Chapter 6

The Riemann-Stieltjes Integral

6.1. Definition and Existence of the Integral

Definition 6.1. Let $a, b \in \mathbf{R}$ and $a < b$.

(a) A **partition** P of interval $[a, b]$ is a finite set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

We write $\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, \cdots, n$, and define the **norm** $\|P\|$ of P by

$$\|P\| = \max_{1 \le i \le n} \Delta x_i.$$

(b) A **refinement** of partition P is a partition Q such that $P \subseteq Q$. In this case, we also say that Q is **finer** than P. Given two partitions P_1 and P_2 of $[a, b]$, we call the union $P_1 \cup P_2$ their **common refinement**.

Definition 6.2. Let α be a monotonically increasing function on $[a, b]$, f be a bounded function on $[a, b]$ and $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Define

$$\Delta \alpha_j = \alpha(x_j) - \alpha(x_{j-1}) \quad (1 \le j \le n)$$

and

$$M_j(f) = \sup_{x \in [x_{j-1}, x_j]} f(x), \quad m_j(f) = \inf_{x \in [x_{j-1}, x_j]} f(x) \quad (1 \le j \le n).$$

We call the numbers

$$U(P, f, \alpha) = \sum_{j=1}^{n} M_j(f) \Delta \alpha_j, \quad L(P, f, \alpha) = \sum_{j=1}^{n} m_j(f) \Delta \alpha_j$$

the **upper (Riemann-Stieltjes) sum** and, respectively, the **lower (Riemann-Stieltjes) sum** of f with partition P over $[a, b]$ with respect to α.

Note that, if $m \le f(x) \le M$ for all $x \in [a, b]$, then $m \le m_j(f) \le M_j(f) \le M$ for each $j = 1, 2, \cdots, n$, and hence

$$m(\alpha(b) - \alpha(a)) \le L(P, f, \alpha) \le U(P, f, \alpha) \le M(\alpha(b) - \alpha(a)).$$

So, the sets $\{U(P,f,\alpha) \mid P \text{ is partition of } [a,b]\}$ and $\{L(P,f,\alpha) \mid P \text{ is partition of } [a,b]\}$ are bounded sets in $\mathbf{R}$.

Define

$$\overline{\int_a^b} f d\alpha = \inf\{U(P,f,\alpha) \mid P \text{ is partition of } [a,b]\}$$

and

$$\underline{\int_a^b} f d\alpha = \sup\{L(P,f,\alpha) \mid P \text{ is partition of } [a,b]\}$$

to be the **upper Riemann-Stieltjes integral** and, respectively, the **lower Riemann-Stieltjes integral** of f over $[a,b]$ with respect to α.

We say that f is **Riemann-Stieltjes integrable** on $[a,b]$ with respect to α, and write $f \in \mathcal{R}(\alpha)[a,b]$, provided that

$$\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha. \tag{6.1}$$

In this case, the common value of the upper and lower Riemann-Stieltjes integrals in (6.1) is called the **Riemann-Stieltjes integral** of f over $[a,b]$ with respect to α and denoted by

$$\int_a^b f d\alpha.$$

Sometimes, we also write

$$\int_a^b f d\alpha = \int_a^b f(x) d\alpha(x);$$

of course, the "dummy variable" x can be replaced by any other letters (except for the letters f, α or d, to avoid obvious confusion).

Definition 6.3. When the function α is the **identity function**, i.e., $\alpha(x) = x$, we define the notations

$$U(P,f), \;\; L(P,f), \;\; \overline{\int_a^b} f dx, \;\; \underline{\int_a^b} f dx, \;\; \int_a^b f dx, \;\; \mathcal{R}[a,b]$$

to be the notations, respectively,

$$U(P,f,\alpha), \;\; L(P,f,\alpha), \;\; \overline{\int_a^b} f d\alpha, \;\; \underline{\int_a^b} f d\alpha, \;\; \int_a^b f d\alpha, \;\; \mathcal{R}(\alpha)[a,b].$$

In this case, if $f \in \mathcal{R}[a,b]$, then we say f is **Riemann integrable** on $[a,b]$ or, simply, **integrable** on $[a,b]$.

Theorem 6.1. *If P, Q are partitions of $[a,b]$ and Q is finer than P, then*

$$L(P,f,\alpha) \le L(Q,f,\alpha) \le U(Q,f,\alpha) \le U(P,f,\alpha).$$

That is, $L(P,f,\alpha)$ increases with P and $U(P,f,\alpha)$ decreases with P.

Proof. Since Q is obtained from P by adding finitely many points, by induction, we only need to prove the case when Q is obtained from P by adding one extra point. So let

$$P = \{x_0, x_1, \ldots, x_{k-1}, x_k, \ldots, x_n\}, \quad Q = \{x_0, x_1, \ldots, x_{k-1}, y, x_k, \ldots, x_n\},$$

where $x_{k-1} < y < x_k$. Then

$$L(P,f,\alpha) = \sum_{j=1}^n m_j(f) \Delta\alpha_j, \quad \text{where} \quad m_j(f) = \inf_{x \in [x_{j-1}, x_j]} f(x),$$

$$L(Q,f,\alpha) = \sum_{j=1}^{k-1} m_j(f)\Delta\alpha_j + \left(\inf_{x\in[x_{k-1},y]} f(x)\right)(\alpha(y)-\alpha(x_{k-1}))$$
$$+ \left(\inf_{x\in[y,x_k]} f(x)\right)(\alpha(x_k)-\alpha(y)) + \sum_{j=k+1}^{n} m_j(f)\Delta\alpha_j.$$

