The Complex Plane

1.1. The Complex Numbers

A **complex number** is an expression of the form

$$z = x + iy = x + yi,$$

where x, y are real numbers and i is a symbol satisfying

$$i^2 = ii = i \cdot i = -1.$$

Here, x is called the **real part** of z and y the **imaginary part** of z and we denote

$$x = \text{Re}z, \quad y = \text{Im}z.$$

We identify two complex numbers z and w if and only if $\text{Re}z = \text{Re}w$ and $\text{Im}z = \text{Im}w$. We also write

$$x + 0i = x, \quad 0 + yi = yi.$$

In this way, real numbers are exactly those complex numbers whose imaginary part is zero.

The **modulus**, or **absolute value**, of z is defined by

$$|z| = \sqrt{x^2 + y^2} \quad \text{if} \quad z = x + iy.$$

The **complex conjugate** of $z = x + iy$ is defined by

$$\bar{z} = x - iy.$$

Obviously,

$$\text{Re}z = \frac{1}{2}(z + \bar{z}) = \text{Re}\bar{z}, \quad \text{Im}z = \frac{1}{2i}(z - \bar{z}) = -\text{Im}\bar{z}$$

and, for $z = x + iy$,

$$|x| \le |z|, \quad |y| \le |z|; \quad |\bar{z}| = |z|.$$

The **addition**, **subtraction**, **multiplication** and **division** of complex numbers are defined as follows: for $z = x + iy$ and $w = s + it$,

(a) $z + w = (x + s) + i(y + t); \quad z - w = (x - s) + i(y - t);$

(b) $zw = (xs - yt) + i(xt + ys);$

(c) if $s^2 + t^2 \neq 0$ then

$$\frac{z}{w} = \frac{\bar{w}z}{\bar{w}w} = \frac{(xs + yt) + i(ys - xt)}{s^2 + t^2}.$$

These operations will follow the same ordinary rules of arithmetic of real numbers. Under these operations, the set of all complex numbers becomes a **field**, with $0 = 0 + 0i$ and $1 = 1 + 0i$.

Note that for all complex numbers z, w

$$z\bar{z} = |z|^2; \quad \overline{z + w} = \bar{z} + \bar{w}; \quad \overline{zw} = \bar{z}\bar{w}; \quad |zw| = |z||w|.$$

Complex Numbers as Vectors in the Complex Plane. A complex number $z = x + iy$ can be identified as a point $P(x, y)$ in the xy-plane, and thus can be viewed as a vector OP in the plane. All the rules for the geometry of the vectors can be recast in terms of complex numbers. For example, let $w = s + it$ be another complex number. Then the point for $z + w$ becomes the **vector sum** of $P(x, y)$ and $Q(s, t)$, and $|z - w|$ is exactly the **distance** between $P(x, y)$ and $Q(s, t)$. Henceforth, then, we refer to the xy-plane as the **complex plane**, and the x-axis as the **real axis**, y-axis the **imaginary axis**. (See the figure below.)

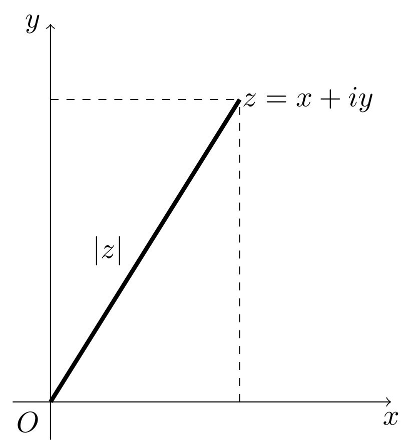

Figure 1.1. Complex numbers as vectors

Polar Representation. The identification of $z = x + iy$ with the point $P(x, y)$ also gives the **polar representation** of z: for $z \neq 0$,

$$z = r \cos\theta + ir \sin\theta = |z|(\cos\theta + i \sin\theta),$$

where θ is any angle verifying this equality. (See the figure below.)

Given a $z \neq 0$, if θ is such a angle then $\theta + 2\pi k$ also verifies this representation for all integers k. The set of all such θ's is called the **argument** of z and is denoted by $\arg z$.

A concrete choice of $\arg z$ can be defined by requiring the angle θ to be in the interval $[-\pi, \pi)$; we define this value to the **principal argument** of z and denote it as

$$\theta = \operatorname{Arg} z \in [-\pi, \pi).$$

This is well-defined for all $z \neq 0$ as the unique angle $\theta \in [-\pi, \pi)$ such that

$$z = |z|(\cos\theta + i \sin\theta).$$

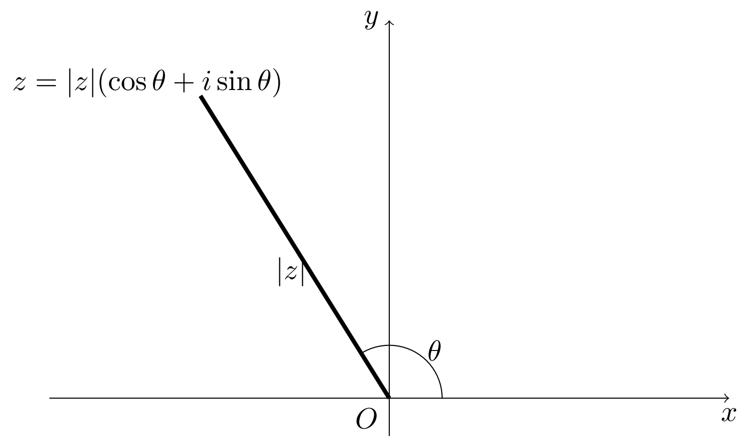

Figure 1.2. Polar representation

The use of polar representation of complex numbers gives a simple and easy way for multiplication and division and powers.

Example 1.

$$-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \quad \text{so } \operatorname{Arg}(-1+i) = \frac{3\pi}{4}.$$

Multiplication and division through polar representation. Let

$$z = |z|(\cos\theta + i\sin\theta), \quad w = |w|(\cos\psi + i\sin\psi).$$

Then, by the definition of multiplication and conjugation, using the trigonometric identities for the sine and cosine of the sum and difference of two angles, we have

$$zw = |z||w|(\cos(\theta+\psi) + i\sin(\theta+\psi)), \quad \bar{z} = |z|(\cos\theta - i\sin\theta),$$

and, if $w \neq 0$,

$$\frac{z}{w} = \frac{|z|}{|w|}\left(\cos(\theta-\psi) + i\sin(\theta-\psi)\right).$$

Theorem 1.1 (De Moivre's Theorem). *Let* $z = |z|(\cos\theta + i\sin\theta) \neq 0$. *Then, for* $n = 0, \pm1, \pm2, \cdots$,

$$z^n = |z|^n(\cos n\theta + i\sin n\theta).$$

Proof. Use induction. □

Example 2. (1) Find the polar representation of $z = -1 + i$ and $w = \sqrt{3} + i$.

(2) Find in polar representation $(-1+i)(\sqrt{3}+i)$ and $\dfrac{-1+i}{\sqrt{3}+i}$.

(3) Find in polar representation $(\sqrt{3}+i)^n$ for all integers n. In particular, find $(\sqrt{3}+i)^6$.

Solution. (1) As above, $-1+i = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$. The polar representation of w is

$$w = \sqrt{3} + i = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}).$$

(2) In polar representation,

$$(-1+i)(\sqrt{3}+i) = 2\sqrt{2}\left[\cos(\frac{3\pi}{4} + \frac{\pi}{6}) + i\sin(\frac{3\pi}{4} + \frac{\pi}{6})\right]$$

$$= 2\sqrt{2}\left[\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right]$$

and

$$\frac{-1+i}{\sqrt{3}+i} = \frac{\sqrt{2}}{2}\left[\cos(\frac{3\pi}{4} - \frac{\pi}{6}) + i\sin(\frac{3\pi}{4} - \frac{\pi}{6})\right] = \frac{\sqrt{2}}{2}\left[\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right].$$

(3) In polar representation,

$$(\sqrt{3}+i)^n = 2^n\left[\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right].$$

In particular, find $(\sqrt{3}+i)^6 = 2^6(\cos\pi + i\sin\pi) = -2^6$. □

Exercises. Problems 1, 2, 3, 5, 6, 11.

