Chapter 2

Basic Properties of Analytic Functions

2.1. Analytic and Harmonic Functions; the Cauchy-Riemann Equations

A function f defined in a complex domain D is **differentiable** at a point $z_0 \in D$ if the limit

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

exists; the limit, if it exists, is called the **complex derivative** of f at z_0 and denoted by $f'(z_0)$; that is,

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}.$$

If f is differentiable at each point of the domain D then f is called **analytic** in D; in this case, the derivative function is defined by

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}.$$

(Note that h is complex number.) A function analytic on the whole complex plane is called an **entire function**.

Note that the limit $f'(z_0)$ above is required to exist (and thus is equal to the same number) *no matter how z approaches z_0 or h approaches* 0 **in the complex plane.** Also, clearly, if f is differentiable at z_0 then it is continuous at z_0.

Like in Calculus, the procedure of finding $f'(z)$ is called **differentiation** of f. Differentiation follows the usual rules for differentiation of real variable functions. For example, if f and g are differentiable at z_0 then so are functions $af+bg$ (a, b are constant complex numbers), fg and f/g (if $g(z_0)\neq 0$) at z_0, and the derivatives can be found by the usual formulas:

$$(af+bg)'(z_0) = af'(z_0)+bg'(z_0);$$

$$(fg)'(z_0) = f(z_0)g'(z_0)+f'(z_0)g(z_0);$$

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0)-f(z_0)g'(z_0)}{(g(z_0))^2} \quad \text{if } g(z_0)\neq 0.$$

Furthermore, the **chain rule** for differentiation is also valid: If f is analytic in the range of a function g defined in a domain D and if g is differentiable at a point $z_0 \in D$, then $f(g(z))$ is differentiable at z_0 and

$$(f(g(z)))'(z_0) = f'(g(z_0))g'(z_0).$$

EXAMPLE 1. (1) Let $f(z) = z^n$, where n is a positive integer. Then f is entire and $f'(z)$ can be found as follows.

$$f'(z) = \lim_{h\to 0} \frac{(z+h)^n - z^n}{h}$$
$$= \lim_{h\to 0} \frac{(z^n + nz^{n-1}h + \cdots + h^n) - z^n}{h} = nz^{n-1}.$$

(2) Let $f(z) = e^z$. Then f is an entire function. We now compute $f'(z)$. Note that

$$f'(z) = \lim_{h\to 0} \frac{e^{z+h} - e^z}{h} = \lim_{h\to 0} \frac{e^z e^h - e^z}{h}$$
$$= \lim_{h\to 0} e^z \frac{e^h - 1}{h} = e^z \lim_{h\to 0} \frac{e^h - 1}{h}.$$

We want to show the limit

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

This is a limit as complex variable $h \to 0$. So let $h = \sigma + i\tau \neq 0$. Then

$$e^h - 1 - h = [e^\sigma \cos\tau - 1 - \sigma] + i[e^\sigma \sin\tau - \tau]$$
$$= e^\sigma(\cos\tau - 1) + (e^\sigma - 1 - \sigma) + ie^\sigma(\sin\tau - \tau) + i\tau(e^\sigma - 1).$$

Since $|h| \geq |\sigma|$ and $|h| \geq |\tau|$, we have by the triangle inequality that

$$\left|\frac{e^h - 1}{h} - 1\right| = \left|\frac{e^h - 1 - h}{h}\right|$$
$$\leq e^\sigma \left|\frac{1 - \cos\tau}{\tau}\right| + \left|\frac{e^\sigma - 1 - \sigma}{\sigma}\right| + e^\sigma \left|\frac{\sin\tau - \tau}{\tau}\right| + |e^\sigma - 1|.$$

As $|h| \to 0$, we have both $\sigma \to 0$ and $\tau \to 0$, and hence the limit of each of the four terms in the last expression is zero by elementary calculus of real variable functions (l'Hǒpital's Rule). Therefore

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Hence

$$(e^z)' = e^z.$$

The Cauchy–Riemann Equations. Let $f(z) = u(x, y) + iv(x, y)$ with $z = x + iy$ in a domain D of complex plane, where u, v are real-valued functions of (x, y). Suppose f is differentiable at a point $z_0 = x_0 + iy_0 \in D$. This means the limit

$$L = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. Let $h = \sigma + i\tau \neq 0$. Then the quotient of difference

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{[u(x_0 + \sigma, y_0 + \tau) - u(x_0, y_0)] + i[v(x_0 + \sigma, y_0 + \tau) - v(x_0, y_0)]}{\sigma + i\tau}.$$

The point is that this quotient of difference has limit L as $h = \sigma + i\tau \to 0$ and there are many ways in the complex plane where $h = \sigma + i\tau$ can approach 0. First, by choosing $h = \sigma \to 0$ $(\tau = 0)$, we have

$$L = \lim_{\sigma \to 0} \left[\frac{u(x_0 + \sigma, y_0) - u(x_0, y_0)}{\sigma} + i \frac{v(x_0 + \sigma, y_0) - v(x_0, y_0)}{\sigma} \right]$$
$$= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0),$$

where the two partial derivatives must also exist. Secondly, by choosing $h = i\tau \to 0$ $(\sigma = 0)$, we have

$$L = \lim_{\tau \to 0} \left[\frac{u(x_0, y_0 + \tau) - u(x_0, y_0)}{i\tau} + i \frac{v(x_0, y_0 + \tau) - v(x_0, y_0)}{i\tau} \right]$$
$$= -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0),$$

where the two partial derivatives must also exist. Therefore, since $L = L$, we must have

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \tag{2.1}$$

This is the **Cauchy-Riemann equations**. Therefore, we have proved the following

Theorem 2.1. *If $f = u + iv$ is differentiable at $z_0 = x_0 + iy_0$ in a domain D, then all the partial derivatives u_x, v_x, u_y and v_y exist at (x_0, y_0) and satisfy the Cauchy-Riemann equations*

$$u_x = v_y, \quad u_y = -v_x \quad \text{at } (x_0, y_0). \tag{2.2}$$

The converse is also valid. We have the following result under some additional conditions.

Theorem 2.2. *Suppose $f = u + iv$ and all of the partial derivatives u_x, u_y, v_x and v_y exist and are continuous in a domain D containing $z_0 = x_0 + iy_0$. If the Cauchy-Riemann equations*

$$u_x = v_y, \quad u_y = -v_x$$

hold at (x_0, y_0), then f is differentiable at z_0 with $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Proof. Let $h = \sigma + i\tau$ with sufficiently small real numbers σ, τ. By Taylor's Theorem,

$$u(x_0 + \sigma, y_0 + \tau) = u(x_0, y_0) + \sigma u_x(x_0, y_0) + \tau u_y(x_0, y_0) + \sigma E_1 + \tau E_2,$$

where E_1, E_2 depend on σ, τ and approach zero as σ and τ approach zero. Likewise,

$$v(x_0 + \sigma, y_0 + \tau) = v(x_0, y_0) + \sigma v_x(x_0, y_0) + \tau v_y(x_0, y_0) + \sigma E_3 + \tau E_4,$$

where E_3, E_4 depend on σ, τ and approach zero as σ and τ approach zero. By the Cauchy–Riemann equations at $z_0 = x_0 + iy_0$, we can write

$$f(z_0 + h) = f(z_0) + [u_x(x_0, y_0) + iv_x(x_0, y_0)]h + \sigma(E_1 + iE_3) + \tau(E_2 + iE_4).$$

From this and the fact that $|\sigma| \leq |h|$, $|\tau| \leq |h|$, we obtain

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

exists. This is exactly the statement that f is differentiable at z_0, with $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$. □

Laplace's Equation, Harmonic Functions and Harmonic Conjugates. Note that from the Cauchy-Riemann equations (2.2) it follows that if the real and imaginary parts u, v of an analytic function $f = u + iv$ have second-order partial derivatives at every point in a domain D, then they must satisfy the **Laplace's equation**:

$$\Delta u \equiv u_{xx} + u_{yy} = 0, \quad \Delta v \equiv v_{xx} + v_{yy} = 0$$

in D. Such functions u, v are called the **harmonic functions** in D.

If u and v satisfy the Cauchy-Riemann equations (2.2) at every point in a domain D, then v is called the **harmonic conjugate** of u in D. Note that the harmonic conjugate is uniquely determined up to an additive constant. Therefore, the imaginary part of an analytic function is uniquely determined by the real part of the function up to additive constants.

EXAMPLE 2. Show $u(x, y) = x^3 - 3xy^2$ is harmonic and find its harmonic conjugate.

Solution. It is easy to check

$$\Delta u = u_{xx} + u_{yy} = 6x - 6x = 0,$$

hence u is harmonic. To find the harmonic conjugate of u, we need to solve for v that satisfies the Cauchy-Riemann equations

$$v_x = -u_y = 6xy, \quad v_y = u_x = 3x^2 - 3y^2.$$

From the first equation, integrate $6xy$ with respect to x and find $v(x, y) = 3x^2y + p(y)$, where $p(y)$ is independent of x. To determine this $p(y)$ we need to use the second equation above. Since $v_y = 3x^2 + p'(y)$, the second equation implies $p'(y) = -3y^2$ and hence $p(y) = -y^3 + C$, where C is a constant. Hence the harmonic conjugate of u is given by

$$v(x, y) = 3x^2y - y^3 + C,$$

where C is any constant. Note that the complex function $f(z) = u + iv = x^3 - 3xy^2 + i(3x^2y - y^3 + C) = (x + iy)^3 + iC = z^3 + iC$ is certainly analytic. □

Theorem 2.3. *Suppose that $f = u + iv$ is analytic on a domain D. If either* Re$f = u$ *is constant on D or* Im$f = v$ *is constant on D, or $|f|^2 = u^2 + v^2$ is constant on D, then f is constant (that is, both u and v are constants) on D.*

Proof. This theorem will be used later; so we give a proof. We first prove that f must be a constant on D if Re$f = u$ is constant on D. To show this, we use the Cauchy–Riemann equations to deduce that $v_x = -u_y = 0$ and $v_y = u_x = 0$ on D, and hence v must be constant on each horizontal and vertical line segments lying in D. Since D is connected, this shows that v must be constant on D and hence $f = u + iv$ is constant on D. Now assume that Im$f = v$ is constant on D. Then Re$(if) = -v$ is constant on D, and hence by the result just proved, if is constant on D.