Note that

$$\inf_{x\in[x_{k-1},y]} f(x) \ge \inf_{x\in[x_{k-1},x_k]} f(x), \qquad \inf_{x\in[y,x_k]} f(x) \ge \inf_{x\in[x_{k-1},x_k]} f(x).$$

Hence, since α is increasing, we have that

$$\left(\inf_{x\in[x_{k-1},y]} f(x)\right)(\alpha(y)-\alpha(x_{k-1})) + \left(\inf_{x\in[y,x_k]} f(x)\right)(\alpha(x_k)-\alpha(y))$$
$$\ge \left(\inf_{x\in[x_{k-1},x_k]} f(x)\right)(\alpha(y)-\alpha(x_{k-1})+\alpha(x_k)-\alpha(y)) = m_k(f)\Delta\alpha_k.$$

Consequently,

$$L(Q,f,\alpha) \ge \sum_{j=1}^{k-1} m_j(f)\Delta\alpha_j + m_k(f)\Delta\alpha_k + \sum_{j=k+1}^{n} m_j(f)\Delta\alpha_j = L(P,f,\alpha).$$

The proof of $U(Q,f,\alpha) \le U(P,f,\alpha)$ is similar. □

Theorem 6.2. $\underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$.

Proof. Let P, Q be arbitrary two partitions of $[a,b]$. Then, since $P \cup Q$ is a refinement of both P and Q, by the previous theorem,

$$L(P,f,\alpha) \le L(P\cup Q,f,\alpha) \le U(P\cup Q,f,\alpha) \le U(Q,f,\alpha).$$

Hence

$$\underline{\int_a^b} f\,d\alpha = \sup_P\{L(P,f,\alpha)\} \le \inf_Q\{U(Q,f,\alpha)\} = \overline{\int_a^b} f\,d\alpha.$$

□

Theorem 6.3 (Criterion for Integrability). *A bounded function f is in $\mathcal{R}(\alpha)[a,b]$ if and only if for each $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that*

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon. \tag{6.2}$$

Proof. (*Sufficiency for Integrability.*) Let $\varepsilon > 0$. Assume that there exists a partition P of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$. Then $U(P,f,\alpha) < L(P,f,\alpha) + \varepsilon$, and thus

$$\overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha) < L(P,f,\alpha) + \varepsilon \le \underline{\int_a^b} f\,d\alpha + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves that $\overline{\int_a^b} f\,d\alpha \le \underline{\int_a^b} f\,d\alpha$, and hence $\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$; so $f \in \mathcal{R}(\alpha)[a,b]$.

(*Necessity for Integrability.*) Assume $f \in \mathcal{R}(\alpha)[a,b]$; namely, $\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$. Let $\varepsilon > 0$. Then there exist partitions P_1, P_2 of $[a,b]$ such that

$$U(P_1,f,\alpha) < \overline{\int_a^b} f\,d\alpha + \varepsilon/2, \quad L(P_2,f,\alpha) > \underline{\int_a^b} f\,d\alpha - \varepsilon/2.$$

Let $P = P_1 \cup P_2$ be the common refinement of P_1 and P_2. Then P is a partition of $[a,b]$, and, using $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$, we have

$$U(P,f,\alpha) - L(P,f,\alpha) \le U(P_1,f,\alpha) - L(P_2,f,\alpha) < \left(\overline{\int_a^b} f d\alpha + \frac{\varepsilon}{2} \right) - \left(\underline{\int_a^b} f d\alpha - \frac{\varepsilon}{2} \right) = \varepsilon.$$

□

Theorem 6.4. *Suppose that* (6.2) *holds for a partition* $P = \{x_0, x_1, \cdots, x_n\}$.

(a) *Then* (6.2) *holds with* P *replaced by any refinement of* P.

(b) *If* s_j, t_j *are points in* $[x_{j-1}, x_j]$ *for each* j, *then*

$$\sum_{j=1}^{n} |f(s_j) - f(t_j)| \Delta\alpha_j < \varepsilon.$$

(c) *If* $f \in \mathcal{R}(\alpha)[a,b]$, *then*

$$\left| \sum_{j=1}^{n} f(t_j)\Delta\alpha_j - \int_a^b f d\alpha \right| < \varepsilon$$

for all $t_j \in [x_{j-1}, x_j]$ *with* $j = 1, 2, \cdots, n$.

Proof. (a) follows easily since $L(P,f,\alpha)$ increases with P and $U(P,f,\alpha)$ decreases with P. (b) follows since both $f(s_j), f(t_j)$ are between $m_j(f)$ and $M_j(f)$ and hence $|f(s_j) - f(t_j)| \le M_j(f) - m_j(f)$. The obvious inequalities

$$L(P,f,\alpha) \le \sum_{j=1}^{n} f(t_j)\Delta\alpha_j \le U(P,f,\alpha), \quad L(P,f,\alpha) \le \int_a^b f d\alpha \le U(P,f,\alpha)$$

prove (c). □

Theorem 6.5. *If* f *is continuous on* $[a,b]$, *then* $f \in \mathcal{R}(\alpha)[a,b]$.

Proof. Let $\varepsilon > 0$ be given, and choose $\eta > 0$ so that $(\alpha(b) - \alpha(a))\eta < \varepsilon$. Since f is *uniformly continuous* on $[a,b]$, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \eta \quad \text{for all } x, y \in [a,b] \text{ with } |x-y| < \delta.$$

Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[a,b]$ with norm $\|P\| < \delta$. Since f is continuous on each subinterval $[x_{j-1}, x_j]$, by the **Extreme Value Theorem**, there exist $c_j, d_j \in [x_{j-1}, x_j]$ such that

$$M_j(f) = f(c_j), \quad m_j(f) = f(d_j).$$

Since $|c_j - d_j| \le \Delta x_j \le \|P\| < \delta$, we have

$$M_j(f) - m_j(f) = f(c_j) - f(d_j) < \eta \quad (1 \le j \le n).$$

Therefore,

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{j=1}^{n} (M_j(f) - m_j(f))\Delta\alpha_j \le \eta \sum_{j=1}^{n} \Delta\alpha_j = \eta(\alpha(b) - \alpha(a)) < \varepsilon.$$

So, by the Criterion for Integrability, $f \in \mathcal{R}(\alpha)[a,b]$. □

Theorem 6.6. *If* α *is continuous on* $[a,b]$, *then every monotonic function on* $[a,b]$ *belongs to* $\mathcal{R}(\alpha)[a,b]$.