1.2. Some Geometry

The Triangle Inequality. From the vector representation of complex numbers, we can easily have the following **triangle inequality**:

$$|z+w| \le |z| + |w|.$$

This inequality can be proved directly as follows. Given z and w,

$$\begin{aligned} |z+w|^2 &= (z+w)(\bar{z}+\bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\ &= |z|^2 + |w|^2 + 2\mathrm{Re}(z\bar{w}) \le |z|^2 + |w|^2 + 2|z\bar{w}| \\ &= |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2. \end{aligned}$$

Hence

$$|z+w| \le |z| + |w|.$$

From this inequality, one also obtains

$$||\zeta| - |\xi|| \le |\zeta - \xi| \quad \forall\ \zeta,\ \xi,$$

which is also called a **triangle inequality**.

Locus of Points. The **locus of points** is the set of points that satisfy a general equation $F(z) = 0$. It is sometime easier to use the xy-coordinates by setting $z = x+iy$ and to study the equations defined by $F(x+iy) = 0$.

Straight Lines and Circles. The equation of a straight line can be written as

$$|z-p| = |z-q|,$$

where p and q are two distinct complex numbers. This line is the bisecting line of the line segment joining p and q. This is the geometric way for the line equation. Also, the algebraic equation for a straight line is

$$\mathrm{Re}(az+b) = 0,$$

where a and b are two complex numbers and $a \ne 0$. Note that a, b are not unique and we can take b to be real.

In the xy-coordinates, the line has an equation of the form

$$Ax + By = C,$$

where A, B, C are real constants and $A^2 + B^2 \ne 0$.

A **circle** is the set (locus) of points equidistant from a given point (center); the distance is called the **radius** of the circle. The equation for a circle of radius r and center z_0 is

$$|z - z_0| = r.$$

A useful characterization of circles and lines. A circle is also a locus of points satisfying the equation

$$|z - p| = \rho |z - q|, \tag{1.1}$$

where p, q are distinct complex numbers and $\rho \neq 1$ is a positive real number. To see this, suppose $0 < \rho < 1$. Let $z = w + q$ and $c = p - q$; then the equation (1.1) becomes $|w - c| = \rho |w|$. Upon squaring and transposing terms, this can be written as

$$|w|^2(1 - \rho^2) - 2\mathrm{Re}(w\bar{c}) + |c|^2 = 0.$$

Dividing by $1 - \rho^2$, completing the square of the left side, and taking the square root will yield that

$$\left| w - \frac{c}{1 - \rho^2} \right| = |c| \frac{\rho}{1 - \rho^2}.$$

Therefore (1.1) is equivalent to

$$\left| z - q - \frac{p - q}{1 - \rho^2} \right| = |p - q| \, \frac{\rho}{1 - \rho^2}.$$

This is the equation of the circle centered at the point $z_0 = \frac{p - \rho^2 q}{1 - \rho^2}$ of radius $R = \frac{\rho |p - q|}{1 - \rho^2}$.

Note that if we allow $\rho = 1$ in (1.1) we have the equation of lines as well. Therefore (1.1) with $\rho > 0$ represents the equation for circles or lines.

Example 3. The locus of points z with $|z - i| = \frac{1}{2}|z - 1|$.

Solution. Here $p = i$, $q = 1$ and $\rho = \frac{1}{2}$; so we know the locus is a circle of center $z_0 = \frac{p - \rho^2 q}{1 - \rho^2} = -\frac{1}{3} + \frac{4}{3}i$ and radius $R = \frac{\rho |p - q|}{1 - \rho^2} = \frac{2\sqrt{2}}{3}$. To confirm this, we multiply the equation by 2 and then square both sides to obtain

$$4[|z|^2 - 2\mathrm{Re}(z\bar{i}) + |i|^2] = |z|^2 - 2\mathrm{Re} z + 1,$$

which simplifies to

$$3|z|^2 - 8y + 2x = -3$$

and to that

$$3x^2 + 2x + 3y^2 - 8y = -3.$$

This is the equation of the circle

$$(x + \frac{1}{3})^2 + (y - \frac{4}{3})^2 = \frac{8}{9}$$

of center $z_0 = -\frac{1}{3} + \frac{4}{3}i$ and radius $\frac{2\sqrt{2}}{3}$. $\square$

Roots of Complex Numbers. Let $n = 1, 2, \cdots$ and $z \neq 0$. Any number w satisfying $w^n = z$ is called an n**th root** of z. We shall see there exist exactly n distinct nth roots of z if $n \geq 2$; the notation $z^{1/n}$ is usually to denote the set of all nth roots of z, not a particular one.

To find all nth roots of $z \neq 0$, write z in polar representation: $z = |z|(\cos\theta + i\sin\theta)$. Let $w = |w|(\cos\psi + i\sin\psi)$ be an nth root of z; that is, $w^n = z$. By De Moivre's Theorem,

$$|w|^n(\cos n\psi + i\sin n\psi) = |z|(\cos\theta + i\sin\theta).$$

Hence $|w|^n = |z|$ and $n\psi = \theta + 2\pi k$ for some integer $k = 0, \pm1, \pm2, \cdots$. Therefore

$$|w| = \sqrt[n]{|z|}; \quad \psi = \psi_k = \frac{\theta}{n} + \frac{2\pi k}{n}.$$

Here, for a positive number $t > 0$ we use $s = \sqrt[n]{t}$ to denote the unique positive number satisfying $s^n = t$. Note that for any integer k the value of ψ_k differs from one of the n values $\{\psi_0, \psi_1, \cdots, \psi_{n-1}\}$ by an integer multiple of 2π. Therefore, there exist exactly n distinct values of w for the nth roots of z given by

$$w_k = \sqrt[n]{|z|}(\cos\psi_k + i\sin\psi_k), \quad k = 0, 1, \cdots, n-1,$$

where ψ_k is defined above.

Example 4. (1) Find all 12th roots of 1.

(2) Find all 5th roots of $1 + i$.

(3) Solve the equation

$$z^4 - 4z^2 + 4 - 2i = 0.$$

Solution. (1) Since $1 = 1(\cos 0 + i\sin 0)$, it follows that all 12th roots of 1 are given by

$$\cos(\frac{2\pi k}{12} + i\sin\frac{2\pi k}{12}) = \cos(\frac{\pi k}{6} + i\sin\frac{\pi k}{6}), \quad k = 0, 1, \cdots, 11,$$

which are 12 different numbers.

(2) Since $1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$, it follows that all 5th roots of $1 + i$ are given by

$$2^{1/10}\left[\cos(\frac{\pi}{20} + \frac{2\pi k}{5}) + i\sin(\frac{\pi}{20} + \frac{2\pi k}{5})\right], \quad k = 0, 1, 2, 3, 4,$$

which are five points on the circle of radius $2^{1/10}$ that start at the angle $\frac{\pi}{20}$ and have equal distances on the circle.

(3) By completing the square, $z^4 - 4z^2 + 4 - 2i = (z^2 - 2)^2 - 2i$ and so the equation is equivalent to $(z^2 - 2)^2 = 2i = (1 + i)^2$, which is equivalent to

$$z^2 - 2 = 1 + i \quad \text{or} \quad z^2 - 2 = -1 - i,$$

that is,

$$z^2 = 3 + i \quad \text{or} \quad z^2 = 1 - i.$$

Hence the solutions are the square roots of $3 + i$ and $1 - i$. They are in the form of

$$z = \pm z_1, \quad z = \pm z_2,$$

where $z_1^2 = 3 + i$ and $z_2^2 = 1 - i$ can be found in polar representation.

□

Exercises. Page 20. Problems 1, 2, 3, 7, 11, 12, 15, 21, 23, 24, 25.