We now assume $|f|^2 = u^2 + v^2$ is constant on D and would like to show f is itself constant on D. If $|f| = 0$ on D then $f = 0$. Assume $|f| = c > 0$ on D. Then $f(z) \neq 0$ for all $z \in D$. Hence $\frac{1}{f(z)}$ is also analytic on D. Note that

$$|f(z)|^2 = f(z)\,\overline{f(z)} = c^2 \neq 0 \quad \text{hence} \quad \overline{f(z)} = \frac{c^2}{f(z)}.$$

Therefore $\bar{f} = c^2/f$ is analytic on D. This implies that $g(z) = f(z) + \overline{f(z)}$ is analytic on D. For this analytic function g, we have Im$g = 0$. By the conclusion just proved, g must be constant on D. However, since $g = 2\text{Re}f$, this implies Ref is constant on D. Again by the result proved above, f itself must be constant on D. □

Exercises. Page 84. 1(a, b), 3, 4, 5, 8, 9, 10, 18, 19, 20(a, b, c), 23.

2.2. Power Series

A **power series** in z is an infinite series of the special form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

where $a_0, a_1, \cdots$, are complex numbers, called the **coefficients** of the series; z_0 is fixed and is called the **center** of the series.

The first major result on power series is as follows.

Theorem 2.4. *Suppose there is some $z_1 \neq z_0$ such that the series $\sum a_n(z_1-z_0)^n$ converges. Then for each z with $|z-z_0| < |z_1-z_0|$, the series $\sum a_n(z-z_0)^n$ is absolutely convergent.*

Proof. Since $\sum a_n(z_1-z_0)^n$ converges, it follows that

$$\lim_{n\to\infty} a_n(z_1-z_0)^n = 0.$$

Hence, there exists a number $M > 0$ such that $|a_n||z_1-z_0|^n \leq M$ for all $n = 1, 2, \cdots$. Note that

$$|a_n(z-z_0)^n| = |a_n||z-z_0|^n$$
$$= |a_n||z_1-z_0|^n \left(\frac{|z-z_0|}{|z_1-z_0|}\right)^n \leq M\rho^n,$$

where $\rho = \frac{|z-z_0|}{|z_1-z_0|}$. Therefore $\sum a_n(z-z_0)^n$ converges absolutely if $0 \leq \rho < 1$; that is, $|z-z_0| < |z_1-z_0|$. □

The Radius of Convergence. Given a power series $\sum a_n(z-z_0)^n$, define

$$S = \{z\colon \sum a_n(z-z_0)^n \text{ converges}\}.$$

Then, there are three mutually exclusive cases:

$$(a)\ S = \{z_0\}; \quad (b)\ S = \mathbf{C}; \quad (c)\ S \neq \{z_0\}, \mathbf{C}.$$

In any cases, define

$$R = \sup\{|z-z_0|\colon z \in S\}.$$

Of course, $R = 0$ in the case (a) and $R = \infty$ in the case (b) above. In the case (c), we have $0 < R < \infty$ and that the series $\sum a_n(z-z_0)^n$ converges **absolutely** on $|z-z_0| < R$ and diverges on $|z-z_0| > R$. For, if $z_1 \in S$ and $z_2 \notin S$ then $0 < |z_1-z_0| \leq R \leq |z_2-z_0| < \infty$; also, for any z with $|z-z_0| < R$, there is $z' \in S$ such that $|z-z_0| < |z'-z_0|$ and hence by Theorem 2.4, $\sum a_n(z-z_0)^n$ converges absolutely. On the other hand, it is easy to see that $S \subset \{|z-z_0| \leq R\}$ and hence any z with $|z-z_0| > R$ does not belong to S; that is $\sum a_n(z-z_0)^n$ diverges.

The number R so defined is called the **radius of convergence** of the power series $\sum a_n(z-z_0)^n$.

Therefore the radius of convergence of a power series is the radius of the open disc inside which the series converges and outside which the series diverges; note that such a disc is unique, and there is no information about the convergence on the boundary of the disc. This disc is sometimes called the **disc of convergence** for the power series.

Theorem 2.5. *Let R be the radius of convergence of the power series $\sum a_n(z-z_0)^n$.*

(a) (**Ratio Test**) *If $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ exists, then $R = \frac{1}{L}$.*

(b) (**Root Test**) *If $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists, then $R = \frac{1}{L}$.*

(c) *In general,*

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

EXAMPLE 3. (1) The power series

$$\sum_{n=0}^{\infty} (z-z_0)^n$$

has the coefficients $a_n = 1$ and hence the radius of convergence $R = 1$ by any of the tests in the above theorem. This means this power series (called the **geometric series** of ratio $z - z_0$) converges (absolutely) for all $|z-z_0| < 1$ and diverges for all $|z-z_0| > 1$. Note that the series diverges also for all $|z - z_0| = 1$. We also know the value of this series is given by

$$\sum_{n=0}^{\infty} (z-z_0)^n = \frac{1}{1-(z-z_0)} = \frac{1}{1+z_0-z} \quad \forall\ |z-z_0| < 1. \tag{2.3}$$

(2) Find the radius of convergence of power series $\sum_0^\infty 5^n(z-1)^n$ and find a **closed form** (that is a simplified value) for the series in its disc of convergence.

Solution. Since $a_n = 5^n$, we have

$$R = 1/\lim_{n\to\infty} \sqrt[n]{5^n} = 1/5.$$

To find a closed form for the convergent series, we have that, for all $|z-1| < \frac{1}{5}$,

$$\sum_{n=0}^{\infty} 5^n(z-1)^n = \sum_{n=0}^{\infty} (5z-5)^n$$

$$= \frac{1}{1-(5z-5)} = \frac{1}{6-5z} \quad \text{by (2.3) above since } |5z-5| < 1.$$

□

(3) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} 4^{n(-1)^n} z^n.$$

Solution. Note that $a_n = 4^{n(-1)^n}$; that is,

$$a_n = \begin{cases} 4^n & \text{if } n = 0, 2, 4, \cdots \\ \frac{1}{4^n} & \text{if } n = 1, 3, 5, \cdots. \end{cases}$$

Note that the ratio and root tests do not work in this case. However the test (c) in the theorem above gives

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{4^{n(-1)^n}}} = \frac{1}{4}.$$

□

(4) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} 4^n z^{3n}.$$

Solution. If we write the series out, it becomes

$$\sum_{n=0}^{\infty} 4^n z^{3n} = 1 + 4z^3 + 4^2 z^6 + 4^3 z^9 + 4^4 z^{12} + \cdots = \sum_{m=0}^{\infty} a_m z^m,$$

where $a_m = 4^{m/3}$ if $m = 0, 3, 6, 9, \cdots$ and $a_m = 0$ for all other m. So technically, we cannot use either tests. But if we let $w = z^3$ then the series becomes a power series for w:

$$\sum_{n=0}^{\infty} 4^n w^n = \sum_{n=0}^{\infty} (4w)^n,$$

which is a geometric series with ratio $r = 4w$. Hence this series converges when $|4w| < 1$ and diverges when $|4w| \geq 1$; therefore the original series converges if $4|z^3| < 1$ that is $|z| < 1/\sqrt[3]{4}$ and diverges if $|z| \geq 1/\sqrt[3]{4}$. Hence the radius of convergence is $R = 1/\sqrt[3]{4}$. □

The Derivative of a Power Series. We can now study some properties of the function defined by a power series on its disc of convergence.

Theorem 2.6. *Let the radius of convergence of power series $\sum a_n(z - z_0)^n$ be $R > 0$. Then the function defined by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is infinitely differentiable within the disc $|z - z_0| < R$, each derivative being again given by a power series which is obtained from the original series by term-by-term differentiation; that is,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (z - z_0)^{n-k}, \quad k = 1, 2, \cdots.$$

In particular, by setting $z = z_0$, we have

$$a_k = \frac{f^{(k)}(z_0)}{k!}, \quad k = 0, 1, 2, \cdots. \tag{2.4}$$

We shall not prove this theorem, but remark that if a function f can be written as a power series around z_0 then this power series must be the **Taylor series** of f:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

Example 4. Show that

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z \quad \text{for all complex number } z.$$

Proof. Let

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Then $f(0) = 1$. Since the radius of this series is $R = \infty$, by the theorem above, $f(z)$ is infinitely differentiable at all z and

$$f'(z) = \sum_{n=1}^{\infty} n\frac{z^{n-1}}{n!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = f(z).$$

Therefore

$$(e^{-z}f(z))' = -e^{-z}f(z) + e^{-z}f'(z) = 0 \quad \forall z.$$

Hence $e^{-z}f(z)$ must be a constant C for all z and so $f(z) = Ce^z$. But $f(0) = 1$ so $C = 1$. Therefore $f(z) = e^z$, as needed. □

EXAMPLE 5. Find the power series of $\sin z$ and $\cos z$ about $z = 0$.

Solution. Since

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

we have

$$\begin{aligned}\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right) \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \quad \forall\ z.\end{aligned}$$

Since $\sin z = -(\cos z)'$, we obtain

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \quad \forall\ z.$$

□

Multiplication and Substitution of Power Series. Sometimes, we can find the power series of product, quotient and composition of power series. This can be done just like polynomials. For example,

$$\left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(z-z_0)^n\right) = \sum_{n=0}^{\infty} c_n(z-z_0)^n,$$

where

$$c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0 = \sum_{k=0}^{n} a_kb_{n-k}.$$

In particular, we may just need the following: for all $k = 1, 2, \cdots$,

$$(z-z_0)^k \sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0} a_n(z-z_0)^{n+k}.$$

When finding the power series (Taylor's series) of a function of type $f(g(z))$ at $z = z_0$, where $g(z)$ has power series at z_0 and $f(w)$ has power series at $w = w_0 = g(z_0)$ as follows:

$$g(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad f(w) = \sum_{n=0}^{\infty} b_n(w-w_0)^n.$$

We can replace $w = g(z)$ into the power series of $f(w)$ to obtain the power series of $f(g(z))$ at z_0. This is especially useful when one of f and g is a polynomial of $z - z_0$.

EXAMPLE 6. (1) Find the power series of $e^z/(1-z)$ about z_0 in $|z| < 1$.

Solution. We know

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad \text{all } z$$

and

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots, \quad |z| < 1.$$

Hence

$$\frac{e^z}{1-z} = 1 + 2z + \frac{5}{2}z^2 + \frac{8}{3}z^3 + \frac{65}{24}z^4 + \cdots, \quad |z| < 1.$$

□

(2) By substitution, we have

$$e^{z^3} = \sum_{n=0}^{\infty} \frac{z^{3n}}{n!}.$$

(3) Find the power series of $(1-z^2)\sin z$ about $z_0 = 0$.

$$(1-z^2)\sin z = (1-z^2)\sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^{2n+1}}{(2n+1)!} = (1-z^2)\left(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots\right)$$

$$= z - \frac{7}{6}z^3 + \frac{21}{120}z^5 - \cdots.$$

Exercises. Page 103. Problems 1, 2, 3, 7, 8, 9, 14, 15, 18

2.3. Cauchy's Theorem and Cauchy's Formula

The **Cauchy's Theorem** and **Cauchy's Formula** are the linchpin of complex variables.