Proof. Without loss of generality, we assume f is a monotonically increasing function on $[a, b]$. Let $\varepsilon > 0$ be given, and choose $\eta > 0$ so that $(f(b) - f(a))\eta < \varepsilon$. Since α is *uniformly continuous* on $[a, b]$, there exists a $\delta > 0$ such that

$$|\alpha(x) - \alpha(y)| < \eta \quad \text{for all } x, y \in [a, b] \text{ with } |x - y| < \delta.$$

Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[a, b]$ with norm $\|P\| < \delta$. Since f is monotonically increasing on each subinterval $[x_{j-1}, x_j]$, we have

$$m_j(f) = f(x_{j-1}), \quad M_j(f) = f(x_j).$$

Since $|x_j - x_{j-1}| = \Delta x_j \leq \|P\| < \delta$, we have

$$\Delta\alpha_j = \alpha(x_j) - \alpha(x_{j-1}) < \eta \quad (1 \leq j \leq n).$$

Therefore,

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))\Delta\alpha_j$$

$$\leq \eta \sum_{j=1}^{n} (f(x_j) - f(x_{j-1})) = \eta(f(b) - f(a)) < \varepsilon.$$

So, by the Criterion for Integrability, $f \in \mathcal{R}(\alpha)[a, b]$. □

Theorem 6.7. *Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathcal{R}(\alpha)[a, b]$.*

Proof. Let $\varepsilon > 0$ be given. Put $M = \sup_{x\in[a,b]} |f(x)|$ and let E be the set of points in $[a, b]$ at which f is discontinuous. Since E is finite and α is continuous at every point of E, we can cover E by finitely many disjoint intervals $[u_j, v_j]$ in $[a, b]$ such that $\sum(\alpha(v_j) - \alpha(u_j)) < \varepsilon$. Furthermore, we can choose these intervals in such a way that every point of $E \cap (a, b)$ lies in the interior of some $[u_j, v_j]$. If $a \in E$, we assume $[a, v_0]$ is one of these intervals; if $b \in E$, assume $[u_0, b]$ is one of these intervals.

Remove all open intervals (u_j, v_j), and possibly $[a, v_0)$ or $(u_0, b]$ if a or b is in E. The remaining set K is then compact, and f is continuous on K. Hence f is uniformly continuous on K, and there exists $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon$ if $s, t \in K$ and $|s - t| < \delta$.

Define a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ as follows. All u_j, v_j belong to P. No points in (u_j, v_j) belong to P, and no points in $[a, v_0)$ or $(u_0, b]$ belong to P. If x_{i-1} is not one of u_j, then $\Delta x_i < \delta$.

Note that $M_i(f) - m_i(f) \leq 2M$ for every i, and that $M_i(f) - m_i(f) \leq \varepsilon$ unless x_{i-1} is one of u_j. Hence

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i(f) - m_i(f))\Delta\alpha_i$$

$$= \sum_{x_{i-1}=u_j} (M_i(f) - m_i(f))\Delta\alpha_i + \sum_{x_{i-1}\neq u_j} (M_i(f) - m_i(f))\Delta\alpha_i \leq 2M\varepsilon + \varepsilon(\alpha(b) - \alpha(a)).$$

Since $\varepsilon > 0$ is arbitrary, this proves $f \in \mathcal{R}(\alpha)[a, b]$. □

Theorem 6.8. *Suppose $f \in \mathcal{R}(\alpha)[a, b]$, $m \leq f \leq M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathcal{R}(\alpha)[a, b]$.*

Proof. Let $\varepsilon > 0$ be given. Since ϕ is uniformly continuous on $[m, M]$, there exists $\delta \in (0, \varepsilon)$ such that $|\phi(s) - \phi(t)| < \varepsilon$ if $|s - t| \le \delta$ and $t, s \in [m, M]$.

Since $f \in \mathcal{R}(\alpha)[a, b]$, there is a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2. \tag{6.3}$$

Let $M_j(f), m_j(f)$ be defined for f as above; similarly, let $M_j(h), m_j(h)$ be defined for function h. Divide the numbers $i = 1, 2, \cdots, n$ into two classes: $i \in A$ if $M_i(f) - m_i(f) < \delta$; $i \in B$ if $M_i(f) - m_i(f) \ge \delta$.