1.3. Subsets of the Plane

The definitions are the same as for the subsets of the Euclidean space $\mathbf{R}^2$, such as **interior points**, **boundary points**, **open sets**, **closed sets**, **closure**, **connected sets**, etc.

An **open disc** in the complex plane is the set of complex numbers defined by

$$\{z \,:\, |z - z_0| < r\}.$$

A **half open plane** is the set defined by

$$\{z \,:\, \mathrm{Re}(az + b) > 0\}.$$

Given two distinct points p and q, the **directed line segment** pq with starting point p and ending point q is the set defined by

$$pq = \{(1 - t)p + tq \colon 0 \le t \le 1\}.$$

Essentially, we need to know the following definitions and facts:

(1) We say that z_0 is an **interior point** of a set D if there is a number $r > 0$ such that the open disc $z : |z - z_0| < r$ is contained in D. We say set D is **open** if every point of D is an interior point of D.

(2) We say that z_0 is a **boundary point** of a set D if every open disc centered at z_0 contains both points in D and points not in D. The set of all boundary points of D is called the **boundary of** D, usually denoted by ∂D.

(3) For a set D in the complex plain, the union $D \cup \partial D$ is called the **closure** of D, usually denoted by $\bar{D}$. Set D is called **closed** if $D = \bar{D}$; that is, if $\partial D \subseteq D$.

(4) **Theorem:** *A set D is open if and only if it contains no boundary points. A set C is closed if and only if its complement $D = \{z : z \notin C\}$ is open.*

(5) A **polygonal curve** is the union of a finite number of directed line segments $P_1P_2, P_2P_3, \cdots, P_{n-1}P_n$, where P_1 is called the starting point and P_n is called the ending point.

(6) An open set D is called **connected** if for each pair of points p, q in D, there exists a polygonal curve lying entirely in D with starting point p and ending point q. (This definition works only for open sets; for general sets, the connectedness is defined differently!)

(7) A **domain** in the complex plane is an open and connected set.

(8) A set S is called **convex** if for each pair of points p, q in S the line segment pq lies also in S.

(9) a set D is said to contain **the point at infinity** in its interior if there exists a number $M > 0$ such that $\{|z| > M\} \subseteq D$.

EXAMPLE 5. (1) Each open disc $D = \{z \colon |z - z_0| < r\}$ is open and connected. The boundary of disc D is the circle $\{z \colon |z - z_0| = r\}$.

(2) The set $R = \{z \colon \mathrm{Re}z > 0\}$ is open. The set $\{z \colon |\mathrm{Im}z| \le 1\}$ is closed, so is the set $\{z \colon \mathrm{Re}z \le 6\}$.

(3) The boundary of the set $\{z = x + iy \colon x^2 < y\}$ is the parabola $\{z = x + iy \colon y = x^2\}$.

(4) The open set $\{z \colon \mathrm{Re}z > 0\}$ is connected, so is the open set $\{z \colon 0 < |z - z_0| < r\}$.

(5) The set $\{z \colon \mathrm{Re}z \ne 0\}$ is open but not connected.

(6) The open set $\{z\colon \mathrm{Re}z > 0\}$ is convex, but the open set $\{z\colon 0 < |z - z_0| < r\}$ is not convex.

(7) The set $\{z\colon \mathrm{Re}z > 0\}$ does not contain the point at infinity in its interior, for, given any $M > 0$, there are points z with $|z| > M$ but $\mathrm{Re}z \le 0$.
But the set $D = \{z\colon |z+1| + |z-1| > 1\}$ does contain the point at infinity in its interior because, for all z with $|z| > 1.5$, it follows from the triangle inequality that $|z+1|+|z-1| \ge (|z|-1) + (|z|-1) = 2|z| - 2 > 1$ and hence these z are contained in the set D.

Exercises. Page 28. Problems 1–8, 10, 11.

1.4. Functions and Limits

A **function** of the complex variable z, written as $w = f(z)$, is a rule that assigns a complex number w to each complex number z in a given subset D of the complex plane. The set D is called the **domain of definition** of the function. The collection of all possible values w of the function is called the **range** of the function.

A function $w = f(z)$ is called **one-to-one** on a set D if from $f(z_1) = f(z_2)$ with $z_1,\, z_2 \in D$ it must follow $z_1 = z_2$. A function $w = f(z)$ is called **onto** a set R if R is a subset of the range of this function.

Example 6. Show the range of the function $w = T(z) = (1+z)/(1-z)$ on the disc $|z| < 1$ is the set of those w whose real part is positive.

Proof. Let $D = \{z\colon |z| < 1\}$ and $S = \{w\colon \mathrm{Re}w > 0\}$. We show $T\colon D \to S$ is onto (surjective).

(1) Given any $z \in D$, let $w = T(z) = \frac{1+z}{1-z}$. Compute

$$\mathrm{Re}w = \mathrm{Re}\frac{1+z}{1-z} = \mathrm{Re}\frac{(1+z)(1-\bar{z})}{|1-z|^2} = \frac{1-|z|^2}{|1-z|^2} > 0$$

since $|z| < 1$. Hence the range of T is inside S.

(2) Given any $w \in S$, we want to show that there is $z \in D$ such that $w = T(z)$. Solve for z from $T(z) = w$ and we have $z = \frac{w-1}{w+1}$ (this shows that T is *one-to-one*). We need to show that this z belongs to D; that is, $|z| < 1$. This is the same as $|w-1| < |w+1|$ or $|w-1|^2 < |w+1|^2$. We expand $|w-1|^2 = (w-1)(\bar{w}-1)$ and $|w+1|^2 = (w+1)(\bar{w}+1)$ to obtain

$$|w-1|^2 = |w|^2 + 1 - 2\mathrm{Re}w; \quad |w+1|^2 = |w|^2 + 1 + 2\mathrm{Re}w.$$

Since $\mathrm{Re}w > 0$, it follows easily that $|w-1|^2 < |w+1|^2$; this proves $|z| < 1$, hence $w = T(z)$ is in the range of T. □

Limits, Continuity and Convergence. The concepts of the limit of a sequence of complex numbers and the limit and continuity of a complex variable function and the convergence of an infinite series of complex numbers are all identical to those for a real variable.

All the rules about the limit and the convergence for a real variable theory are also true for the complex variable theory. We mention some of these below as a review of such materials learned in the calculus.

A sequence $\{z_n\}$ is a list of complex numbers, usually starting with $n = 1, 2, 3, \cdots$. We say $\{z_n\}$ **converges** to a complex number A and write

$$\lim_{n\to\infty} z_n = A \quad \text{or simply} \quad z_n \to A,$$

if, given any positive number ϵ, there exists an integer N such that

$$|z_n - A| < \epsilon \quad \forall\, n \geq N.$$

Fact. *If $z_n = x_n + iy_n$ and $A = s + it$, then $z_n \to A$ if and only if $x_n \to s$ and $y_n \to t$.*

Fact. *If $z_n \to A$ then $|z_n| \to |A|$.*

Theorem. *Let $z_n \to A$ and $w_n \to B$. Then, for any constants λ, μ,*

$$\lambda z_n + \mu w_n \to \lambda A + \mu B,$$
$$z_n w_n \to AB,$$
$$\frac{z_n}{w_n} \to \frac{A}{B} \quad \textit{if } B \neq 0.$$

Suppose next that f is a function defined on a subset S of the complex plane. Let z_0 be a point either in S or in the boundary of S. We say that f has **limit** L at the point z_0 and we write

$$\lim_{z \to z_0} f(z) = L \quad \text{or} \quad f(z) \to L \text{ as } z \to z_0$$

if, given any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - L| < \epsilon \quad \text{whenever } z \in S \text{ and } 0 < |z - z_0| < \delta.$$

We say that a function f has limit L at ∞, and we write

$$\lim_{z \to \infty} f(z) = L$$

if

$$\lim_{z \to 0} f(\frac{1}{z}) = L,$$

which means that, given any $\epsilon > 0$, there is a large number M such that

$$|f(z) - L| < \epsilon \quad \text{whenever } z \in S \text{ and } |z| \geq M.$$

It is easy to see

$$\lim_{z \to \infty} \frac{1}{z^m} = 0$$

for all $m = 1, 2, \cdots$.