Theorem 2.7 (**Cauchy's Theorem**). *Let f be analytic in a domain D and let γ be a piecewise smooth simple closed curve in D whose inside Ω also lies in D. Then*

$$\int_\gamma f(z)\,dz = 0. \tag{2.5}$$

Proof. The proof of this theorem under the additional assumption that f' **is continuous in** D follows directly from the Green's Theorem above. For example, in this case, one can apply Green's Theorem to obtain

$$\int_\gamma f(z)\,dz = i\iint_\Omega (f_x + if_y)\,dxdy.$$

Now, if $f = u + iv$ then, by the Cauchy-Riemann equations,

$$f_x + if_y = (u_x + iv_x) + i(u_y + iv_y) = (u_x - v_y) + i(u_y + v_x) = 0 + i0 = 0 \quad \text{in } D.$$

Hence (2.5) holds.

However, without the hypothesis of continuity of f', we cannot apply Green's Theorem; so we must use a more sophisticated technique to prove this theorem. This starts with the following Cauchy-Goursat Theorem. □

Theorem 2.8 (**Cauchy-Goursat Theorem**). *Suppose f is analytic in a domain D. If Γ is a triangle in D whose inside Δ is also in D, then*

$$\int_\Gamma f(z)\,dz = 0.$$

Proof*. Let $I = |\int_\Gamma f(z)\,dz|$. So $I \geq 0$, and we want to show $I = 0$. Divide the solid triangle $\Gamma \cup \Delta$ into four triangles by joining the midpoints of the three sides of Γ. Orient the boundaries of all triangles positively, call the four small solid triangles $\Omega_1, \cdots, \Omega_4$ and let their boundaries be $\gamma_1, \cdots, \gamma_4$, respectively. Then

$$I = \left|\int_\Gamma f(z)\,dz\right| = \left|\sum_{j=1}^4 \int_{\gamma_j} f(z)\,dz\right|$$

$$\leq \sum_{j=1}^4 \left|\int_{\gamma_j} f(z)\,dz\right| \leq 4\left|\int_{\gamma_m} f(z)\,dz\right|,$$

where $m \in \{1,2,3,4\}$ is chosen so that

$$I_1 = \left|\int_{\gamma_m} f(z)\,dz\right| = \max\left\{\left|\int_{\gamma_j} f(z)\,dz\right| : j = 1,2,3,4\right\}.$$

Let then $\Delta_1 = \Omega_m$ and $\Gamma_1 = \gamma_m$. Then divide the solid triangle Δ_1 into four triangles by joining the midpoints of its sides as above to obtain a solid triangle Δ_2 with positively oriented boundary Γ_2 such that

$$I_1 \leq 4I_2 = 4\left|\int_{\Gamma_2} f(z)\,dz\right|.$$

Continuing in this way, we obtain a sequence of solid triangles $\{\Delta_j\}$ with positively oriented boundary Γ_j, $j = 1, 2, \cdots$, such that

$$(i) \quad \Delta_{j+1} \text{ is a subset of } \Delta_j$$

$$(ii) \quad \text{length}(\Gamma_{j+1}) = \frac{1}{2}\text{length}(\Gamma_j)$$

$$(iii) \quad \text{the diameter of } \Delta_{j+1} = \frac{1}{2}\{\text{the diameter of } \Delta_j\}$$

$$(iv) \quad \text{if } I_j = |\textstyle\int_{\Gamma_j} f(z)\,dz|, \text{ then } I_j \leq 4I_{j+1}.$$

In particular, for $j = 1, 2, \cdots$

$$(ii)' \quad \text{length}(\Gamma_j) = \frac{1}{2^j}\text{length}(\Gamma)$$

$$(iii)' \quad \text{diameter}(\Delta_j) = \frac{1}{2^j}\text{diameter}(\Delta)$$

$$(iv)' \quad I \leq 4^j I_j.$$

Therefore, there exists a unique point z_0 in D that lies in all the Δ_j. Since f is differentiable at z_0 so, given $\epsilon > 0$, there is a small $\delta > 0$ such that the disc $\{|z - z_0| < \delta\}$ is in D and

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon \quad \text{for all} \quad 0 < |z - z_0| < \delta.$$

Equivalently,

$$|f(z) - [f(z_0) + f'(z_0)(z - z_0)]| \leq \epsilon|z - z_0| \quad \forall\ |z - z_0| < \delta.$$

We know there exists j_0 such that the triangles Δ_j lies within the disc $\{|z - z_0| < \delta\}$ for all $j \geq j_0$. By the direct computation or Green's Theorem, we know that

$$\int_{\Gamma_j} dz = \int_{\Gamma_j} z\, dz = 0$$

(or by the Cauchy's Theorem just proved under the additional assumption). Therefore, for all $j \geq j_0$,

$$\begin{aligned} I_j = \left| \int_{\Gamma_j} f(z)\, dz \right| &= \left| \int_{\Gamma_j} \big(f(z) - [f(z_0) + f'(z_0)(z - z_0)] \big)\, dz \right| \\ &\leq \epsilon \max_{z \in \Gamma_j} |z - z_0| \cdot \mathrm{length}(\Gamma_j) \\ &\leq \epsilon\, \mathrm{diameter}(\Delta_j) \cdot \mathrm{length}(\Gamma_j) \\ &\leq \epsilon \frac{1}{4^j} \mathrm{diameter}(\Delta) \cdot \mathrm{length}(\Gamma). \end{aligned}$$

Finally by $(iv)'$ above, we find, for all $j \geq j_0$,

$$0 \leq I \leq 4^j I_j \leq \epsilon\, \mathrm{diameter}(\Delta) \cdot \mathrm{length}(\Gamma)$$

for arbitrary $\epsilon > 0$. This proves $I = 0$. □

We now introduce a class of special domains. A domain D is called **simply-connected** if, whenever γ is a simple closed curve in D, the inside of γ is also in D.

Theorem 2.9. *Let D be a simply-connected domain and let f be an analytic function in D. Then*

$$\int_\gamma f(z)\, dz = 0$$

for any closed polygonal curve γ in D.

Proof. By connecting some vertices to form certain triangles, γ can be separated into the sum of closed triangles with line integrals on overlapping sides counted and thus cancelled in opposite directions. Therefore this theorem follows from the Cauchy–Goursat theorem proved above. □

Theorem 2.10. *Let D be a simply-connected domain and let f be an analytic function in D. Then there exists an analytic function F in D such that $F'(z) = f(z)$ for all $z \in D$.*

Proof. Fix a point z_0 in D and define a function F in D as follows. Given any $z \in D$ let γ be any polygonal curve connecting z_0 to z. Define

$$F(z) = \int_\gamma f(\zeta)\, d\zeta.$$

First, we claim $F(z)$ is independent of the choice of the polygonal curve γ and hence F is well-defined. To see this, let δ be another polygonal curve joining z_0 to z in D. Let $\Gamma = \gamma \cup (-\delta)$ be the closed polygonal curve consisting of γ and the reversed curve of δ. Then, by the previous theorem,

$$0 = \int_\Gamma f(\zeta)\, d\zeta = \int_\gamma f(\zeta)\, d\zeta - \int_\delta f(\zeta)\, d\zeta.$$

Hence $\int_\gamma f(\zeta)\, d\zeta = \int_\delta f(\zeta)\, d\zeta$; this proves that $F(z)$ is independent of the curve γ.

Next we show that F is differentiable at every point $z \in D$ and $F'(z) = f(z)$. This is to prove

$$\lim_{h\to 0}\left[\frac{F(z+h)-F(z)}{h} - f(z)\right] = 0.$$

To prove this, assume the disc $\{|\zeta - z| < \delta\}$ is contained in D for some $\delta > 0$. Let $z = x+iy$ and $h = \sigma + i\tau$ with $0 < |h| < \delta$. Let L be the line segment joining z to $z+h$. Then L is in D since it is in the disc $\{|\zeta - z| < \delta\}$. Let γ be any polygonal curve γ joining z_0 to z in D and let $\gamma \cup L$ be the polygonal curve consisting of γ and L. Then since $F(z)$ and $F(z+h)$ are independent of the curves, we have

$$F(z+h) = \int_{\gamma\cup L} f(\zeta)\, d\zeta; \quad F(z) = \int_\gamma f(\zeta)\, d\zeta.$$

Hence

$$F(z+h) - F(z) = \int_L f(\zeta)\, d\zeta.$$

Note that, by an easy computation,

$$\int_L f(z)\, d\zeta = f(z)\int_L d\zeta = f(z)h.$$

Therefore

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_L [f(\zeta) - f(z)]\, d\zeta.$$

So

$$\begin{aligned}\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| &\le \frac{1}{|h|}\max_{|\zeta - z|\le |h|} |f(\zeta) - f(z)|\cdot \text{length}(L)\\ &= \max_{|\zeta - z|\le |h|} |f(\zeta)-f(z)| \to 0 \quad \text{as } |h|\to 0.\end{aligned}$$

Therefore, $F'(z) = f(z)$ for all $z \in D$. □

Theorem 2.11. *Let f and F be analytic in a domain D and $F'(z) = f(z)$ for all $z \in D$. Let Γ be any piecewise smooth curve in D with starting point A and ending point B. Then*

$$\int_\Gamma f(z)\, dz = F(B) - F(A).$$

In particular, if γ is a piecewise smooth closed curve in D, then

$$\int_\gamma f(z)\, dz = 0.$$

Proof. Assume Γ itself is smooth with parameterization $\gamma(t)$ with $a \le t \le b$ and $\gamma(a) = A$, $\gamma(b) = B$. It is easy to see

$$[F(\gamma(t))]' = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t) \quad \forall\ t \in [a,b].$$

Hence, by definition of the line integral and the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_\Gamma f(z)\, dz &= \int_a^b f(\gamma(t))\gamma'(t)\, dt\\ &= \int_a^b [F(\gamma(t))]'\, dt = F(\gamma(b)) - F(\gamma(a)) = F(B) - F(A).\end{aligned}$$

This integral would be zero if $B = A$, which is the case when Γ is closed. □

Final Proof of Cauchy's Theorem. Let D be any domain and a simple closed curve γ along with its inside be contained in D. Then there exists a simply-connected subdomain D' in D that contains γ and its inside. Since D' is simply-connected, by the theorem above, there exists an analytic function F in D' such that $F'(z) = f(z)$ for all $z \in D'$. Hence by the previous theorem

$$\int_\gamma f(z)\,dz = 0$$

since γ is a closed curve in D'. This completes the proof of Cauchy's Theorem. □

Theorem 2.12 (**Cauchy's Formula**). *Let f be analytic in a domain D and γ a piecewise smooth positively oriented simple closed curve in D whose inside Ω also lies in D. Then*

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz \quad \textit{for all } a \in \Omega. \tag{2.6}$$

Proof. Let $B_\epsilon = \{|z-a| < \epsilon\} \subset\subset \Omega$ and $\Omega_\epsilon = \Omega \setminus \bar{B}_\epsilon$. Let CD be a fixed diameter on the circle $|z-a| = \epsilon$; let E, F be two other points on the circle so that the closed arc $CEDFC$ represents the positively oriented circle $\delta_\epsilon = \{|z-a| = \epsilon\}$. Let A, B be the first intersecting point with γ of ray aC and ray aD, respectively; let G, H be two other points on γ so that the closed loop $AGBHA$ along γ represents the positively oriented simple closed curve γ. Finally, let $\Gamma_1 = AGBDECA$ and $\Gamma_2 = ACFDBHA$ be two closed piece-wise smooth curves. (See the figure below.)