For $i \in A$, we have $|f(x) - f(y)| \le M_i(f) - m_i(f) < \delta$ for all $x, y \in [x_{i-1}, x_i]$; hence, our choice of δ shows that

$$M_i(h) - m_i(h) = \sup_{x \in [x_{i-1}, x_i]} \phi(f(x)) - \inf_{y \in [x_{i-1}, x_i]} \phi(f(y)) = \sup_{x, y \in [x_{i-1}, x_i]} (\phi(f(x)) - \phi(f(y))) \le \varepsilon.$$

For $i \in B$, we have $M_i(h) - m_i(h) \le 2K$, where $K = \sup_{t \in [m, M]} |\phi(t)| < +\infty$. By (6.3),

$$\delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B} (M_i(f) - m_i(f)) \Delta\alpha_i \le U(P, f, \alpha) - L(P, f, \alpha) < \delta^2,$$

so that $\sum_{i \in B} \Delta\alpha_i < \delta$. Thus it follows that

$$U(P, h, \alpha) - L(P, h, \alpha) = \sum_{i \in A} (M_i(h) - m_i(h)) \Delta\alpha_i + \sum_{i \in B} (M_i(h) - m_i(h)) \Delta\alpha_i$$

$$\le \varepsilon \sum_{i \in A} \Delta\alpha_i + 2K \sum_{i \in B} \Delta\alpha_i \le \varepsilon[\alpha(b) - \alpha(a)] + 2K\delta < \varepsilon[\alpha(b) - \alpha(a) + 2K].$$

Since $\varepsilon > 0$ is arbitrary, by the **Criterion of integrability**, this proves $h \in \mathcal{R}(\alpha)[a, b]$. $\square$

6.2. Further Properties of the Integral

Theorem 6.9. *The Riemann-Stieltjes integral has the following properties.*

(a) **(Linear property)** *If $f_1, f_2 \in \mathcal{R}(\alpha)[a, b]$, then $c_1 f_1 + c_2 f_2 \in \mathcal{R}(\alpha)[a, b]$ for all real numbers c_1, c_2, and*

$$\int_a^b (c_1 f_1 + c_2 f_2) d\alpha = c_1 \int_a^b f_1 d\alpha + c_2 \int_a^b f_2 d\alpha.$$

(b) **(Order property)** *If $f_1, f_2 \in \mathcal{R}(\alpha)[a, b]$ and $f_1(x) \le f_2(x)$ on $[a, b]$, then*

$$\int_a^b f_1 d\alpha \le \int_a^b f_2 d\alpha.$$

(c) **(Additivity)** *If $f \in \mathcal{R}(\alpha)[a, b]$ and $a < c < b$, then $f \in \mathcal{R}(\alpha)[a, c]$ and $f \in \mathcal{R}(\alpha)[c, b]$; moreover,*

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha. \tag{6.4}$$

Conversely, if $a < c < b$ and if $f \in \mathcal{R}(\alpha)[a, c]$ and $f \in \mathcal{R}(\alpha)[c, b]$, then $f \in \mathcal{R}(\alpha)[a, b]$, and (6.4) holds.

(d) **(Positive combination)** *If $f \in \mathcal{R}(\alpha_1)[a, b]$ and $f \in \mathcal{R}(\alpha_2)[a, b]$, and k_1, k_2 are* nonnegative *constants, then $f \in \mathcal{R}(k_1\alpha_1 + k_2\alpha_2)[a, b]$, and*

$$\int_a^b f d(k_1\alpha_1 + k_2\alpha_2) = k_1 \int_a^b f d\alpha_1 + k_2 \int_a^b f d\alpha_2.$$

(e) **(Absolute integrability)** *If $f \in \mathcal{R}(\alpha)[a,b]$, then $|f| \in \mathcal{R}(\alpha)[a,b]$, and*

$$\left| \int_a^b f d\alpha \right| \le \int_a^b |f| d\alpha.$$

Proof. Let's prove (c) and (e) only.

(c) Assume $f \in \mathcal{R}(\alpha)[a,b]$. Let $[c,d]$ be a subinterval of $[a,b]$. Let $\varepsilon > 0$. Choose a partition P of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon.$$

Let $P' = P \cup \{c,d\}$ and $P_1 = P' \cap [c,d]$. Then P' is a refinement of P on $[a,b]$ and P_1 is a partition of $[c,d]$, which is part of partition P' of $[a,b]$. Therefore, we have

$$U_c^d(P_1,f,\alpha) - L_c^d(P_1,f,\alpha) \le U(P',f,\alpha) - L(P',f,\alpha) \le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon,$$

where $U_c^d(P_1,f,\alpha)$, $L_c^d(P_1,f,\alpha)$ are for f defined on $[c,d]$, with P_1 being a partition on $[c,d]$. Hence, by the **Criterion for integrability**, $f \in \mathcal{R}(\alpha)[c,d]$.

Now assume $a < c < b$ and $f \in \mathcal{R}(\alpha)[a,c]$ and $f \in \mathcal{R}(\alpha)[c,b]$. Let $\varepsilon > 0$. Choose partitions P_1 of $[a,c]$ and P_2 of $[c,b]$ such that

$$U_a^c(P_1,f,\alpha) - L_a^c(P_1,f,\alpha) < \varepsilon/2, \quad U_c^b(P_2,f,\alpha) - L_c^b(P_2,f,\alpha) < \varepsilon/2.$$

Let $P = P_1 \cup P_2$. Then P is a partition of $[a,b]$, and we have

$$U(P,f,\alpha) - L(P,f,\alpha) = [U_a^c(P_1,f,\alpha) + U_c^b(P_2,f,\alpha)] - [L_a^c(P_1,f,\alpha) + L_c^b(P_2,f,\alpha)] < \varepsilon.$$

Hence, $f \in \mathcal{R}(\alpha)[a,b]$.