Example 7. 1) The function $f(z) = |z|^2$ has limit 4 at the point $z_0 = 2i$.

2) The function $g(z) = \frac{1}{z-1}$ has limit $L = \frac{1+i}{2}$ at $z_0 = i$.

3) The function $f(z) = \frac{z^4-1}{z-i}$ is not defined at $z = i$, but has limit $-4i$ at $z_0 = i$, since

$$f(z) = \frac{z^4 - 1}{z - i} = \frac{(z+1)(z-1)(z+i)(z-i)}{z - i}$$
$$= (z+1)(z-1)(z+i) = (z^2 - 1)(z + i),$$

and so $f(z) = (z^2 - 1)(z + i) \to (i^2 - 1)(2i) = -4i$ as $z \to i$.

4) The function $f(z) = \frac{z}{\bar{z}}$ has no limit at $z_0 = 0$. For if z is real, $f(z) = 1$, while if $z = iy$ (purely imaginary), then $f(z) = f(iy) = -1$. Such a function cannot have limit at $z_0 = 0$.

5)

$$\lim_{z \to \infty} \frac{z^4 + 1}{2z^4 + 5z^2 + 3} = \lim_{z \to \infty} \frac{1 + \frac{1}{z^4}}{2 + \frac{5}{z^2} + \frac{3}{z^4}} = \frac{1}{2}.$$

6) Let $z = x + iy$ and $f(z) = \frac{x+y^3}{x^2+y^3}$. Then $\lim_{z\to\infty} f(z)$ does not exist; for

$$\lim_{z=x\to\infty} f(z) = \lim_{x\to\infty} \frac{1}{x} = 0; \qquad \lim_{z=iy\to\infty} f(z) = \lim_{y\to\infty} 1 = 1.$$

Suppose f is a function defined on a set S. Let $z_0 \in S$. Then we say that f is **continuous** at z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0).$$

If f is continuous at every point of S then we say f is **continuous on** S. The function f is continuous at ∞ if $f(\infty)$ is defined and $\lim_{z\to\infty} f(z) = f(\infty)$.

Polynomials are continuous functions on the whole plane. **Rational functions** are quotients of two polynomials. All rational functions are continuous wherever the denominator is not zero.

Infinite Series. Like real variables, an infinite series of complex numbers is written as

$$\sum_{j=1}^{\infty} z_j.$$

We can define the **partial sums**, **convergence**, **sum**, **divergence**, **absolute convergence** of such series in the same way as the real variable theories.

A special kind of infinite series is the **power series** of the form

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

Facts. *1. If* $\sum |z_n|$ *converges, then* $\sum z_n$ *converges.*

2. Let $z_n = x_n + iy_n$. *Then* $\sum z_n$ *converges if and only if both* $\sum x_n$ *and* $\sum y_n$ *converge, and the sum is given by*

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n.$$

In this way, the convergence problems for $\sum z_n$ become the corresponding problems for two real series $\sum x_n$ and $\sum y_n$, or become the problem for one real series $\sum |z_n|$. For example, we can use the **ratio test** and **root test** for these real series.

However, the ratio and root tests can also be applied directly to the complex series $\sum z_n$ (see Exercises 42 and 43 in this section).

A useful identity is the **geometric series formula**:

$$\sum_{n=0}^{k} \alpha^n = \frac{1 - \alpha^{k+1}}{1 - \alpha} \quad \forall\ \alpha \neq 1.$$

Therefore

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1 - \alpha} \quad \forall\ |\alpha| < 1.$$

EXAMPLE 8. 1) The series

$$\sum_{n=1}^{\infty} n\left(\frac{1+2i}{3}\right)^n$$

converges, since

$$\left| n \left(\frac{1+2i}{3} \right)^n \right| = n \left(\frac{\sqrt{5}}{3} \right)^n$$

and hence $\sum |z_n|$ converges.

2) The series

$$\sum_{n=1}^{\infty} \frac{i^n}{n}$$

can be written as

$$\sum_{n=1}^{\infty} \frac{i^n}{n} = (-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \cdots) + i(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$$

and hence converges by the **alternating-series test**.

3) If $\sum z_n$ converges then $z_n \to 0$. This is very useful for showing the divergence of a series.

Exercises. Page 41. Problems 1–10, 13, 14, 31, 33, 36.

1.5. The Exponential, Logarithm, and Trigonometric Functions

The Exponential Function. The **exponential function** e^z is one of the most important functions in complex analysis and is defined as follows: Given $z = x + iy$,

$$e^z = e^x(\cos y + i \sin y).$$

We also use $\exp(z)$ to denote e^z, especially when z itself is a complicated expression. From the definition, it is direct to verify the following important properties for the exponential function:

$$e^{z+w} = e^z e^w \quad \text{(using the sum formulas for sine and cosine);}$$

$$|e^z| = e^{\mathrm{Re} z} > 0 \quad \text{(in particular, } |e^{iy}| = 1 \text{ for all real numbers } y\text{);}$$

$$e^{z+2\pi k i} = e^z \quad \forall k = 0, \pm 1, \pm 2, \cdots \quad \text{(that is, } e^z \text{ is periodic of period } 2\pi i\text{).}$$

We also have $e^{it} = \cos t + i \sin t$ for all real t; hence we have the interesting formula relating five most important numbers: $e^{i\pi} + 1 = 0$. Also for $z \neq 0$,

$$z = |z|e^{i\theta}, \text{ where } \theta \in \arg z, \quad \text{ or } \quad z = |z|e^{i \operatorname{Arg} z}.$$

The function $w = f(z) = e^z$ is never zero and it maps the z-plane onto the w-plane with the origin 0 removed but is not one-to-one (see the logarithm function below). The function $w = e^z$ maps the vertical line $x = \mathrm{Re} z = x_0$ onto the circle $|w| = e^{x_0}$ and maps the horizontal line $y = \mathrm{Im} z = y_0$ onto a ray from origin with fixed argument y_0.

The function $w = e^z$ carries each strip $y_0 \leq \mathrm{Im} z < y_0 + 2\pi$, $-\infty < \mathrm{Re} z < \infty$, **one-to-one** and **onto** the w-plane with the origin removed. For, if $e^{z_1} = e^{z_2}$ and z_1, z_2 are in the strip, then $z_1 - z_2 = 2\pi i k$ for some integer k; but then $2\pi|k| = |\mathrm{Im} z_1 - \mathrm{Im} z_2| < 2\pi$ for z_1, z_2 in the strip, so k must be zero and hence $z_1 = z_2$.

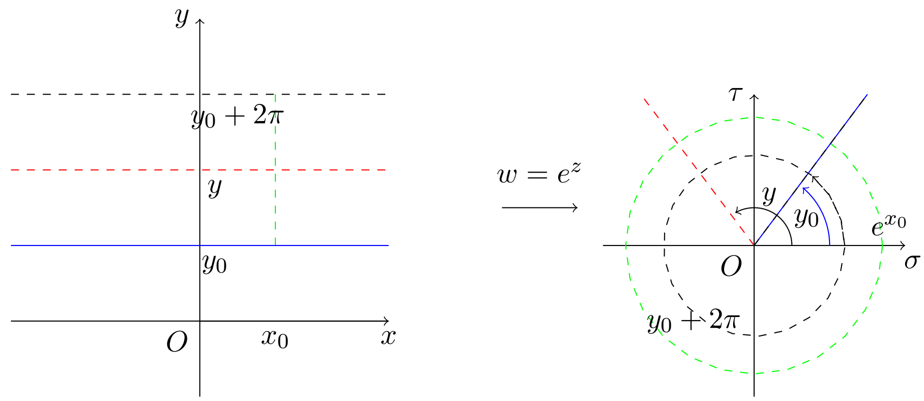

Figure 1.3. The mapping $w = e^z$

The Logarithmic Function. The **logarithm function** is the inverse of the exponential function (but it is **not** a well-defined function). For a nonzero complex number z, any number w satisfying $e^w = z$ is called a **logarithm** of z; the set of all logarithms of z is denoted by $\log z$ (this is not a single valued function).