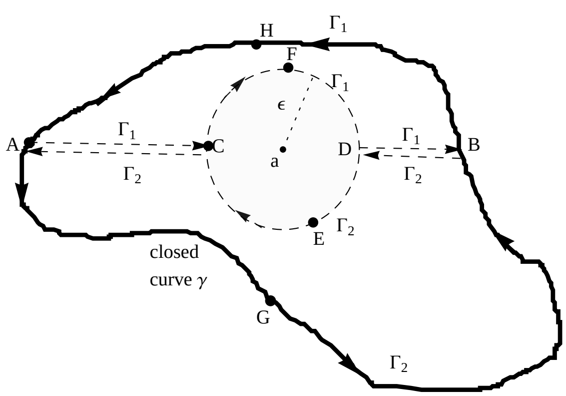

Figure 2.1. The proof of Cauchy's Formula

Let $g(z) = \frac{f(z)}{z-a}$. Then, by Cauchy's Theorem above,

$$\int_{\Gamma_1} g(z)\,dz = 0, \quad \int_{\Gamma_2} g(z)\,dz = 0.$$

Add the two integrals, cancel the line integrals on segments AC with CA and BD with DB, and note that the line integrals on arc DEC and CFD are combined to the line integral on the reversed circle $-\delta_\epsilon$. Hence we have

$$\int_\gamma g(z)\,dz = \int_{\delta_\epsilon} g(z)\,dz.$$

Therefore

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz &= \frac{1}{2\pi i}\int_{\delta_\epsilon} \frac{f(z)}{z-a}\,dz \\ &\to f(a) \quad \text{as } \epsilon \to 0, \text{ by Example (3) of §1.6.}\end{aligned}$$

□

Applications of Cauchy's Theorem/Formula. Cauchy's Theorem and Formula can be used to evaluate certain line integrals and definite integrals. See more applications later in §2.6.

Example 7. Evaluate the integral

$$\int_\gamma \frac{z}{z+1}\,dz,$$

where γ is any curve in the domain $\{z : \mathrm{Im}z > 0\}$ joining the points $-1+2i$ and $1+2i$. (Or replace this example by any of Exercises 9, 11, 12 to avoid the logarithmic function.)

Solution. The integrand

$$f(z) = \frac{z}{z+1} = 1 - \frac{1}{z+1}$$

is analytic in the domain $\{z : \mathrm{Im}z > 0\}$, which is simply connected. Therefore, there exists an analytic function $F(z)$ on the domain such that $F'(z) = f(z)$. But what is $F(z)$ then? In fact, let

$$F(z) = z - \mathrm{Log}(z+1),$$

where $\mathrm{Log}(z+1)$ is defined and is analytic on the whole plane with the ray $(-\infty, -1]$ deleted, and hence $F(z)$ is analytic on the given domain $\{z : \mathrm{Im}z > 0\}$, and it can be seen that $F'(z) = f(z)$ on the domain. Therefore,

$$\begin{aligned}\int_\gamma f(z)\,dz &= F(B) - F(A) = F(1+2i) - F(-1+2i) \\ &= [1+2i - \mathrm{Log}(2+2i)] - [-1+2i - \mathrm{Log}(2i)] \\ &= 2 - [\ln(\sqrt{8}) + i\frac{\pi}{4}] + [\ln 2 + i\frac{\pi}{2}] = 2 - \frac{\ln 2}{2} + i\frac{\pi}{4}.\end{aligned}$$

□

Example 8. Evaluate

$$\int_{|z+1|=2} \frac{z^2}{z^2-4}\,dz.$$

Solution. The integrand $\frac{z^2}{z^2-4}$ can be written as $\frac{f(z)}{z+2}$, where $f(z) = \frac{z^2}{z-2}$. Since $f(z)$ is analytic on $|z+1| \le 2$ and -2 is inside the circle $|z+1| = 2$, by Cauchy's Formula, we have

$$\int_{|z+1|=2} \frac{z^2}{z^2-4}\,dz = \int_{|z+1|=2} \frac{f(z)}{z+2}\,dz = 2\pi i f(-2) = 2\pi i(-1) = -2\pi i.$$

□

Example 9. Evaluate

$$\int_0^{2\pi} \frac{1}{2+\sin\theta}\, d\theta.$$

Solution. We try to write this integral as the defining integral for a line integral $\int_\gamma g(z)\,dz$ for some complex function g and a closed curve γ. If the curve γ has the smooth parameterization $\gamma(\theta)$ with $\theta \in [0, 2\pi]$, then

$$\int_\gamma g(z)\,dz = \int_0^{2\pi} g(\gamma(\theta))\gamma'(\theta)\,d\theta.$$

We would like to choose $\gamma(\theta)$ so that the right-hand side definite integral is exactly the one we need to compute. Usually, for the definite integral involving $\sin\theta$ or $\cos\theta$, we always choose $\gamma(\theta) = e^{i\theta}$ and hence γ is the unit circle. Introduce

$$z = \gamma(\theta) = e^{i\theta}, \quad 0 \le \theta \le 2\pi,$$

then

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - \frac{1}{z}),$$
$$dz = \gamma'(\theta)\,d\theta = ie^{i\theta}\,d\theta = iz\,d\theta$$

and hence

$$d\theta = \frac{1}{iz}\,dz.$$

The interval $\theta \in [0, 2\pi]$ is transformed to the simple closed positively oriented unit circle $|z| = 1$. Hence

$$\int_0^{2\pi} \frac{1}{2+\sin\theta}\,d\theta = \int_{|z|=1} \frac{1}{2+\frac{1}{2i}(z-\frac{1}{z})}\frac{1}{iz}\,dz = \int_{|z|=1} \frac{2}{z^2+4iz-1}\,dz.$$

Therefore, we need to evaluate the line integral

$$\int_{|z|=1} \frac{2}{z^2+4iz-1}\,dz.$$

We try to use Cauchy's Formula. Note that

$$z^2 + 4iz - 1 = (z-p)(z-q); \quad p = i(\sqrt{3}-2), \quad q = -i(\sqrt{3}+2),$$

and that p **is inside the circle** $|z| = 1$ **and** q **is outside** $|z| = 1$. Hence

$$\frac{2}{z^2+4iz-1} = \frac{2}{(z-p)(z-q)} = \frac{f(z)}{z-p},$$

where $f(z) = \frac{2}{z-q}$ is analytic on and inside the circle $|z| = 1$. Therefore, by Cauchy's Formula,

$$\int_{|z|=1} \frac{2}{z^2+4iz-1}\,dz = \int_{|z|=1} \frac{f(z)}{z-p}\,dz = 2\pi i f(p)$$
$$= 2\pi i \frac{2}{p-q} = 2\pi i \frac{2}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

Finally we have

$$\int_0^{2\pi} \frac{1}{2+\sin\theta}\,d\theta = \int_{|z|=1} \frac{2}{z^2+4iz-1}\,dz = \frac{2\pi}{\sqrt{3}}.$$

□

Example 10. Let $0 < a < 1$ be a given real number. Evaluate

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}.$$

Solution. Use the same substitution $z = e^{i\theta}$ as above and we have

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad d\theta = \frac{1}{iz}\,dz$$

and hence

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} = \int_{|z|=1} \frac{1}{1 - a(z + \frac{1}{z}) + a^2}\frac{1}{iz}\,dz$$

$$= -\frac{1}{ai}\int_{|z|=1} \frac{1}{(z-a)(z-\frac{1}{a})}\,dz = -\frac{1}{ai}\int_{|z|=1} \frac{f(z)}{z-a}\,dz \quad (\text{where } f(z) = \tfrac{1}{z - \frac{1}{a}})$$

$$= -\frac{1}{ai}\left[2\pi i f(a)\right] = \frac{2\pi}{1-a^2}.$$

The function

$$P_a(\theta) = \frac{1}{2\pi}\frac{1-a^2}{1 - 2a\cos\theta + a^2}, \quad 0 < a < 1, \; 0 \le \theta \le 2\pi,$$

is called the **Poisson kernel** and satisfies

$$\int_0^{2\pi} P_a(\theta)\,d\theta = 1, \quad 0 < a < 1.$$

□

Exercises. Page 116. Problems 1–7, 9, 11, 12.

2.4. Consequences of Cauchy's Theorem

Theorem 2.13. *Suppose that f is analytic in a domain D and $z_0 \in D$. If the disc $\{z : |z - z_0| < R\}$ lies in D, then*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \forall \; |z - z_0| < R,$$

where the coefficients a_k are given by

$$a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,d\zeta \quad k = 0, 1, 2, \cdots,$$

where γ is the positively oriented circle $\{\zeta : |\zeta - z_0| = r\}$ for any $0 < r < R$.

Proof. First, by Cauchy's Theorem, the number a_k defined above is independent of r for all $0 < r < R$. Given any z with $|z - z_0| < R$, let $r > 0$ be a fixed number such that $|z - z_0| < r < R$, and let γ be the positively oriented circle $|\zeta - z_0| = r$. Then by Cauchy's Formula,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Note that, for all $\zeta \in \gamma$; that is $|\zeta - z_0| = r > |z - z_0|$, it follows by the geometric series that

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\,\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}$$

$$= \frac{1}{\zeta - z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{\zeta - z_0}\right)^k = \sum_{k=0}^{\infty}\frac{(z-z_0)^k}{(\zeta - z_0)^{k+1}}.$$

Since this infinite series converges absolutely and uniformly on $\zeta \in \gamma$, the interchange of orders of integration and summation is legitimate and yields that

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_\gamma \left(\sum_{k=0}^{\infty} f(\zeta)\frac{(z-z_0)^k}{(\zeta - z_0)^{k+1}}\right) d\zeta$$

$$= \sum_{k=0}^{\infty}(z-z_0)^k \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,d\zeta = \sum_{k=0}^{\infty} a_k (z-z_0)^k.$$

□

From this theorem, we obtain the following important facts about the analytic functions.

FACT 1. *If f is analytic in a domain D then f has all orders of derivatives in D and each higher order derivative $f^{(k)}$ is also analytic in D.*

This is due to the fact that an analytic function can be written as a power series at each of the points in the domain. In fact, by the formula for a_k and the property of Taylor's formula, we have the following **generalized Cauchy's Formula**: If f is analytic in a domain D and $z_0 \in D$, then

$$f^{(k)}(z_0) = \frac{k!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,d\zeta \quad k = 0, 1, 2, \cdots$$

for all piecewise smooth positively oriented simple closed curve γ in D whose inside containing z_0 is also in D.