To verify the additivity property (6.4), assume P is a partition of $[a,b]$. Let $P_0 = P \cup \{c\}$, $P_1 = P_0 \cap [a,c]$, and $P_2 = P_0 \cap [c,b]$. Then $P_0 = P_1 \cup P_2$ and

$$U(P,f,\alpha) \ge U(P_0,f,\alpha) = U_a^c(P_1,f,\alpha) + U_c^b(P_2,f,\alpha) \ge \int_a^c f d\alpha + \int_c^b f d\alpha,$$

$$L(P,f,\alpha) \le L(P_0,f,\alpha) = L_a^c(P_1,f,\alpha) + L_c^b(P_2,f,\alpha) \le \int_a^c f d\alpha + \int_c^b f d\alpha.$$

Hence

$$\overline{\int_a^b} f d\alpha \ge \int_a^c f d\alpha + \int_c^b f d\alpha, \quad \underline{\int_a^b} f d\alpha \le \int_a^c f d\alpha + \int_c^b f d\alpha;$$

but, $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$, and this proves

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha.$$

(e) Assume $f \in \mathcal{R}(\alpha)[a,b]$. Then, with $\phi(t) = |t|$ in Theorem 6.8, we have $|f| \in \mathcal{R}(\alpha)[a,b]$. Since

$$-|f(x)| \le f(x) \le |f(x)| \quad (x \in [a,b]),$$

by part (a) and (b), we have

$$-\int_a^b |f| d\alpha \le \int_a^b f d\alpha \le \int_a^b |f| d\alpha,$$

which proves $|\int_a^b f d\alpha| \le \int_a^b |f| d\alpha$. $\square$

Remark 6.4. The converse of (e) in the theorem is false. Indeed, consider the function

$$f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ -1 & x \notin \mathbf{Q}. \end{cases}$$

Then $|f(x)| = 1$ is constant and hence $f \in \mathcal{R}[a,b]$; however, it is easily seen that

$$\overline{\int_a^b} f dx = b - a > 0, \quad \underline{\int_a^b} f dx = a - b < 0,$$

and hence $f \notin \mathcal{R}[a,b]$.

Theorem 6.10. *If $f, g \in \mathcal{R}(\alpha)[a,b]$, then $fg \in \mathcal{R}(\alpha)[a,b]$.*

Proof. Using $\phi(t) = t^2$ in Theorem 6.8, we have $f^2, g^2, (f+g)^2 \in \mathcal{R}(\alpha)[a,b]$, and hence

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2} \in \mathcal{R}(\alpha)[a,b].$$

□

Theorem 6.11. *Assume α is monotonically increasing and differentiable on $[a,b]$ and $\alpha' \in \mathcal{R}[a,b]$. Let f be bounded on $[a,b]$. Then $f \in \mathcal{R}(\alpha)[a,b]$ if and only if $f\alpha' \in \mathcal{R}[a,b]$. In this case,*

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx.$$

Proof. Let $\varepsilon > 0$. Since $\alpha' \in \mathcal{R}[a,b]$, there is a partition $P = \{x_0, \cdots, x_n\}$ of $[a,b]$ such that

$$U(P, \alpha') - L(P, \alpha') < \varepsilon. \tag{6.5}$$

By the MVT, for each $j = 1, 2, \ldots, n$, there exists $t_j \in (x_{j-1}, x_j)$ such that $\Delta\alpha_j = \alpha'(t_j)\Delta x_j$.

Let $s_j \in [x_{j-1}, x_j]$ be arbitrary. Then, by (6.5) and Theorem 6.4, we have

$$\sum_{j=1}^n |\alpha'(t_j) - \alpha'(s_j)|\Delta x_j < \varepsilon.$$

Put $M = \sup_{[a,b]} |f| < +\infty$. Since $\Delta\alpha_j = \alpha'(t_j)\Delta x_j$, it follows that

$$\left| \sum_{j=1}^n f(s_j)\Delta\alpha_j - \sum_{j=1}^n f(s_j)\alpha'(s_j)\Delta x_j \right| \le M\varepsilon.$$

In particular,

$$\sum_{j=1}^n f(s_j)\Delta\alpha_j \le U(P, f\alpha') + M\varepsilon; \quad \sum_{j=1}^n f(s_j)\alpha'(s_j)\Delta x_j \le U(P, f, \alpha) + M\varepsilon.$$

Since $s_j \in [x_{j-1}, x_j]$ is arbitrary, these two inequalities imply that

$$U(P, f, \alpha) \le U(P, f\alpha') + M\varepsilon; \quad U(P, f\alpha') \le U(P, f, \alpha) + M\varepsilon,$$

and hence that

$$\overline{\int_a^b} f d\alpha \le U(P, f\alpha') + M\varepsilon, \quad \overline{\int_a^b} f\alpha' dx \le U(P, f, \alpha) + M\varepsilon. \tag{6.6}$$

Now note that (6.5) remains true if P is replaced by the refinement $P \cup Q$, where Q is any given partition of $[a, b]$. Hence (6.6) also remains true with P replaced by $P \cup Q$, which yields that

$$\overline{\int_a^b} f d\alpha \leq U(Q, f\alpha') + M\varepsilon, \quad \overline{\int_a^b} f\alpha' dx \leq U(Q, f, \alpha) + M\varepsilon \tag{6.7}$$

for all partitions Q of $[a, b]$. Taking the infima over partitions Q, we have

$$\overline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f\alpha' dx + M\varepsilon, \quad \overline{\int_a^b} f\alpha' dx \leq \overline{\int_a^b} f d\alpha + M\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies

$$\overline{\int_a^b} f d\alpha = \overline{\int_a^b} f\alpha' dx,$$

which is valid for *all bounded* functions f.