We now compute $\log z$. Write $z = |z|(\cos\theta + i\sin\theta)$ with any θ being an argument of z and let $w = s + it$ be such that $e^w = z$. Then

$$e^s(\cos t + i\sin t) = |z|(\cos\theta + i\sin\theta).$$

Hence

$$s = \ln|z|, \quad t = \theta + 2\pi k \in \arg z.$$

Therefore all values of $\log z$ are given as

$$\log z = \ln|z| + i\arg z.$$

Since $\arg z$ is not single-valued, neither is $\log z$; but we can define the **principal logarithm function** using Arg$z$ to be

$$\text{Log}z = \ln|z| + i\text{Arg}z.$$

This is a well-defined function for all $z \neq 0$.

There is a way to make $\log z$ a well-defined (and nice, say, continuous) function if we delete a ray starting the origin from the the z-plane and use a continuous range of arguments for $\arg z$ in the definition of $\log z$. For example, if D is the open domain in the complex plane with the origin and the negative x-axis deleted, then Log$z$ becomes a continuous (even later analytic) function in D.

Given two complex numbers z and a with $z \neq 0$, the **power** z^a is defined by

$$z^a = e^{a\log z},$$

which is not single-valued (unless a is an integer). When $a = \frac{1}{n}$ for positive integers n this definition agrees with the nth roots of z defined before.

Example 9. Find $(-1)^i$.

Solution. Since $\log(-1) = \ln|-1| + i\arg(-1) = (2n+1)\pi i$, $n = 0, \pm1, \pm2, \cdots$, we obtain

$$(-1)^i = e^{i\log(-1)} = e^{-(2n+1)}, \quad n = 0, \pm1, \pm2, \cdots.$$

□

EXAMPLE 10. Solve $z^{1+i} = 4$.

Solution. Write the equation as

$$e^{(1+i)\log z} = 4,$$

so $(1+i)\log z = \log 4 = \ln 4 + 2\pi n i$, $n = 0, \pm 1, \cdots$. Hence

$$\log z = \frac{\log 4}{1+i} = \frac{1-i}{2}[\ln 4 + 2\pi n i]$$

$$= (1-i)[\ln 2 + \pi n i] = (\ln 2 + \pi n) + i(\pi n - \ln 2).$$

Thus

$$z = e^{\log z} = e^{\ln 2 + \pi n}[\cos(\pi n - \ln 2) + i\sin(\pi n - \ln 2)]$$

$$= 2e^{\pi n}[(-1)^n \cos\ln 2 + i(-1)^{n+1}\sin\ln 2]$$

$$= (-1)^n 2e^{\pi n}[\cos\ln 2 - i\sin\ln 2], \quad n = 0, \pm 1, \pm 2, \cdots .$$

□

EXAMPLE 11. Establish the formula

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = e^z \quad \forall\, z.$$

Proof. Look at $n\,\mathrm{Log}(1+z/n)$ for large n. Write

$$n\,\mathrm{Log}(1+\frac{z}{n}) = n\ln|1+\frac{z}{n}| + in\mathrm{Arg}(1+\frac{z}{n}).$$

The real part satisfies

$$n\ln|1+\frac{z}{n}| = \frac{1}{2}n\ln(1+\frac{2x}{n}+\frac{x^2+y^2}{n^2}) \to x$$

as $n \to \infty$. Next, to handle the imaginary part, let $z = r(\cos\theta + i\sin\theta)$ and $\psi_n = \mathrm{Arg}(1+z/n)$; then from the geometry,

$$\tan\psi_n = \frac{\frac{r}{n}\sin\theta}{1+\frac{r}{n}\cos\theta},$$

from which it follows that $\psi_n \to 0$ and

$$n\tan\psi_n \to r\sin\theta = y$$

as $n \to \infty$. Hence $n\psi_n \to y$ as $n \to \infty$. Consequently,

$$n\,\mathrm{Log}\left(1+\frac{z}{n}\right) \to x + iy = z$$

as $n \to \infty$. This proves that

$$\left(1+\frac{z}{n}\right)^n = \exp\left[n\,\mathrm{Log}\left(1+\frac{z}{n}\right)\right] \to e^z$$

as $n \to \infty$, as intended. □

Trigonometric Functions. The **trigonometric functions** of z are defined in terms of the exponential function e^z as follows:

$$\cos z = \frac{1}{2}\left(e^{iz}+e^{-iz}\right); \quad \sin z = \frac{1}{2i}\left(e^{iz}-e^{-iz}\right).$$
$$\tan z = \frac{\sin z}{\cos z}; \quad \cot z = \frac{\cos z}{\sin z}; \quad \sec z = \frac{1}{\cos z}; \quad \csc z = \frac{1}{\sin z},$$

wherever the denominator is not zero.

It is easy to see that

$$\cos(z+2\pi k) = \cos z, \quad \sin(z+2\pi k) = \sin z$$

for z and all $k = 0, \pm 1, \pm 2, \cdots$. Furthermore, $2\pi k$ is the only numbers α for which $\cos(z+\alpha) = \cos z$ holds for all z; the same is true for $\sin z$. For if$\cos(z+\alpha) = \cos z$ for all complex numbers z then

$$e^{iz}e^{i\alpha} + e^{-iz}e^{-i\alpha} = e^{iz} + e^{-iz} \quad \forall\ z.$$

So $e^{iz}(e^{i\alpha}-1) = e^{-iz}e^{-i\alpha}(e^{i\alpha}-1)$ for all z. Set $z = 0$ to obtain $(e^{i\alpha}-1)^2 = 0$ and hence $e^{i\alpha} = 1$, which gives $\alpha = 2k\pi$ for some integer k.

For this reason, 2π is called the **basic period** of $\sin z$ and $\cos z$.

The mapping property of $w = \sin z$. Note that

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y,$$

where, $\sinh u$ and $\cosh u$ are **hyperbolic-sine** and **hyperbolic-cosine** functions for real numbers u,

$$\sinh u = \frac{1}{2}(e^u - e^{-u}), \quad \cosh u = \frac{1}{2}(e^u + e^{-u}).$$

These hyperbolic functions can also be defined for all complex numbers u by the same formulas.

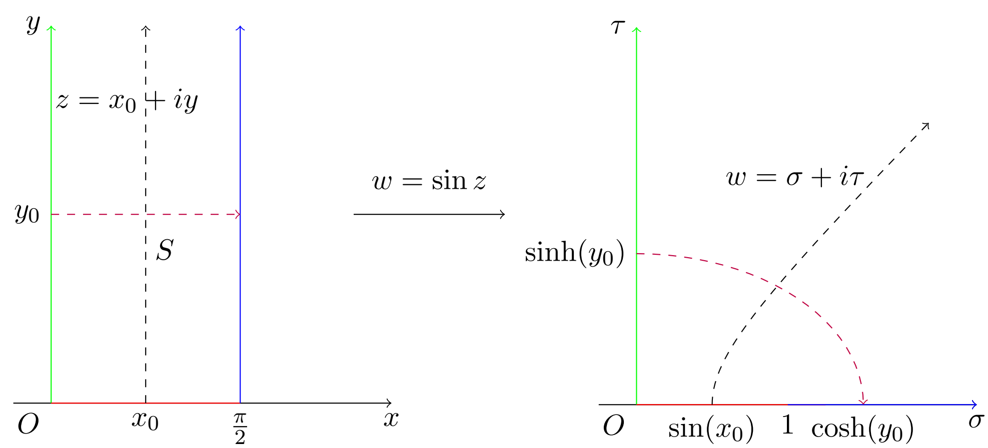

Figure 1.4. The mapping $w = \sin z$

We now restrict $z = x+iy$ to the half-strip $S = \{z : 0 < x < \pi/2,\ y > 0\}$. Note that