FACT 2. *If f is analytic in a domain D and $f^{(k)}(z_0) = 0$ for all $k = 0, 1, 2, \cdots$ at some point $z_0 \in D$, then $f(z) = 0$ for all $z \in D$.*

Proof. Let z_1 be any point in D. Let Γ be a polygonal curve in D joining z_0 to z_1. One can finds a finite number of closed discs in D, say, $\Delta_1, \cdots, \Delta_N$ with the property: the center of Δ_1 is z_0 and the centers of all Δ_k are on the polygonal curve and the center of Δ_k is inside Δ_{k-1} for all $k = 2, 3, \cdots, N$ and $z_1 \in \Delta_N$. Since Δ_1 is a disc in D with center z_0, by the theorem above, $f(z) = 0$ for all $z \in \Delta_1$, and hence $f^{(k)}$ is zero for all $k = 0, 1, 2, \cdots$ at the center of Δ_2 hence $f \equiv 0$ on Δ_2. Repeating in this way we have $f \equiv 0$ on Δ_N; in particular, $f(z_1) = 0$. □

FACT 3: The order of zero. Suppose that f is analytic in a domain D and $f(z_0) = 0$ at some $z_0 \in D$. Suppose that f is not identically zero in D. By Fact 2 above, there must be a positive integer m such that

$$f^{(k)}(z_0) = 0 \quad k = 0, 1, \cdots, m-1; \quad f^{(m)}(z_0) \neq 0.$$

In this case, near z_0 in D

$$f(z) = \sum_{k=m}^{\infty} a_k (z-z_0)^k = (z-z_0)^m [a_m + a_{m+1}(z-z_0) + \cdots],$$

where $a_m \neq 0$. In this case, we say f has a **zero of order** m at z_0.

It is easy to see that an analytic function f in D has a zero of order m at $z_0 \in D$ if and only if

$$f(z) = (z - z_0)^m g(z) \quad \text{where } g \text{ is analytic in } D \text{ and } g(z_0) \neq 0.$$

Morera's Theorem. The following theorem shows that Cauchy-Goursat Theorem exactly characterizes the analytic functions.

Theorem 2.14 (**Morera's Theorem**). *Let f be continuous in a domain D. If*

$$\int_\gamma f(z)\, dz = 0$$

holds for all triangles γ in D whose inside also lies in D, then f is analytic in D.

Proof. Let $z_0 \in D$ and let disc $\Delta = \{\zeta : |\zeta - z_0| < R\}$ be also in D. Define a function $F(z)$ in Δ as follows:

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta,$$

where the notation means the line integral on the line segment joining z_0 to z. By the definition, we can easily show that F is analytic in Δ and $F'(z) = f(z)$ in Δ. This proves that F is analytic in Δ; so by Fact 1 above f is also analytic in Δ. Hence f is analytic in D. □

Theorem 2.15 (**Liouville's Theorem**). *Let f be an entire and bounded function. Then f must be a constant.*

Proof. By Theorem 2.13 above,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{for all } z,$$

where

$$a_k = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{k+1}}\, dz \quad k = 0, 1, 2 \cdots,$$

where R is any positive number. Let $|f(z)| \leq M$ for all z. Then, for all $k = 1, 2, \cdots$,

$$|a_k| = \frac{1}{2\pi} \left| \int_{|z|=R} \frac{f(z)}{z^{k+1}}\, dz \right|$$

$$\leq \frac{1}{2\pi} \max_{|z|=R} \left| \frac{f(z)}{z^{k+1}} \right| \cdot (\text{Length of circle } |z| = R)$$

$$\leq \frac{1}{2\pi} \frac{M}{R^{k+1}} 2\pi R = \frac{M}{R^k} \to 0$$

as $R \to \infty$. Hence $a_k = 0$ for all $k = 1, 2, \cdots$. Therefore $f(z) = a_0$ is constant. □

Analytic Logarithms. *Let f be analytic in a simply-connected domain D and $f(z) \neq 0$ for all $z \in D$. Then there exists an analytic function g in D such that*

$$f(z) = e^{g(z)} \quad \forall\ z \in D.$$

Proof. Since the function $\frac{f'(z)}{f(z)}$ is analytic in the simply-connected domain D, by Theorem 2.10 above, there exists an analytic function F in D such that

$$F'(z) = \frac{f'(z)}{f(z)} \quad \forall\ z \in D.$$

Hence

$$[e^{-F(z)}f(z)]' = e^{-F(z)}f'(z) - e^{-F(z)}F'(z)f(z) = 0 \quad \forall\ z \in D.$$

Therefore $e^{-F(z)}f(z)$ is constant in D; let $e^{-F(z)}f(z) = C$ be a constant. Hence

$$f(z) = Ce^{F(z)} = e^{\mathrm{Log}C}e^{F(z)} = e^{g(z)}, \quad g(z) = \mathrm{Log}C + F(z),$$

and $g(z)$ is analytic in D and satisfies $f(z) = e^{g(z)}$ in D. Note that function $g(z)$ is not unique, for $g(z) + 2\pi ki$ with any integer k will provide another analytic function satisfying the same requirement. Any such function $g(z)$ is called an **analytic logarithm** of $f(z)$. $\square$

Exercises. Page 133. Problems 1, 2, 3, 9, 12, 17, 18, 21.

2.5. Isolated Singularities

A function f is said to have an **isolated singularity** at a point z_0 if f is analytic in a *punctured disc* $\{0 < |z - z_0| < r\}$ for some $r > 0$. The meaning of "isolated" is that in the full disc $\{|z - z_0| < r\}$, z_0 is the only (possible) point at which f is **not** differentiable. The isolated singularities play an important role in the applications of complex variables.

There are precisely three possible distinct cases for a function f to have an isolated singularity at z_0:

(a) $|f(z)|$ is bounded in $0 < |z - z_0| < r' < r$ for some $0 < r' < r$.

(b) $\lim_{z \to z_0} |f(z)| = +\infty$.

(c) Neither (a) nor (b) holds. This is equivalent to the following conditions:

(c)'

$$\limsup_{z \to z_0} |f(z)| = +\infty; \quad \liminf_{z \to z_0} |f(z)| < \infty.$$

In case (a), we say f has a **removable singularity** at z_0 (see below for the reason); in case (b) we say f has a **pole** at z_0; while in case (c), we say f has an **essential singularity** at z_0.

We shall only study the first two cases; the essential singularities will not be studied.

Removable Singularities. Assume case (a) holds; that is, assume $|f(z)| \leq M$ for all $0 < |z - z_0| < r'$, where $r' \in (0, r)$ is a constant. Let

$$g(z) = \begin{cases} (z - z_0)^2 f(z) & 0 < |z - z_0| < r \\ 0 & z = z_0. \end{cases}$$

Then g is defined in the full disc $D = \{|z - z_0| < r\}$, and we show that g is analytic in D. This is to say that g is differentiable at every point in D. Certainly, g is differentiable at each $z \in D \setminus \{z_0\}$. At z_0

$$g'(z_0) = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} (z - z_0) f(z) = 0$$

since $|f(z)|$ is bounded near z_0. Therefore g is analytic in D and $g(z_0) = g'(z_0) = 0$. Let m be the order of zero of g at z_0. Then $m \geq 2$ and

$$g(z) = (z - z_0)^m h(z) \quad \text{where } h \text{ is analytic in } D \text{ and } h(z_0) \neq 0.$$

Let

$$\tilde{f}(z) = (z - z_0)^{m-2} h(z) \quad z \in D.$$

Then $\tilde{f}$ is analytic in D and for all $z \in D \setminus \{z_0\}$

$$\tilde{f}(z) = (z - z_0)^{m-2} \frac{g(z)}{(z - z_0)^m} = \frac{g(z)}{(z - z_0)^2} = f(z).$$

Therefore, f can be extended to the whole disc as an analytic function $\tilde{f}$. Hence, in this case, z_0 is called a **removable singularity** for f.

Poles. In case (b) above, $|f(z)| \to \infty$ as $z \to z_0$; without loss of generality, assume

$$|f(z)| \geq 1 \qquad \forall \; 0 < |z - z_0| < r.$$

In this case, function $g(z) = \frac{1}{f(z)}$ is a bounded analytic function in $0 < |z - z_0| < r$; that is, g has a removable singularity at z_0. Therefore there exists an analytic function $\tilde{g}$ in the full disc $D = \{|z - z_0| < r\}$ such that $\tilde{g}(z) = g(z)$ for $z \in D \setminus \{z_0\}$. Note that

$$|\tilde{g}(z_0)| = \lim_{z \to z_0} |g(z)| = \lim_{z \to z_0} \frac{1}{|f(z)|} = 0$$

by the assumption in case (b). Hence $\tilde{g}$ has a zero at z_0 and let m be the order of zero of $\tilde{g}$ at z_0. Then

$$\tilde{g}(z) = (z - z_0)^m h(z) \quad \text{where } h \text{ is analytic in } D \text{ and } h(z_0) \neq 0.$$

Therefore, for all $0 < |z - z_0| < r$, we have

$$f(z) = \frac{1}{g(z)} = \frac{1}{(z - z_0)^m} \frac{1}{h(z)} = \frac{H(z)}{(z - z_0)^m},$$

where $H(z)$ is analytic in $D = \{|z - z_0| < r\}$ and $H(z_0) \neq 0$.

In this case, we say f has a **pole of order** m at the singularity z_0. Therefore, *the order of pole of f at z_0 is the same as the order of zero of $\frac{1}{f}$ at z_0.*

The Residue at a Singularity. Assume f is analytic in a punctured disc $\{0 < |z - z_0| < r\}$ for some $r > 0$. Then the number

$$\frac{1}{2\pi i} \int_{|z - z_0| = s} f(z)\, dz$$

is independent of the number s as long as $0 < s < r$. This can be easily seen from the Green's Theorem or an application of Cauchy's Theorem. We define this number to be the **residue** of f at z_0 and denote it by the notation

$$\text{Res}(f; z_0) = \frac{1}{2\pi i} \int_{|z - z_0| = s} f(z)\, dz. \tag{2.7}$$

This is the definition of residue and does not give the most effective way to compute it. We shall study some more effective ways to compute residues.