The equality $\underline{\int_a^b} f d\alpha = \underline{\int_a^b} f\alpha' dx$ follows in exactly the same manner. Hence the theorem is proved. □

Theorem 6.12 (**Change of Variables**). *Suppose ϕ is a strictly increasing continuous from an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathcal{R}(\alpha)[a, b]$. Define β and g on $[A, B]$ by*

$$\beta(y) = \alpha(\phi(y)), \quad g(y) = f(\phi(y)).$$

Then $g \in \mathcal{R}(\beta)[A, B]$, and

$$\int_A^B g d\beta = \int_a^b f d\alpha. \tag{6.8}$$

Proof. To each partition $P = \{x_0, \cdots, x_n\}$ of $[a, b]$ corresponds a unique partition $Q = \{y_0, \cdots, y_n\}$ of $[A, B]$ such that $x_j = \phi(y_j)$, and vice versa. Then

$$\Delta\alpha_j = \alpha(x_j) - \alpha(x_{j-1}) = \alpha(\phi(y_j)) - \alpha(\phi(y_{j-1})) = \beta(y_j) - \beta(y_{j-1}) = \Delta\beta_j$$

for each $j = 1, 2, \ldots, n$. Since the values taken by f on $[x_{j-1}, x_j]$ are exactly the same as those taken by g on $[y_{j-1}, y_j]$, we see that

$$U(Q, g, \beta) = U(P, f, \alpha), \quad L(Q, g, \beta) = L(P, f, \alpha).$$

If $f \in \mathcal{R}(\alpha)[a, b]$, then, for every $\varepsilon > 0$, it follows that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ for some partition P of $[a, b]$; hence, with the corresponding partition Q of $[A, B]$, we have $U(Q, g, \beta) - L(Q, g, \beta) < \varepsilon$. Hence $g \in \mathcal{R}(\beta)[A, B]$, and (6.8) holds. □

Combing the previous two theorems, we obtain the following change of variable theorem for Riemann integrals.

Theorem 6.13. *If $f \in \mathcal{R}[a, b]$ and if ϕ: $[A, B] \to [a, b]$ is strictly increasing and differentiable with $\phi' \in \mathcal{R}[A, B]$, then*

$$\int_a^b f(x)\, dx = \int_A^B f(\phi(y))\phi'(y)\, dy,$$

where $a = \phi(A)$, $b = \phi(B)$.

6.3. Integration and Differentiation

Theorem 6.14. *Let $f \in \mathcal{R}[a,b]$. Define*

$$F(x) = \int_a^x f(t)dt \quad (a \le x \le b).$$

Then F is continuous on $[a,b]$; furthermore, if f is continuous at a point $c \in [a,b]$, then F is differentiable at c, with $F'(c) = f(c)$.

Proof. Suppose $M = \sup_{[a,b]} |f| < +\infty$. If $a \le x < y \le b$, then

$$|F(y) - F(x)| = \left| \int_x^y f(t)\,dt \right| \le M(y-x).$$

Hence F is uniformly continuous on $[a,b]$.

Now assume f is continuous at a point $c \in [a,b]$. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$|f(t) - f(c)| < \varepsilon \quad \forall\, t \in [a,b],\ |t-c| < \delta.$$

Then, for any $x \in [a,b]$ with $c < x < c + \delta$,

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| = \frac{1}{x-c} \left| \int_c^x (f(t) - f(c))\,dt \right| \le \frac{1}{x-c} \int_c^x |f(t) - f(c)|\,dt < \varepsilon,$$

and similarly, for for any $x \in [a,b]$ with $c - \delta < x < c$,

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| = \frac{1}{c-x} \left| \int_x^c (f(t) - f(c))\,dt \right| \le \frac{1}{c-x} \int_x^c |f(t) - f(c)|\,dt < \varepsilon.$$

Hence, whenever $x \in [a,b]$ and $0 < |x - c| < \delta$, it follows that

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| < \varepsilon;$$

so $F'(c) = f(c)$. □

Theorem 6.15 (Fundamental Theorem of Calculus). *If $f \in \mathcal{R}[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Let $\varepsilon > 0$ be given. Choose a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$. The MVT furnishes points $t_j \in (x_{j-1}, x_j)$ such that

$$F(x_j) - F(x_{j-1}) = f(t_j)\Delta x_j \quad (j = 1, 2, \ldots, n).$$

Thus

$$\sum_{j=1}^n f(t_j)\Delta x_j = \sum_{j=1}^n (F(x_j) - F(x_{j-1})) = F(b) - F(a).$$

But

$$L(P,f) \le \sum_{j=1}^n f(t_j)\Delta x_j \le U(P,f), \quad L(P,f) \le \int_a^b f(t)\,dt \le U(P,f);$$

hence

$$\left| F(b) - F(a) - \int_a^b f(t)\,dt \right| \le U(P,f) - L(P,f) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof. □

Theorem 6.16 (Integration by Parts). *Suppose F, G are differentiable on $[a, b]$, $F' = f \in \mathcal{R}[a, b]$, and $G' = g \in \mathcal{R}[a, b]$. Then*

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx.$$

Proof. Clearly $Fg, fG \in \mathcal{R}[a, b]$. Let $H(x) = F(x)G(x)$ on $[a, b]$. Then $H'(x) = f(x)G(x)+ g(x)F(x)$ and hence $H' \in \mathcal{R}[a, b]$. So, by Theorem 6.15,

$$H(b) - H(a) = \int_a^b H'(x)\,dx = \int_a^b f(x)G(x)\,dx + \int_a^b g(x)F(x)\,dx,$$

proving the theorem. □

6.4. Integration of Vector-Valued Functions

Definition 6.5. Let $\mathbf{f}\colon [a, b] \to \mathbf{R}^k$ and $\alpha\colon [a, b] \to \mathbf{R}$ be monotonically increasing on $[a, b]$. Let $\mathbf{f}(x) = (f_1(x), \cdots, f_k(x))$, which each coordinate function f_j is real valued. We say $\mathbf{f} \in \mathcal{R}(\alpha)[a, b]$ if each of its coordinate functions $f_j \in \mathcal{R}(\alpha)[a, b]$. In this case, we define

$$\int_a^b \mathbf{f}\,d\alpha = \left(\int_a^b f_1 d\alpha, \cdots, \int_a^b f_k d\alpha\right).$$

Many of the results on real-valued functions also hold for these vector-valued functions. To illustrate, we state the analogue of the fundamental theorem of calculus.