$$\sin(iy) = i \sinh y, \quad \sin(\frac{\pi}{2} + iy) = \cosh y.$$

The function $w = \sin z$ is *one-to-one* from S *onto* the open first quadrant of the w-plane, with the boundary of S being mapped onto the boundary of the first quadrant in an interesting way. The vertical half line $\{z = x_0 + iy,\ y > 0\}$ in S is mapped onto the portion of the hyperbola

$$\frac{\sigma^2}{\sin^2 x_0} - \frac{\tau^2}{\cos^2 x_0} = 1$$

in the first quadrant of the w plane, while the horizontal line segment $\{z = x + iy_0,\ 0 < x < \pi/2\}$ in S is mapped onto the portion of the ellipse

$$\frac{\sigma^2}{\cosh^2 y_0} + \frac{\tau^2}{\sinh^2 y_0} = 1$$

in the first quadrant of the w plane. Note that the hyperbola and ellipse given above is *perpendicular* at their intersection point $z_0 = x_0 + iy_0$.

Exercises. Page 53. Problems 1, 3, 4, 10, 23, 27, 28.

1.6. Line Integrals and Green's Theorem

The fundamental theorems of complex variables are built on Cauchy's Theorem and Formula, which depend on **line integrals** and Green's Theorem. So we review and discuss these materials now.

A **curve** γ is defined to be a continuous function from an finite closed interval $[a, b]$ to the complex plane. This function $\gamma(t)$ is sometime also called a **parameterization** of the curve. The natural **orientation** of the curve γ is defined by tracing the point $\gamma(t)$ starting with $t = a$ and ending with $t = b$.
A curve γ is called **simple** if $\gamma(t_1) \neq \gamma(t_2)$ for all $a \leq t_1 < t_2 < b$.
A curve γ is called **closed** if $\gamma(a) = \gamma(b)$; that is, if its starting and ending points coincide.

Theorem 1.2 (**Jordan's Theorem**). *Let γ be a simple and closed curve. Then the complement of its range consists of two domains, one of which is bounded and the other is unbounded. The bounded one is called the* **inside** *of γ and the unbounded one the* **outside** *of γ.*

EXAMPLE 12. (1) Given any two distinct complex numbers z_0 and z_1, the (directed) line segment from z_0 to z_1 is a curve with parameterization

$$\gamma(t) = (1 - t)z_0 + tz_1, \quad 0 \leq t \leq 1.$$

(2) The curve with function

$$\gamma(t) = z_0 + Re^{it}, \quad 0 \leq t \leq 2\pi$$

represents the simple closed positively oriented circle with center z_0 and radius R.

However, if we change the parameter interval to $0 \leq t \leq 4\pi$, this curve is also closed but not simple; but the range (imagine) of the curve is the same circle.

Let $f(t) = x(t)+iy(t)$ be a complex-valued function defined on $[a, b]$, where $x(t), y(t)$ are the real and imaginary parts of $f(t)$. If both x and y are differentiable at a point $t_0 \in [a, b]$ then we say f is differentiable at t_0, and define

$$f'(t_0) = x'(t_0) + iy'(t_0).$$

We say f is **differentiable on** $[a, b]$ if f is differentiable at every point of $[a, b]$; we say f is **smooth on** $[a, b]$ or C^1 **on** $[a, b]$ if f is differentiable on $[a, b]$ and f' is continuous on $[a, b]$.

The differentiation of complex-valued functions of one real variable is consistent with the differentiation of vector-valued functions studied in the vector calculus. Many differentiation rules also hold for this differentiation. But we want to emphasize the following product rule:

$$(f(t)g(t))' = f(t)g'(t) + f'(t)g(t)$$

holds for differentiable complex-valued functions f and g. Therefore we have by induction

$$[(f(t))^m]' = m(f(t))^{m-1}f'(t), \quad m = 1, 2, \cdots .$$

A curve γ is said to be a **smooth curve** if its parametrization $\gamma(t)$ is smooth on its interval $[a, b]$.

A curve can also be defined by joining a finite number of curves together; this is a **piecewisely defined** curve. The parametrization may have several independent parameters on different intervals, but the only requirement is that the end point of one piece coincides with the starting point of the following piece. Therefore, if γ is a piece-wisely defined curve if there exist intervals $[a_k, b_k]$ $(k = 1, 2, \cdots, n)$ and continuous function γ_k on $[a_k, b_k]$ such that

$$\gamma(b_k) = \gamma(a_{k+1}) \quad \forall\ k = 1, 2, \cdots, n-1.$$

The starting point of the curve is $\gamma_1(a_1)$ and the ending point is $\gamma_n(b_n)$. A curve is called **piecewise smooth** if each piece of the curve $\gamma_k|_{[a_k,b_k]}$ is smooth for all $k = 1, 2, \cdots, n$.

If $\gamma(t)$, $a \le t \le b$, is a curve, then the curve $-\gamma$ defined by

$$-\gamma(t) = \gamma(a + b - t), \quad a \le t \le b,$$

is called the **reversed curve** of γ. This is the curve with the same range as γ but starting with $\gamma(b)$ and ending with $\gamma(a)$; the orientation is exactly reversed.

A simple closed curve γ is called **positively oriented** if you trace the points $\gamma(t)$ while increasing the parameter from a to b the inside of γ is always on your left; this is equivalent to saying that the γ is traced **counterclockwise**.

Example 13. (1) The square with vertices at points z_0, iz_0, $-z_0$ and $-iz_0$ can be made a simple closed curve by the following parameterization:

$$\gamma(t) = \begin{cases} tiz_0 + (1-t)z_0, & 0 \le t \le 1 \\ (t-1)(-z_0) + (2-t)iz_0, & 1 \le t \le 2 \\ (t-2)(-iz_0) + (3-t)(-z_0), & 2 \le t \le 3 \\ (t-3)z_0 + (4-t)(-iz_0), & 3 \le t \le 4. \end{cases}$$

Of course, one can also define the same square as piecewisely defined curve as follows:

$$\gamma(t) = \begin{cases} tiz_0 + (1-t)z_0, & 0 \le t \le 1 \\ t(-z_0) + (1-t)iz_0, & 0 \le t \le 1 \\ t(-iz_0) + (1-t)(-z_0), & 0 \le t \le 1 \\ tz_0 + (1-t)(-iz_0), & 0 \le t \le 1. \end{cases}$$

(2) The curve

$$\gamma : \quad \begin{cases} z = Re^{i\theta}, & 0 \le \theta \le \pi \\ z = t, & -R \le t \le R \end{cases}$$

is piece-wisely defined and represents the simple closed piecewise smooth curve which consists a semicircle and a diameter.

(3) The curve shown in the figure below has a parametrization given by

$$\gamma : \begin{cases} z = Re^{i\theta}, & 0 \le \theta \le \pi \\ z = x, & -R \le x \le -\epsilon \\ z = \epsilon e^{i(\pi-\theta)}, & 0 \le \theta \le \pi \\ z = x, & \epsilon \le x \le R. \end{cases}$$

Note that the third formula is different from the text because we trace the curve always along the increasing direction of the parameter.