The Computation of Residues. Assume

$$f(z) = \frac{H(z)}{(z-z_0)^m},$$

where H is analytic in a disc $\{|z-z_0| < r\}$ and m is a positive integer. We claim

$$\text{(2.8)} \qquad \operatorname{Res}(f; z_0) = \operatorname{Res}\left(\frac{H(z)}{(z-z_0)^m}; z_0\right) = \frac{H^{(m-1)}(z_0)}{(m-1)!}.$$

To see this, since H is analytic in $\{|z-z_0| < r\}$ we have

$$H(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k, \quad |z-z_0| < r,$$

where

$$c_k = \frac{H^{(k)}(z_0)}{k!} \quad \forall\ k = 0, 1, 2, \cdots.$$

Therefore

$$\begin{aligned} f(z) &= \frac{c_0}{(z-z_0)^m} + \cdots + \frac{c_{m-1}}{z-z_0} + c_m + c_{m+1}(z-z_0) + \cdots \\ &= \frac{c_0}{(z-z_0)^m} + \cdots + \frac{c_{m-1}}{z-z_0} + g(z), \end{aligned}$$

where $g(z)$ is analytic in $|z-z_0| < r$. Hence by the definition of residue and generalized Cauchy's Formula,

$$\begin{aligned} \operatorname{Res}(f; z_0) &= \frac{1}{2\pi i} \int_{|z-z_0|=s} \left(\frac{c_0}{(z-z_0)^m} + \cdots + \frac{c_{m-1}}{z-z_0} \right) dz \\ &= c_{m-1} = \frac{H^{(m-1)}(z_0)}{(m-1)!}. \end{aligned}$$

In particular, when $m = 1$, we have

$$\text{(2.9)} \qquad \operatorname{Res}\left(\frac{H(z)}{z-z_0}; z_0\right) = H(z_0).$$

Laurent Series. *Suppose that $f(z)$ is analytic in an annulus $D = \{z \colon r < |z-z_0| < R\}$, where $0 \le r < R$ are given numbers. Then f can be written as the following* **Laurent series** *at z_0 :*

$$\text{(2.10)} \qquad f(z) = \sum_{k=1}^{\infty} b_k (z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k, \qquad r < |z-z_0| < R,$$

where $b_k = a_{-k}$ for $k = 1, 2, \cdots$ and all a_k's are given by

$$a_k = \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{f(z)}{(z-z_0)^{k+1}}\, dz \quad \forall k = 0, \pm 1, \pm 2, \cdots$$

with $s \in (r, R)$ being any given number.

Proof. Given z with $r < |z - z_0| < R$, select two numbers r_1, R_1 with

$$r < r_1 < |z - z_0| < R_1 < R.$$

Let Γ be the circle $|w - z_0| = R_1$ oriented counterclockwise and let γ be the circle $|w - z_0| = r_1$ oriented clockwise. Choose a radius of the circle $|w - z_0| = R$ that does not contain z and let P, Q be the intersection points of the radius with Γ and γ, respectively. Cut the annulus $\{r_1 < |w - z_0| < R_1\}$ along PQ, and this makes two simple closed curves. From this and using Cauchy's Formula, we have

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\, dw + \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw.$$

Then, on γ and Γ, we express $\frac{1}{w-z}$ as series around z_0. This will eventually prove the theorem. See Text for details. □

Now assume $f(z)$ is analytic in the punctured disc $\{0 < |z - z_0| < R\}$. Then

$$\operatorname{Res}(f; z_0) = b_1 = \text{the coefficient of } (z - z_0)^{-1} \text{ in the Laurent series (2.10).}$$

It can also be shown that

(1) f has a removable singularity at z_0 if and only if the coefficient b_k of $(z - z_0)^{-k}$ in the Laurent series equals zero for all $k \geq 1$.

(2) f has a pole of order m at z_0 if and only if the coefficient b_k of $(z - z_0)^{-k}$ in the Laurent series equals zero for all $k \geq m + 1$ and $b_m \neq 0$. In this case the sum involving all negative powers of $z - z_0$ in the Laurent series is called the **principal part of f at z_0**. This is exactly a polynomial of order m without constant term in $\frac{1}{z-z_0}$:

$$P\left(\frac{1}{z - z_0}\right) = \sum_{k=1}^{m} \frac{b_k}{(z - z_0)^k}.$$

(3) f has an essential singularity at z_0 if and only if there are infinitely many nonzero coefficients b_k of negative power terms $(z - z_0)^{-k}$ in the Laurent series.

Example 11. (1)

$$\operatorname{Res}\left(\frac{z^2 + 3z - 1}{z + 2}; -2\right) = (-2)^2 + 3(-2) - 1 = -3.$$

(2)

$$\operatorname{Res}\left(\frac{e^z}{(z - 1)^3}; 1\right) = \frac{(e^z)''}{2!}\Big|_{z=1} = \frac{e}{2}.$$

(3) Find the residues of

$$R(z) = \frac{z + 1}{(z^2 + 4)(z - 1)^3}$$

at each of its poles.

Solution. The poles are the zeros of the denominator and thus are $-2i$, $2i$ and 1 with the order $1, 1$ and 3, respectively. Note that

$$R(z) = \frac{z + 1}{(z - 2i)(z + 2i)(z - 1)^3} = \frac{f(z)}{z - 2i} = \frac{g(z)}{z + 2i} = \frac{h(z)}{(z - 1)^3},$$

where

$$f(z) = \frac{z+1}{(z+2i)(z-1)^3}, \quad g(z) = \frac{z+1}{(z-2i)(z-1)^3}, \quad h(z) = \frac{z+1}{z^2+4}.$$

Hence

$$\mathrm{Res}(R; 2i) = f(2i) = \frac{2i+1}{(2i+2i)(2i-1)^3} = \frac{24-7i}{500};$$

$$\mathrm{Res}(R; -2i) = g(-2i) = \frac{-2i+1}{(-2i-2i)(-2i-1)^3} = \frac{24+7i}{500};$$

$$\mathrm{Res}(R; 1) = \frac{h''(1)}{2!} = -\frac{12}{125}.$$

□

(4) Suppose that F and G are analytic in the disc $\{|z-z_0| < r\}$ with $G(z_0) = 0$ but $G'(z_0) \neq 0$. Then

$$\mathrm{Res}\left(\frac{F}{G}; z_0\right) = \frac{F(z_0)}{G'(z_0)}.$$

To see this, write $G(z) = (z-z_0)g(z)$, where g is analytic in the disc and $g(z_0) = G'(z_0) \neq 0$. Hence

$$\mathrm{Res}\left(\frac{F}{G}; z_0\right) = \mathrm{Res}\left(\frac{F(z)}{(z-z_0)g(z)}; z_0\right)$$

$$= \frac{F(z_0)}{g(z_0)} = \frac{F(z_0)}{G'(z_0)}.$$

(5) Find the Laurent series at $z_0 = 0$ for $f(z) = (\sin z)/z^3$.

Solution.

$$f(z) = \frac{\sin z}{z^3} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z^3}$$

$$= \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots, \qquad z \neq 0.$$

This is the Laurent series of $f(z)$ and the principal part is $\frac{1}{z^2}$ and coeffient b_1 of term $\frac{1}{z}$ is zero and hence the residue of f at $z_0 = 0$ is zero. □

(6) The Laurent series of $e^{1/z}$ can be easily obtained by the power series expansion of e^z as follows:

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}(1/z)^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots.$$

Therefore

$$\mathrm{Res}(e^{1/z}; 0) = 1.$$

Note that $e^{1/z}$ has an essential singularity at $z_0 = 0$ since the Laurent series has infinitely many negative powers.

(7) (Exercise #12.) Find the Laurent series of $\dfrac{1}{e^z - 1}$ at $z_0 = 0$ with first four terms.

Solution. Since $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$, we have

$$\frac{1}{e^z - 1} = \frac{1}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots}$$

$$= \frac{1}{z}\,\frac{1}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots} = \frac{1}{z}\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right),$$

where

$$\frac{1}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots .$$

We can determine $a_0, a_1, a_2, a_3, \cdots$ by multiplication of power series

$$\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \cdots\right)\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right) = 1.$$

For example, this equality becomes

$$a_0 + \left(a_1 + \frac{a_0}{2}\right) z + \left(a_2 + \frac{a_1}{2} + \frac{a_0}{6}\right) z^2 + \left(a_3 + \frac{a_2}{2} + \frac{a_1}{6} + \frac{a_0}{24}\right) z^3 + \cdots = 1.$$

Hence

$$a_0 = 1, \quad a_1 = -\frac{1}{2}, \quad a_2 = \frac{1}{12}, \quad a_3 = 0.$$

Therefore,

$$\frac{1}{e^z - 1} = \frac{1 - \frac{1}{12}z + \frac{1}{12}z^2 + \cdots}{z} = \frac{1}{z} - \frac{1}{12} + \frac{z}{12} + \cdots .$$

This is the Laurent series for $\dfrac{1}{e^z - 1}$ at $z_0 = 0$. Easily we see the residue of the function at the pole is 1. □

Exercises. Page 150. Problems 1, 3, 5–8, 11, 14, 15, 22(a, b).

2.6. The Residue Theorem and Its Applications

Theorem 2.16 (**The Residue Theorem**). *Suppose f is analytic in a simply-connected domain D except for a finite number of isolated singularities at points $z_1, z_2, \cdots, z_N$ of D. Let γ be a piecewise smooth positively oriented simple closed curve in D that does not pass through any of the points $z_1, z_2, \cdots, z_N$. Then*

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{\text{all } z_k \text{ inside } \gamma} \operatorname{Res}(f; z_k), \tag{2.11}$$

where the sum is taken over all those singularities z_k lying inside γ.

Proof. Let Ω be the inside of γ and $z_1', z_2', \cdots, z_n'$ are the isolated singularities that lie in Ω. Let Δ_j be the closed disc with center z_j' that is contained in Ω and be disjoint for $j = 1, \cdots, n$. Let U be the domain $\Omega \setminus \cup_{j=1}^n \Delta_j$. Then the positively oriented boundary of U is

$$\partial U = \gamma \cup (-\partial\Delta_1) \cup \cdots \cup (-\partial\Delta_n).$$

By Green's Theorem

$$\int_{\partial U} f(z)\, dz = 0.$$

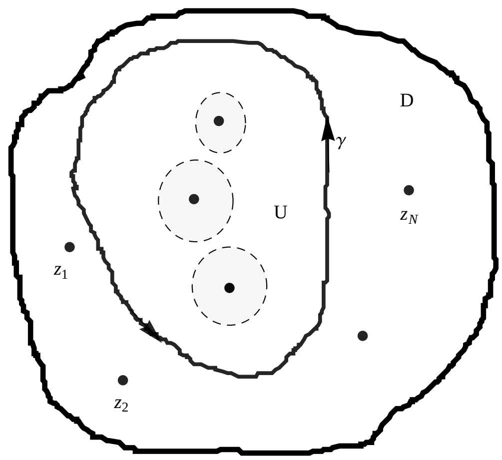

Figure 2.2. The proof of Residue Theorem

Hence

$$\int_\gamma f(z)\,dz = \sum_{j=1}^{n} \int_{\partial\Delta_j} f(z)\,dz = \sum_{j=1}^{n} (2\pi i)\,\mathrm{Res}(f; z_j').$$

□

Applications of The Residue Theorem for Evaluating Integrals. We will apply the Residue Theorem to compute real variable integrals of the following type.

