

$$\alpha'(t) = \int_{C_t} \beta_t(x,t)\, dx - \int_{\partial C_t} \frac{\beta(x,t)}{|Dq(x)|}\, dS_x.$$

Proof. This follows from the **co-area formula**. □

Theorem 5.21 (Domain of dependence)**.** *Let u be a smooth solution to $u_{tt} + Lu = 0$ in $\mathbb{R}^n \times [0, \infty)$. If $u = u_t = 0$ on C_0, then $u \equiv 0$ with the cone C.*

Proof. Define the local energy

$$e(t) = \frac{1}{2}\int_{C_t} \Big(u_t^2 + \sum_{i,j=1}^{n} a^{ij} D_i u D_j u\Big) dx \quad (0 \le t \le t_0).$$

Then the lemma above implies

$$\begin{aligned} e'(t) &= \int_{C_t} \Big(u_t u_{tt} + \sum_{i,j=1}^{n} a^{ij} D_i u D_j u_t\Big) dx - \frac{1}{2}\int_{\partial C_t} \Big(u_t^2 + \sum_{i,j=1}^{n} a^{ij} D_i u D_j u\Big) \frac{1}{|Dq|} dS \\ &:= A - B. \end{aligned}$$

Note that $a^{ij} D_i u D_j u_t = D_j(a^{ij} u_t D_i u) - u_t D_j(a^{ij} D_i u)$ (no sum); hence, integrating by parts and by the equation $u_{tt} + Lu = 0$, we have

$$\begin{aligned} A &= \int_{C_t} u_t \Big(u_{tt} - \sum_{i,j=1}^{n} D_j(a^{ij} D_i u)\Big) dx + \int_{\partial C_t} \sum_{i,j=1}^{n} u_t a^{ij} (D_i u) \nu_j dS \\ &= -\int_{C_t} u_t \sum_{i,j=1}^{n} D_i u D_j a^{ij}\, dx + \int_{\partial C_t} \sum_{i,j=1}^{n} u_t a^{ij} (D_i u) \nu_j dS, \end{aligned}$$

where $\nu = (\mu_1, \nu_2, \cdots, \nu_n)$ is the outer unit normal on ∂C_t. In fact,

$$\nu_j = \frac{D_j q}{|Dq|} \quad (j = 1, 2, \cdots, n) \quad \text{on } \partial C_t.$$

Since $\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i,j=1}^{n} a^{ij} v_i w_j$ defines an inner product on $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, by **Cauchy-Schwartz's inequality**,

$$\Big| \sum_{i,j=1}^{n} a^{ij} (D_i u) \nu_j \Big| \le \Big(\sum_{i,j=1}^{n} a^{ij} D_i u D_j u\Big)^{1/2} \Big(\sum_{i,j=1}^{n} a^{ij} \nu_i \nu_j\Big)^{1/2}.$$

Therefore

$$\begin{aligned} |A| &\le Ce(t) + \int_{\partial C_t} |u_t| \Big(\sum_{i,j=1}^{n} a^{ij} D_i u D_j u\Big)^{1/2} \Big(\sum_{i,j=1}^{n} a^{ij} \nu_i \nu_j\Big)^{1/2} dS \\ &\le Ce(t) + \frac{1}{2}\int_{\partial C_t} \Big(u_t^2 + \sum_{i,j=1}^{n} a^{ij} D_i u D_j u\Big) \Big(\sum_{i,j=1}^{n} a^{ij} \nu_i \nu_j\Big)^{1/2} dS. \end{aligned}$$

However, since $\nu_j = (D_j q)/|Dq|$, by (5.25), we have

$$\Big(\sum_{i,j=1}^{n} a^{ij} \nu_i \nu_j\Big)^{1/2} \le \frac{1}{|Dq|} \quad \text{on } \partial C_t.$$

Consequently, we derive that $|A| \le Ce(t) + B$ and thus

$$e'(t) \le Ce(t) \quad (0 < t < t_0).$$

Sine $e(0) = 0$, this gives $e(t) \equiv 0$. Hence $u \equiv 0$ within C. $\square$