Theorem 6.17. *If $\mathbf{f}$ and $\mathbf{F}$ map $[a, b]$ into $\mathbf{R}^k$, $\mathbf{f} \in \mathcal{R}[a, b]$, and $\mathbf{F}' = \mathbf{f}$ on $[a, b]$, then*

$$\int_a^b \mathbf{f}(x)dx = \mathbf{F}(b) - \mathbf{F}(a).$$

Theorem 6.18. *If $\mathbf{f} \in \mathcal{R}(\alpha)[a, b]$, then $\|\mathbf{f}\| \in \mathcal{R}(\alpha)[a, b]$, and*

$$\left\|\int_a^b \mathbf{f}\,d\alpha\right\| \le \int_a^b \|\mathbf{f}\|\,d\alpha.$$

Proof. If $\mathbf{f} = (f_1, \cdots, f_k)$ then $\|\mathbf{f}\| = (f_1^2 + \cdots + f_k^2)^{1/2}$. Hence if each $f_j \in \mathcal{R}[a, b]$, then $\|\mathbf{f}\| \in \mathcal{R}[a, b]$. To show the inequality on the integrals, we assume $\mathbf{y} = \int_a^b \mathbf{f}\,d\alpha \ne \mathbf{0}$; otherwise there is nothing to prove. By linearity and order properties and the Cauchy-Schwarz inequality,

$$\|\mathbf{y}\|^2 = \mathbf{y}\cdot\int_a^b \mathbf{f}\,d\alpha = \int_a^b \mathbf{y}\cdot\mathbf{f}\,d\alpha \le \int_a^b \|\mathbf{y}\|\|\mathbf{f}\|\,d\alpha = \|\mathbf{y}\|\int_a^b \|\mathbf{f}\|\,d\alpha.$$

Cancelling $\|\mathbf{y}\| > 0$ proves the inequality. □

6.5. Rectifiable Curves in $\mathbf{R}^k$

Definition 6.6. A continuous function γ from an interval $[a, b]$ into $\mathbf{R}^k$ is called a **curve** in $\mathbf{R}^k$. To emphasize the parameter interval $[a, b]$, we may also say that γ is a curve on $[a, b]$.

If γ is one-to-one, then γ is called an **arc**.

If $\gamma(a) = \gamma(b)$ then γ is said to be a **closed curve**.

Note that a curve in $\mathbf{R}^k$ is a function, not the range of γ, which is a point set in $\mathbf{R}^k$; different curves may have the same range.

Let $\gamma\colon [a,b] \to \mathbf{R}^k$ be a curve. We associate to each partition $P = \{x_0, \cdots, x_n\}$ of $[a,b]$ the number

$$\Lambda(P,\gamma) = \sum_{j=1}^{n} \|\gamma(x_j) - \gamma(x_{j-1})\|.$$

(This could be defined for curves in any metric space X, with $\|\gamma(x_j) - \gamma(x_{j-1})\|$ replaced by $d(\gamma(x_j), \gamma(x_{j-1}))$.) Define the **length** of γ to be the number (including $+\infty$)

$$\Lambda(\gamma) = \sup \Lambda(P,\gamma),$$

where the supremum is taken over all partitions P of $[a,b]$. We say that γ is **rectifiable** if $\Lambda(\gamma) < +\infty$.

Theorem 6.19. *If γ' is continuous on $[a,b]$, then γ is rectifiable, and*

$$\Lambda(\gamma) = \int_a^b \|\gamma'(t)\|\, dt.$$

Proof. If $a \le x_{j-1} < x_j \le b$, then

$$\|\gamma(x_{j-1} - \gamma(x_j)\| = \left\| \int_{x_{j-1}}^{x_j} \gamma'(t)\, dt \right\| \le \int_{x_{j-1}}^{x_j} \|\gamma'(t)\|\, dt.$$

Hence

$$\Lambda(P,\gamma) \le \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} \|\gamma'(t)\|\, dt = \int_a^b \|\gamma'(t)\|\, dt$$

for all partitions P of $[a,b]$. Consequently, $\Lambda(\gamma) \le \int_a^b \|\gamma'(t)\|\, dt$.

To show the opposite inequality, let $\varepsilon > 0$ be given. Since γ' is uniformly continuous on $[a,b]$, there exists $\delta > 0$ such that

$$\|\gamma'(t) - \gamma'(s)\| < \varepsilon \quad \text{if } |s-t| < \delta,\ s,t \in [a,b].$$

Let $P = \{x_0, \cdots, x_n\}$ be a partition of $[a,b]$ with $\|P\| < \delta$. If $t \in [x_{j-1}, x_j]$ then $\|\gamma'(t)\| \le \|\gamma'(x_j)\| + \varepsilon$. Hence, by Theorem 6.17,

$$\begin{aligned}
\int_{x_{j-1}}^{x_j} \|\gamma'(t)\|\, dt \le \|\gamma'(x_j)\|\Delta x_j + \varepsilon\Delta x_j &= \left\| \int_{x_{j-1}}^{x_j} [\gamma'(t) + \gamma'(x_j) - \gamma'(t)]\, dt \right\| + \varepsilon\Delta x_j \\
&\le \left\| \int_{x_{j-1}}^{x_j} \gamma'(t)\, dt \right\| + \left\| \int_{x_{j-1}}^{x_j} [\gamma'(x_j) - \gamma'(t)]\, dt \right\| + \varepsilon\Delta x_j \\
&\le \|\gamma(x_j) - \gamma(x_{j-1})\| + \int_{x_{j-1}}^{x_j} \|\gamma'(x_j) - \gamma'(t)\|\, dt + \varepsilon\Delta x_j \\
&\le \|\gamma(x_j) - \gamma(x_{j-1})\| + 2\varepsilon\Delta x_j.
\end{aligned}$$