Application 1: *Suppose that P, Q are polynomials of real coefficients and deg$Q \geq$ deg$P$+2. Assume $Q(x) \neq 0$ for all real numbers x. Then*

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{Q(z_j)=0;\mathrm{Im} z_j>0} \mathrm{Res}\left(\frac{P(z)}{Q(z)}; z_j\right). \tag{2.12}$$

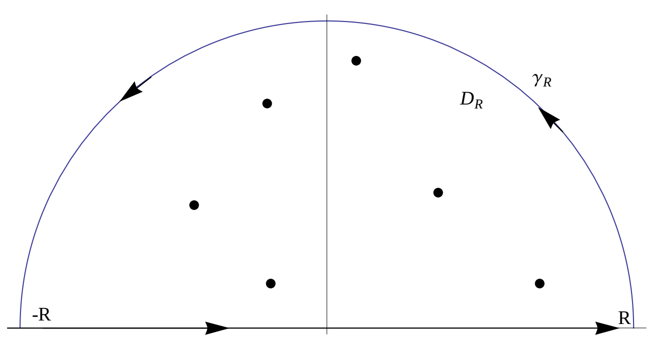

Figure 2.3. The integral path in Application 1

Proof. Since $Q(z)$ has finitely many zeros (a property of polynomials) and has no zeros on the real axis, we assume that all zeros of Q in the upper half plane are inside the semi-disc $D_R = \{z : \mathrm{Im} z > 0,\ |z| < R\}$, for all sufficiently large $R > 0$. (See Figure 2.3 given.) Let γ_R be the positively oriented boundary of D_R. Then by the Residue Theorem

$$\int_{\gamma_R} \frac{P(z)}{Q(z)}\, dz = 2\pi i \sum_{Q(z_j)=0; \mathrm{Im} z_j>0} \mathrm{Res}\left(\frac{P(z)}{Q(z)}; z_j\right). \tag{2.13}$$

However,

$$\int_{\gamma_R} \frac{P(z)}{Q(z)}\, dz = \int_{-R}^{R} \frac{P(x)}{Q(x)}\, dx + \int_{|z|=R, \mathrm{Im} z>0} \frac{P(z)}{Q(z)}\, dz. \tag{2.14}$$

Let $P(z) = a_n z^n + \cdots + a_0$ $(a_n \neq 0)$ and $Q(z) = b_m z^m + \cdots + b_0$ $(b_m \neq 0)$. Then, by the assumption on degrees, $m - n \geq 2$. Therefore

$$\frac{P(z)}{Q(z)} = \frac{a_n z^n + \cdots + a_0}{b_m z^m + \cdots + b_0} = \frac{z^n(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n})}{z^m(b_m + \frac{b_{m-1}}{z} + \cdots + \frac{b_0}{z^m})} = z^{n-m} f(z),$$

where

$$f(z) = \frac{a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}}{b_m + \frac{b_{m-1}}{z} + \cdots + \frac{b_0}{z^m}} \to \frac{a_n}{b_m} \qquad \text{as } |z| \to \infty.$$

Hence there is a constant $M > 0$ such that, for R sufficiently large,

$$\left|\frac{P(z)}{Q(z)}\right| = |z|^{n-m} |f(z)| \leq M\, R^{n-m} \qquad \forall\ |z| = R.$$

Therefore, for such R's,

$$\left| \int_{|z|=R, \mathrm{Im} z>0} \frac{P(z)}{Q(z)}\, dz \right| \leq M R^{n-m}(\pi R) = \pi M R^{n-m+1}.$$

From this, since $n - m + 1 \leq -1$, we have

$$\lim_{R\to\infty} \int_{|z|=R, \mathrm{Im} z>0} \frac{P(z)}{Q(z)}\, dz = 0.$$

In (2.14), letting $R \to \infty$, we have

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{P(x)}{Q(x)}\, dx = 2\pi i \sum_{Q(z_j)=0; \mathrm{Im} z_j>0} \mathrm{Res}\left(\frac{P(z)}{Q(z)}; z_j\right).$$

However, we also know that $\frac{P(x)}{Q(x)} \approx |x|^{n-m}$ and thus is bounded by x^{-2}, which has finite improper integral at $\pm\infty$, for all large $|x|$, so the imporper integral $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx$ converges and

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{P(x)}{Q(x)}\, dx = 2\pi i \sum_{Q(z_j)=0; \mathrm{Im} z_j>0} \mathrm{Res}\left(\frac{P(z)}{Q(z)}; z_j\right).$$

□

Example 12. (1) Compute

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}\, dx = \frac{\pi}{3}.$$

Solution. Here $P(z) = z^2$ and $Q(z) = (1+z^2)(4+z^2)$, and $Q(z) = 0$ has two solutions $z_1 = i$ and $z_2 = 2i$ in the upper half-plane $\text{Im } z > 0$. Note that

$$\frac{P(z)}{Q(z)} = \frac{z^2}{(z-i)(z+i)(4+z^2)} = \frac{z^2}{(1+z^2)(z-2i)(z+2i)}.$$

Hence

$$\text{Res } (\frac{P}{Q}; i) = \frac{i^2}{(i+i)(4+i^2)} = -\frac{1}{6i};$$

$$\text{Res } (\frac{P}{Q}; 2i) = \frac{(2i)^2}{(1+(2i)^2)(2i+2i)} = \frac{1}{3i}.$$

Therefore, by (2.12),

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx = 2\pi i(-\frac{1}{6i} + \frac{1}{3i}) = \frac{\pi}{3}.$$

□

(2)

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}.$$

Proof. Note that $1 + z^2 = 0$ has one solution $z_0 = i$ in the upper half-plane and that

$$\frac{1}{(1+z^2)^2} = \frac{1}{(z-i)^2(z+i)^2} = \frac{H(z)}{(z-i)^2},$$

where

$$H(z) = \frac{1}{(z+i)^2}; \quad H'(z) = -\frac{2}{(z+i)^3}.$$

Hence

$$\text{Res } (\frac{1}{(1+z^2)^2}; i) = H'(i) = -\frac{2}{(i+i)^3} = \frac{1}{4i}.$$

Therefore, by (2.12),

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i \,\text{Res}\, \left(\frac{1}{(1+z^2)^2}; i\right) = 2\pi i\, \frac{1}{4i} = \frac{\pi}{2}.$$

□

Application 2: *Compute the integrals of the type*

(2.15) $$\int_{-\infty}^{\infty} \frac{P(x)\cos x}{Q(x)}\, dx; \quad \int_{-\infty}^{\infty} \frac{P(x)\sin x}{Q(x)}\, dx,$$

where P, Q are polynomials of real coefficients and $Q(x) \neq 0$ for all real x.

Note that if x is real, then

$$\cos x = \text{Re}(e^{ix}); \quad \sin x = \text{Im}(e^{ix}).$$

Therefore to compute (2.15) we can compute the value of the integral

(2.16) $$\int_{-\infty}^{\infty} \frac{P(x)e^{ix}}{Q(x)}\, dx$$

and then take the real and imaginary parts of this value to get the integrals in (2.15). Note that the improper integral (2.16) converges if $\deg Q \geq \deg P + 2$ and

$$\int_{-\infty}^{\infty} \frac{P(x)e^{ix}}{Q(x)}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{P(x)e^{ix}}{Q(x)}\, dx.$$

Theorem. *If* $degQ \geq degP + 1$, *then*

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{P(x)e^{ix}}{Q(x)}\, dx = 2\pi i \sum_{Q(z_j)=0;\mathrm{Im} z_j>0} \mathrm{Res}\left(\frac{P(z)e^{iz}}{Q(z)}; z_j\right). \tag{2.17}$$

We define this value to be the **Cauchy's principal value** *of the improper integral (2.16) and denote it by*

$$(P.V.)\int_{-\infty}^{\infty} \frac{P(x)e^{ix}}{Q(x)}\, dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{P(x)e^{ix}}{Q(x)}\, dx.$$

Note that if an improper integral $\int_{-\infty}^{\infty} f(x)\, dx$ converges then

$$\int_{-\infty}^{\infty} f(x)\, dx = (P.V.)\int_{-\infty}^{\infty} f(x)\, dx.$$

Proof. Again, we use the integral path γ_R in Figure 2.3 inside which all zeros of $Q(z)$ in the upper half plane lie. Let $\delta_R = \{z : \mathrm{Im} z > 0,\ |z| = R\}$. As above, we only need to show

$$\lim_{R\to\infty}\int_{\delta_R} \frac{P(z)e^{iz}}{Q(z)}\, dz = 0. \tag{2.18}$$

We parameterize δ_R using $z(\theta) = Re^{i\theta}$ with $0 \leq \theta \leq \pi$. Then $dz = iRe^{i\theta}\, d\theta$ and

$$|e^{iz(\theta)}| = e^{\mathrm{Re}(iz(\theta))} = e^{-R\sin\theta}.$$

Therefore

$$\begin{aligned}\left|\int_{\delta_R} \frac{P(z)e^{iz}}{Q(z)}\, dz\right| &= \left|\int_0^{\pi} \frac{P(z(\theta))}{Q(z(\theta))} e^{iz(\theta)} iRe^{i\theta}\, d\theta\right| \\ &\leq \int_0^{\pi} \left|\frac{P(Re^{i\theta})}{Q(Re^{i\theta})}\right| Re^{-R\sin\theta}\, d\theta.\end{aligned}$$

As above, there is a constant M such that, for all sufficiently large R,

$$\left|\frac{P(Re^{i\theta})}{Q(Re^{i\theta})}\right| \leq MR^{n-m}.$$

Also,

$$\int_0^{\pi} e^{-R\sin\theta}\, d\theta = \int_0^{\pi/2} e^{-R\sin\theta}\, d\theta + \int_{\pi/2}^{\pi} e^{-R\sin\theta}\, d\theta$$

$$= 2\int_0^{\pi/2} e^{-R\sin\theta}\, d\theta \qquad \text{(using substitution } \theta = \pi - \tau \text{ in the second integral).}$$

Note that the function $h(\theta) = \frac{\sin\theta}{\theta}$ is decreasing on $(0, \pi/2)$ (checking $h'(\theta) \leq 0$) and hence

$$\sin\theta \geq \frac{2}{\pi}\theta \qquad \forall\ 0 \leq \theta \leq \pi/2.$$

Hence

$$e^{-R\sin\theta} \leq e^{-(2R/\pi)\theta} \qquad \forall\ 0 \leq \theta \leq \pi/2$$

and thus

$$\int_0^{\pi/2} e^{-R\sin\theta}\, d\theta \leq \int_0^{\pi/2} e^{-(2R/\pi)\theta}\, d\theta = \frac{e^{-(2R/\pi)\theta}}{-2R/\pi}\Big|_0^{\pi/2} = \frac{\pi}{2R}\left(1 - e^{-R}\right).$$