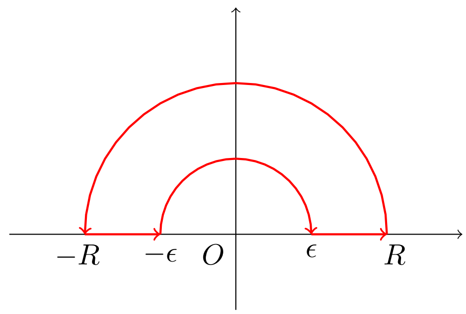

Figure 1.5. A piecewise simple closed curve with separate parameterizations

Definite Integrals. Suppose $g(t) = \sigma(t) + i\tau(t)$ is a continuous complex-valued function on the interval $[a, b]$. We define the **definite integral of g over** $[a, b]$ by

$$\int_a^b g(t)\,dt = \int_a^b \sigma(t)\,dt + i\int_a^b \tau(t)\,dt.$$

This definition is consistent with the definition of the definite integral of a vector-valued function of the real variable t in the vector calculus, and so many rules of integration also hold. From this definition, it is easy to have

$$\mathrm{Re}\left(\int_a^b g(t)\,dt\right) = \int_a^b \mathrm{Re}(g(t))\,dt, \quad \mathrm{Im}\left(\int_a^b g(t)\,dt\right) = \int_a^b \mathrm{Im}(g(t))\,dt.$$

We also have the following estimate:

$$\left|\int_a^b g(t)\,dt\right| \le \int_a^b |g(t)|\,dt. \tag{1.2}$$

Proof. The inequality is obviously true if $\int_a^b g(t)\,dt = 0$, so we may assume $z_0 = \int_a^b g(t)dt \ne 0$ and let $\theta = \mathrm{Arg} z_0$. Define $h(t) = e^{-i\theta}g(t)$, $a \le t \le b$. Then

$$\begin{aligned} \left|\int_a^b g(t)\,dt\right| &= |z_0| = e^{-i\theta}z_0 \\ &= e^{-i\theta}\left(\int_a^b g(t)\,dt\right) = \int_a^b h(t)\,dt \\ &= \mathrm{Re}\left(\int_a^b h(t)\,dt\right) = \int_a^b (\mathrm{Re} h(t))\,dt \\ &\le \int_a^b |h(t)|\,dt = \int_a^b |g(t)|\,dt. \end{aligned}$$

□

Line Integrals. Suppose that γ is a piece-wise smooth curve with smooth parameterizations on intervals $[a_k, b_k]$, $k = 1, \cdots, n$, and that u is a continuous function on the range of γ. We define the **line integral along** γ of u by

$$\int_\gamma u(z)\, dz = \sum_{k=1}^{n} \int_{a_k}^{b_k} u(\gamma(t))\, \gamma'(t)\, dt. \tag{1.3}$$

Recall that if a smooth plane curve γ is parametrized by $\gamma(t) = (x(t), y(t))$ on $[a, b]$ then for a real valued continuous function $p = p(x, y)$ we have the definition of **line integral**:

$$\int_\gamma p\, dx = \int_a^b p(x(t), y(t))x'(t)\, dt, \qquad \int_\gamma p\, dy = \int_a^b p(x(t), y(t))y'(t)\, dt.$$

Then it is easy to see that the definition (1.3) above agrees with this definition under the convention that $dz = dx + idy$ and $\gamma'(t) = x'(t) + iy'(t)$.

Line integrals have the familiar properties of definite integrals studied in calculus. For example

$$\int_\gamma [Au(z) + Bv(z)]\, dz = A \int_\gamma u(z)\, dz + B \int_\gamma v(z)\, dz \tag{1.4}$$

if A, B are constants of complex numbers and u, v are continuous functions on the range of curve γ.

For the reversed curve $-\gamma$ we have

$$\int_{-\gamma} u(z)\, dz = -\int_\gamma u(z)\, dz.$$

Furthermore, we have the following important estimate:

$$\left| \int_\gamma u(z)\, dz \right| \le \left(\max_{z\in\gamma} |u(z)| \right) \text{Length } (\gamma). \tag{1.5}$$

Proof. We can assume γ consists of simply one smooth curve with parameterization $\gamma(t) = x(t) + iy(t)$, $t \in [a, b]$. Then, by (1.2) and definition (1.3),

$$\begin{aligned} \left| \int_\gamma u(z)\, dz \right| &= \left| \int_a^b u(\gamma(t))\gamma'(t)\, dt \right| \\ \le \int_a^b |u(\gamma(t))||\gamma'(t)|\, dt &\le \left(\max_{z\in\gamma} |u(z)| \right) \int_a^b |\gamma'(t)|\, dt \\ &= \left(\max_{z\in\gamma} |u(z)| \right) \text{Length } (\gamma), \end{aligned}$$

because

$$\int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \text{Length of } \gamma.$$

$\square$

EXAMPLE 14. (1) Compute $\int_\gamma (z^2 - 3|z| + \text{Im} z)\, dz$, where $\gamma(t) = 2e^{it}$, $0 \le t \le \pi/2$.

Let $u(z) = z^2 - 3|z| + \text{Im}z$. Then

$$u(\gamma(t)) = 4e^{2it} - 3 \cdot 2 + 2 \sin t; \quad \gamma'(t) = 2ie^{it} (\text{informally, but true}).$$

Hence

$$\begin{aligned} u(\gamma(t))\gamma'(t) &= 8ie^{3it} - 12ie^{it} + 4ie^{it} \sin t \\ = 8ie^{3it} - 12ie^{it} + 2ie^{it}(e^{it} - e^{-it}) &= 8ie^{3it} - 12ie^{it} + 2e^{2it} - 2. \end{aligned}$$

So, by definition,

$$\int_\gamma u(z)\,dz = \int_0^{\frac{\pi}{2}} u(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} (8ie^{3it} - 12ie^{it} + 2e^{2it} - 2)\,dt$$

$$= \left(\frac{8}{3}e^{3it} - 12e^{it} + \frac{1}{i}e^{2it} - 2t\right)\Big|_0^{\frac{\pi}{2}} = \frac{28}{3} - \pi - \frac{38}{3}i.$$

(2) Show

$$\left|\int_{|z|=R} \frac{1}{z^2+4}\,dz\right| \le \frac{2\pi R}{R^2-4}$$

if $R > 2$.

Proof. On the circle $|z| = R$, by the triangle inequality, $|z^2+4| \ge |z|^2 - 4 = R^2 - 4$, so

$$\max_{|z|=R}\left|\frac{1}{z^2+4}\right| \le \frac{1}{R^2-4}$$

and the length of the circle is $2\pi R$. Hence this estimate follows from (1.5) above. □

(3) Let u be continuous in $|z - z_0| < r$ and γ_ϵ be the simple closed positively oriented circle $|z - z_0| = \epsilon$ with $0 < \epsilon < r$. Show that

$$\lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\gamma_\epsilon} \frac{u(z)}{z - z_0}\,dz = u(z_0).$$

Proof. The curve γ_ϵ is parameterized by $z = z_0 + \epsilon e^{it}$, $0 \le t \le 2\pi$; hence $\gamma_\epsilon'(t) = \epsilon i e^{it}$. So

$$\int_{\gamma_\epsilon} \frac{u(z)}{z-z_0}\,dz = \int_0^{2\pi} \frac{u(z_0+\epsilon e^{it})}{\epsilon e^{it}} \cdot \epsilon i e^{it}\,dt$$

$$= i\int_0^{2\pi} u(z_0 + \epsilon e^{it})\,dt$$

$$= i\int_0^{2\pi} [u(z_0+\epsilon e^{it}) - u(z_0)]\,dt + i\int_0^{2\pi} u(z_0)\,dt$$

$$\to 2\pi i u(z_0) \quad \text{as } \epsilon \to 0.$$

□

(4) Let γ be any piece-wise smooth curve with starting point A and ending point B. Then, for all $m = 0, 1, \cdots$,

$$\int_\gamma z^m\,dz = \frac{1}{m+1}(B^{m+1} - A^{m+1}).$$

Therefore, the line integral is independent of the curve γ but only dependent on the endpoints of the curve.