5.7.1. Energy method for mixed-value problems. Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and let $\Omega_T = \Omega \times (0,T]$. Let

$$\Gamma_T = \partial'\Omega_T = \overline{\Omega_T} \setminus \Omega_T.$$

We are interested in the uniqueness of initial-boundary value problem (mixed-value problem)

$$\begin{cases} u_{tt} + Lu + B(x,t) \cdot Du + c(x,t)u = f \text{ in } \Omega_T, \\ u = g \text{ on } \Gamma_T, \quad\quad u_t = h \text{ on } \Omega \times \{t=0\}, \end{cases} \tag{5.26}$$

where Lu is defined as above, B and c are bounded functions on $\overline{\Omega_T}$, and g, h are given functions.

Theorem 5.22 (Uniqueness of mixed-value problem)**.** *There exists at most one solution $u \in C^2(\overline{\Omega_T})$ of* **mixed-value problem** (5.26)*.*

Proof. Let u_1, u_2 be any two solutions; then $v = u_1 - u_2$ is in $C^2(\overline{\Omega_T})$ and solves

$$\begin{cases} v_{tt} + Lv + B(x,t) \cdot Dv + c(x,t)v = 0 \quad \text{in } \Omega_T, \\ v = 0 \quad \text{on } \Gamma_T, \qquad v_t = 0 \quad \text{on } \Omega \times \{t=0\}. \end{cases}$$

Define the energy

$$e(t) = \frac{1}{2}\int_\Omega \Big[v_t^2(x,t) + v^2(x,t) + \sum_{i,j=1}^n a^{ij}(x) v_{x_i} v_{x_j} \Big]\, dx \quad (0 \le t \le T).$$

Then

$$e'(t) = \int_\Omega \Big(v_t v_{tt} + v v_t + \sum_{i,j=1}^n a^{ij} v_{x_i} v_{x_j t} \Big)\, dx. \tag{5.27}$$

Since $a^{ij} v_{x_i} v_{x_j t} = (a^{ij} v_{x_i} v_t)_{x_j} - a^{ij}_{x_j} v_{x_i} v_t - a^{ij} v_{x_i x_j} v_t$ and since $v = 0$ on $\partial\Omega$ for all $0 < t < T$ implies that $v_t = 0$ on $\partial\Omega$ for all $0 < t < T$, it follows by the divergence theorem that

$$\begin{aligned} \int_\Omega a^{ij}(x) v_{x_i} v_{x_j t}\, dx &= \int_{\partial\Omega} a^{ij} v_{x_i} v_t \nu_j\, dS - \int_\Omega (a^{ij}_{x_j} v_{x_i} v_t + a^{ij} v_{x_i x_j} v_t)\, dx \\ &= -\int_\Omega (a^{ij}_{x_j} v_{x_i} v_t + a^{ij} v_{x_i x_j} v_t)\, dx. \end{aligned}$$

Therefore, using the equation $v_{tt} + Lv = -B \cdot Dv - cv$, by (5.27), we obtain that

$$\begin{aligned} e'(t) &= \int_\Omega v_t \Big(v - B \cdot Dv - cv - \sum_{i,j=1}^n a^{ij}_{x_j} v_{x_i} \Big)\, dx \\ &\le C \int_\Omega (v_t^2 + v^2 + |Dv|^2)\, dx \le C \int_\Omega \Big(v_t^2 + v^2 + \sum_{i,j=1}^n a^{ij} v_{x_i} v_{x_j} \Big) dx, \end{aligned}$$

where the last inequality follows from the uniform positivity of matrix (a^{ij}). Hence $e'(t) \le Ce(t)$ on $(0,T)$. Note that $e(t) \ge 0$ and $e(0) = 0$; this implies $e(t) \equiv 0$ on $[0,T]$, proving $v \equiv 0$ on Ω_T. $\square$