So

$$\int_0^{\pi} e^{-R\sin\theta}\, d\theta < \frac{\pi}{R}. \tag{2.19}$$

Combining these estimates, we have

$$\int_0^\pi \left|\frac{P(Re^{i\theta})}{Q(Re^{i\theta})}\right| Re^{-R\sin\theta}\, d\theta < \pi M R^{n-m} \to 0 \quad \text{as } R \to \infty$$

since $n-m \leq -1$. This completes the proof. □

EXAMPLE 13. (1) Compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+\alpha^2}\, dx = \pi \frac{e^{-\alpha}}{\alpha}, \quad \alpha > 0.$$

Solution. First of all,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+\alpha^2}\, dx = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+\alpha^2}\, dx$$

and the integral is convergent; second, $z^2+\alpha^2=0$ has only one solution $z_0=\alpha i$ in the upper half-plane and

$$\mathrm{Res}\left(\frac{e^{iz}}{z^2+\alpha^2}; \alpha i\right) = \frac{e^{iz_0}}{2z_0} = \frac{e^{-\alpha}}{2\alpha i}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+\alpha^2}\, dx = \mathrm{Re}\left[2\pi i\, \frac{e^{-\alpha}}{2\alpha i}\right] = \pi \frac{e^{-\alpha}}{\alpha}.$$

□

(2) Compute

$$(P.V.) \int_{-\infty}^{\infty} \frac{x^3 \sin x}{x^4+16}\, dx = \pi e^{-\sqrt{2}} \cos(\sqrt{2}).$$

Solution. We set

$$f(z) = e^{iz} \frac{z^3}{z^4+16} = \frac{e^{iz} z^3}{z^4+16}.$$

The function f has two poles in the upper half plane U at $z_1 = \sqrt{2}(1+i) = 2e^{i\pi/4}$ and $z_2 = \sqrt{2}(-1+i) = 2e^{3i\pi/4}$ (by finding the 4-th roots of $-16 = 2^4 e^{i\pi}$). Therefore, the residues are

$$\mathrm{Res}(f; z_1) = \left.\frac{e^{iz} z^3}{(z^4+16)'}\right|_{z=z_1} = \left.\frac{e^{iz} z^3}{4z^3}\right|_{z=z_1} = \frac{1}{4} e^{iz_1} = \frac{1}{4} e^{\sqrt{2}(-1+i)}$$

and

$$\mathrm{Res}(f; z_2) = \left.\frac{e^{iz} z^3}{(z^4+16)'}\right|_{z=z_2} = \left.\frac{e^{iz} z^3}{4z^3}\right|_{z=z_2} = \frac{1}{4} e^{iz_2} = \frac{1}{4} e^{\sqrt{2}(-1-i)}$$

By the Residue Theorem and the result above,

$$\begin{aligned}(P.V.) \int_{-\infty}^{\infty} \frac{x^3 \sin x}{x^4+16}\, dx &= \mathrm{Im}\left[(P.V.) \int_{-\infty}^{\infty} \frac{e^{ix} x^3}{x^4+16}\, dx\right] \\ &= \mathrm{Im}[2\pi i(\mathrm{Res}(f; z_1) + \mathrm{Res}(f; z_2)] \\ &= \pi e^{-\sqrt{2}} \cos(\sqrt{2}).\end{aligned}$$

□

Application 3: *Compute the integrals of type*

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta,$$

where $R(z) = \frac{P(z)}{Q(z)}$ *are certain rational functions.*

We have done such problems directly using Cauchy's Formula in some special cases before. As before, we use the substitution: $z = e^{i\theta}$ so that

$$d\theta = \frac{1}{iz}\, dz, \quad \cos\theta = \frac{1}{2}(z + \frac{1}{z}), \quad \sin\theta = \frac{1}{2i}(z - \frac{1}{z}).$$

Therefore

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \int_{|z|=1} R\left(\frac{1}{2}(z+\frac{1}{z}), \frac{1}{2i}(z-\frac{1}{z})\right) \frac{1}{iz}\, dz. \tag{2.20}$$

Then one can use the Residue Theorem to compute this line integral by studying the poles of function

$$f(z) = R\left(\frac{1}{2}(z+\frac{1}{z}), \frac{1}{2i}(z-\frac{1}{z})\right) \frac{1}{iz}$$

inside the unit disc $|z| = 1$.

EXAMPLE 14. Compute

$$\int_0^{2\pi} \frac{1}{2+\cos^2\theta}\, d\theta = \frac{2\pi}{\sqrt{6}}.$$

Solution. With $z = e^{i\theta}$, it follows that $d\theta = dz/iz$ and $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$. Therefore

$$\begin{aligned}\int_0^{2\pi} \frac{1}{2+\cos^2\theta}\, d\theta &= \int_{|z|=1} \frac{dz}{iz\left(2 + \frac{1}{4}(z^2 + 2 + \frac{1}{z^2})\right)} \\ &= \int_{|z|=1} \frac{4z}{i(z^4+10z^2+1)}\, dz \\ &= \int_{|z|=1} f(z)\, dz,\end{aligned}$$

where

$$f(z) = \frac{4z}{i(z^4+10z^2+1)} = \frac{F(z)}{iG(z)}$$

has two poles inside the circle $\{|z| = 1\}$ at

$$z_1 = i\sqrt{5-2\sqrt{6}}; \quad z_2 = -i\sqrt{5-2\sqrt{6}}$$

by solving $G(z) = z^4 + 10z^2 + 1 = (z^2+5)^2 - 24 = 0$. (Note that $G(z) = 0$ has four solutions $\pm i\sqrt{5 \pm 2\sqrt{6}}$, but only z_1, z_2 defined above are inside the unit circle; the other two are outside the circle.) The residues are

$$\text{Res}(f; z_1) = \frac{F(z_1)}{iG'(z_1)} = \frac{4z_1}{i(4z_1^3 + 20z_1)} = \frac{1}{i(z_1^2+5)} = \frac{1}{i2\sqrt{6}}$$

and

$$\text{Res}(f; z_2) = \frac{F(z_2)}{iG'(z_2)} = \frac{4z_2}{i(4z_2^3 + 20z_2)} = \frac{1}{i(z_2^2+5)} = \frac{1}{i2\sqrt{6}}.$$

Hence

$$\int_0^{2\pi} \frac{1}{2+\cos^2\theta}\, d\theta = \int_{|z|=1} f(z)\, dz$$

$$= 2\pi i[\mathrm{Res}(f;z_1) + \mathrm{Res}(f;z_2)] = \frac{2\pi}{\sqrt{6}}.$$

□

Theorem 2.17.

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

Proof. The proof is a little tricky. We use the identity $2\sin^2 x = 1 - \cos 2x = \mathrm{Re}(1 - e^{2ix})$ to write

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{1}{2}\mathrm{Re}\left(\int_0^\infty \frac{1-e^{2ix}}{x^2}\,dx\right)$$

and compute the following improper integral using the complex variable method discussed above

$$\int_0^\infty \frac{1-e^{2ix}}{x^2}\,dx.$$

Let

$$f(z) = \frac{1-e^{2iz}}{z^2}.$$

This function has only the pole at $z = 0$. We integrate $f(z)$ along the path $\gamma_{\epsilon,R}$ given below: Since f is analytic inside $\gamma_{\epsilon,R}$, we have

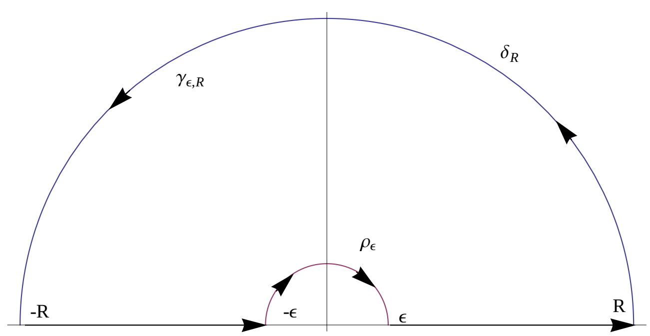

Figure 2.4. The path $\gamma_{\epsilon,R}$

$$(2.21) \qquad 0 = \int_{\gamma_{\epsilon,R}} f(z)\,dz = \int_{-R}^{-\epsilon} f(x)\,dx + \int_\epsilon^R f(x)\,dx + \int_{\delta_R} f(z)\,dz + \int_{\rho_\epsilon} f(z)\,dz.$$

Note that on the semi-circle $\delta_R = \{|z| = R, \mathrm{Im} z > 0\}$, $z = Re^{i\theta}$ with $0 \le \theta \le \pi$, and hence

$$|f(Re^{i\theta})| \le \frac{1+e^{-2R\sin\theta}}{R^2} \le \frac{2}{R^2}$$

and so

$$\left|\int_{\delta_R} f(z)dz\right| \le \frac{2}{R^2}\pi R \to 0 \quad \text{as } R \to \infty.$$

Also note that

$$f(z) = \frac{1 - \left[1 + 2iz + \frac{(2iz)^2}{2!} + \cdots\right]}{z^2}$$
$$= \frac{-2i}{z} + g(z),$$

where $g(z) = 2 - \frac{4i}{3}z + \cdots$ is an entire function. Therefore

$$\begin{aligned}\int_{\rho_\epsilon} f(z)dz &= -\int_0^\pi f(\epsilon e^{it}) i\epsilon e^{it}\,dt \\ &= -\int_0^\pi \frac{-2i}{\epsilon e^{it}} i\epsilon e^{it}\,dt - \int_0^\pi g(\epsilon e^{it}) i\epsilon e^{it}\,dt \\ &= -2\pi + \text{a term that goes to zero as } \epsilon \to 0.\end{aligned}$$

In (2.21), taking the real part first and then letting $R \to \infty$ and $\epsilon \to 0$ we have

$$0 = 2\mathrm{Re}\left(\int_0^\infty f(x)dx\right) - 2\pi$$

and hence

$$\mathrm{Re}\left(\int_0^\infty f(x)dx\right) = \pi$$

and so

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{1}{2}\mathrm{Re}\left(\int_0^\infty \frac{1-e^{2ix}}{x^2}\,dx\right) = \frac{\pi}{2}.$$

□

Exercises. Page 167. Problems 1–11.

Homework Problems for Chapter 2. (From Textbook)

2.1	1(a, b), 3, 4, 5, 8, 9, 10, 18, 19, 20(a, b, c), 23
2.2	1, 2, 3, 7, 8, 9, 14, 15, 18
2.3	1–7, 9, 11, 12
2.4	1, 2, 3, 9, 12, 17, 18, 21
2.5	1, 3, 5–8, 11, 14, 15, 22(a, b)
2.6	1–11