Proof. Note that for complex-valued differentiable functions $z(t)$ and $w(t)$ the product rule of differentiation is also valid:

$$[z(t)w(t)]' = z'(t)w(t) + z(t)w'(t).$$

Therefore the power chain rule is valid:

$$[z^m(t)]' = mz^{m-1}(t)\, z'(t), \quad m = 1, 2, \cdots .$$

Therefore, if $\gamma(t)$ is smooth, then

$$\gamma^m(t)\gamma'(t) = \left[\frac{1}{m+1}\gamma^{m+1}(t)\right]', \quad m = 0, 1, 2, \cdots .$$

Let γ be a piece-wise smooth curve with smooth parameterizations on intervals $[a_k, b_k]$, $k = 1, 2, \cdots, n$, and $\gamma(a_1) = A$ and $\gamma(b_n) = B$. Then

$$\begin{aligned}\int_\gamma z^m\, dz &= \sum_{k=1}^n \int_{a_k}^{b_k} \gamma^m(t)\, \gamma'(t)\, dt \\ &= \sum_{k=1}^n \int_{a_k}^{b_k} \left[\frac{1}{m+1}\gamma^{m+1}(t)\right]' dt \\ &= \sum_{k=1}^n \left[\frac{1}{m+1}\gamma^{m+1}(t)\right]_{a_k}^{b_k} = \frac{1}{m+1}(B^{m+1} - A^{m+1}).\end{aligned}$$

□

Green's Theorem. The most important result on line integrals is **Green's Theorem**, which is a reformulation of the theorems in vector calculus. To state the theorem, we need to consider domains with certain properties.

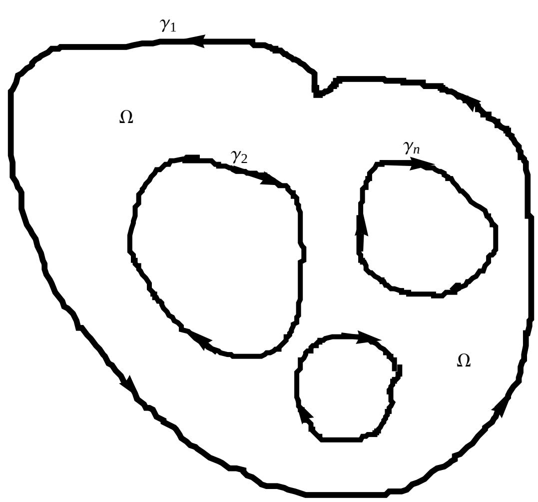

Figure 1.6. The domain Ω with positively oriented boundaries.

Let Ω be a domain whose boundary $\Gamma = \partial\Omega$ consists of a finite number of disjoint, piece-wise smooth, simple and closed curves $\gamma_1, \gamma_2, \cdots, \gamma_n$. We say that the boundary Γ is **positively oriented** if Ω remains always on our left if we walk along Γ. Thus, if positively oriented, the *outer* piece of Γ is oriented *counterclockwise*, and each of the *inner* pieces of Γ is oriented *clockwise*. In this case, if f is a continuous complex-valued function on Γ, we define the line integral of f over Γ by

$$\int_\Gamma f(z)\,dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\,dz,$$

where, of course, we already know how to compute each $\int_{\gamma_j} f(z)\,dz$.

Green's Theorem relates the line integral of a function f over Γ to the (area) integral of a certain combination of partial derivatives of f over domain Ω. In order to state it properly, we introduce the following notation.

Let $f(z) = p(z) + iq(z)$ be a function of complex variable $z = x + iy$ in some domain D containing $\bar{\Omega} = \Gamma \cup \Omega$, where p, q are real and imaginary parts of f, considered as functions of (x, y). We then define

$$f_x = p_x + iq_x, \quad f_y = p_y + iq_y,$$

where p_x, p_y, q_x, and q_y are partial derivatives, that are assumed to exist.

Theorem 1.3 (**Green's Theorem**). *Let $f(z)$ be a function of complex variable which has continuous partial derivatives in some domain D containing $\bar{\Omega} = \Gamma \cup \Omega$, where Ω is a domain as described above with positively oriented boundary Γ. Then*

$$\int_\Gamma f(z)\,dz = i \iint_\Omega (f_x + i f_y)\,dxdy. \tag{1.6}$$

Recall that in multi-variable calculus, Green's Theorem was stated as

$$\int_\Gamma (u\,dx + v\,dy) = \iint_\Omega (v_x - u_y)\,dxdy \tag{1.7}$$

for any two continuously differentiable real-valued functions u, v of two real variables x, y. It is a good exercise to verify that the complex variable formula (1.6) in Green's Theorem above is equivalent to Formula (1.7). (See **Exercises #10, #11**.)

Example 15. Let γ be a piece-wise smooth positively oriented simple closed curve and p be a point not on γ. Compute

$$\frac{1}{2\pi i} \int_\gamma \frac{1}{z - p}\,dz.$$

Proof. Let Ω be the inside of γ. Let $f(z) = \frac{1}{z-p}$ for all $z \neq p$. If $z = x + iy \neq p$ then $f(z) = \frac{1}{x+iy-p}$ and one can compute

$$f_x = -\frac{1}{(x + iy - p)^2}, \quad f_y = -\frac{i}{(x + iy - p)^2};$$

hence $f_x + i f_y = 0$ at all points $z \neq p$.

If $p \notin \Omega$ and thus $p \notin \bar{\Omega}$ (since $p \notin \gamma$), then by Green's Theorem,

$$\int_\gamma f(z)\,dz = i \iint_\Omega (f_x + i f_y)\,dxdy = 0.$$

Therefore

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-p}\,dz = 0 \quad \text{if } p \notin \Omega.$$

Now assume $p \in \Omega$. Let D_ϵ be the closed disc centered p of radius $\epsilon > 0$. Assume ϵ is sufficiently small so that $D_\epsilon \subset \Omega$. Let $\Omega_\epsilon = \Omega \setminus D_\epsilon$. The positively oriented boundary Γ_ϵ of Ω_ϵ consists of two parts: γ and circle $\delta_\epsilon = \{|z-p| = \epsilon\}$ oriented clockwise. Since $p \notin \bar{\Omega}_\epsilon$, by Green's Theorem,

$$\int_{\gamma_\epsilon} f(z)\,dz = i\iint_{\Omega_\epsilon} (f_x + if_y)\,dxdy = 0.$$

Hence

$$\int_\gamma f(z)\,dz = -\int_{\delta_\epsilon} f(z)\,dz = \int_{-\delta_\epsilon} f(z)\,dz.$$

We can parameterize $-\delta_\epsilon$ (which is the circle oriented counter clockwise) by $z = p + \epsilon e^{it}$ with $0 \le t \le 2\pi$, and hence

$$\int_{-\delta_\epsilon} f(z)\,dz = \int_0^{2\pi} \frac{1}{\epsilon e^{it}}\, i\epsilon e^{it}\,dt = 2\pi i.$$

Therefore

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-p}\,dz = 1 \quad \text{if } p \in \Omega.$$

□

Example 16. Let γ be the simple positively oriented unit circle $|z| = 1$. Compute

$$\int_\gamma \bar{z}\,dz.$$

Solution. Let Ω be the unit disc $|z| < 1$ and $f(z) = \bar{z} = x - iy$. Then

$$f_x + if_y = (x-iy)_x + i(x-iy)_y = 1 - i^2 = 2.$$

Hence, by Green's Theorem,

$$\int_\gamma f(z)\,dz = i\iint_\Omega (f_x + if_y)\,dxdy = i\iint_\Omega 2\,dxdy = 2i\,\text{Area of }\Omega = 2\pi i.$$

We can also compute the line integral directly by definition. Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. Then

$$\int_\gamma \bar{z}\,dz = \int_0^{2\pi} \overline{e^{it}}\,(e^{it})'\,dt = \int_0^{2\pi} e^{-it} i e^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

As expected, we got the same answer. □

Exercises. Page 73. Problems. 1, 2, 3, 7, 8, 9, 12.

Homework Problems for Chapter 1.

1.1	1, 2, 3, 5, 6, 11
1.2	1, 2, 3, 7, 11, 12, 15, 21, 23, 24, 25
1.3	1–8, 10, 11
1.4	1-10, 13, 14, 31, 33, 36
1.5	1, 3, 4, 10, 23, 27, 28
1.6	1, 2, 3, 7, 8, 9, 12