Adding these inequalities, we obtain

$$\int_a^b \|\gamma'(t)\|\, dt \le \Lambda(P,\gamma) + 2\varepsilon(b-a).$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\int_a^b \|\gamma'(t)\|\, dt \le \Lambda(P,\gamma)$. This completes the proof. □

Suggested Homework Problems

Pages 138–142

Problems: 1–5, 7–9, 15, 17, 19

6.6. Improper Riemann Integrals*

Definition 6.7. Let $(a,b) \subseteq \mathbf{R}$, where $-\infty \le a < b \le +\infty$, and $f\colon (a,b) \to \mathbf{R}$.

We say that f is **locally integrable on** (a,b) if $f \in \mathcal{R}[c,d]$ for each finite closed subinterval $[c,d]$ of (a,b).

We say that f is **improperly (Riemann) integrable** on (a,b) if f is locally integrable on (a,b) and the limit

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\left(\lim_{d\to b^-}\int_c^d f(x)\,dx\right) \tag{6.9}$$

exists and is finite. In this case, this limit is called the **improper (Riemann) integral** of f on (a,b).

Sometimes we also use the notation

$$\int_a^b f(x)\,dx = \int_{a^+}^{b^-} f(x)\,dx$$

to distinguish the improper integrals from the Riemann integrals defined earlier.

Lemma 6.20. *The order of limits in* (6.9) *does not matter. In particular, if the limit in* (6.9) *exists and is finite, then the limit*

$$\lim_{d\to b^-}\left(\lim_{c\to a^+}\int_c^d f(x)\,dx\right)$$

exists and equals the limit in (6.9).

Proof. Let $x_0 \in (a,b)$. Then

$$\lim_{c\to a^+}\left(\lim_{d\to b^-}\int_c^d f(x)\,dx\right) = \lim_{c\to a^+}\left(\int_c^{x_0} f(x)\,dx + \lim_{d\to b^-}\int_{x_0}^d f(x)\,dx\right)$$

$$= \lim_{c\to a^+}\int_c^{x_0} f(x)\,dx + \lim_{d\to b^-}\int_{x_0}^d f(x)\,dx. \tag{6.10}$$

Since, for each c, $\lim\limits_{d\to b^-}\int_c^d f(x)dx$ exists, we have

$$\begin{aligned}
\lim_{x_0\to b^-}\left(\lim_{d\to b^-}\int_{x_0}^d f(x)\,dx\right) &= \lim_{x_0\to b^-}\left[\lim_{d\to b^-}\left(\int_c^d f(x)\,dx - \int_c^{x_0} f(x)\,dx\right)\right]\\
&= \lim_{x_0\to b^-}\left[\lim_{d\to b^-}\int_c^d f(x)\,dx - \int_c^{x_0} f(x)\,dx\right]\\
&= \lim_{d\to b^-}\int_c^d f(x)\,dx - \lim_{x_0\to b^-}\int_c^{x_0} f(x)\,dx = 0.
\end{aligned}$$

Therefore, in (6.10) letting $x_0 \to b^-$, we obtain that

$$\lim_{x_0\to b^-}\left(\lim_{c\to a^+}\int_c^{x_0} f(x)\,dx\right) = \lim_{c\to a^+}\left(\lim_{d\to b^-}\int_c^d f(x)\,dx\right).$$

□

Remark 6.8. (i) If f is integrable on $[c,b]$ for all $c \in (a,b)$, then the improper Riemann integral of f on (a,b) is also given by

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx := \int_{a^+}^b f(x)\,dx.$$

If this limit exists and is finite, we also say that f is **improperly integrable on** $(a,b]$. The similar situation applies at the endpoint b, in which case we say that f is **improperly integrable on** $[a,b)$.

(ii) It is easily seen that f is improperly integrable on (a,b) if and only if f is improperly integrable on $(a,c]$ and on $[c,b)$ for all $c \in (a,b)$. In this case, we have that

$$\int_{a^+}^{b^-} f(x)\,dx = \int_{a^+}^{c} f(x)\,dx + \int_{c}^{b^-} f(x)\,dx.$$

Theorem 6.21. *The function $f(x) = 1/x^p$ is improperly integrable on $(0,1]$ if and only if $p < 1$, and is improperly integrable on $[1,+\infty)$ if and only if $p > 1$.*

Proof. Exercise! □

Theorem 6.22 (Linear Property). *If f, g are improperly integrable on (a,b) and $k, l \in \mathbf{R}$, then $kf + lg$ is improperly integrable on (a,b), and*

$$\int_a^b (kf(x) + lg(x))\,dx = k\int_a^b f(x)\,dx + l\int_a^b g(x)\,dx.$$

Proof. Use the **Linear Property** of integrals on each subinterval $[c,d]$ of (a,b). □

Theorem 6.23 (Comparison Theorem for Improper Integrals). *Suppose that f, g are locally integrable on (a,b) and $0 \le f(x) \le g(x)$ for all $x \in (a,b)$. If g is improperly integrable on (a,b), then f is also improperly integrable on (a,b) and*

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Proof. Fix $c \in (a,b)$. Let $F(d) = \int_c^d f(x)\,dx$ and $G(d) = \int_c^d g(x)dx$ for $d \in [c,b)$. Then by the **Order Property**, $F(d) \le G(d)$. Note that F and G are increasing on $[c,b)$ and $G(b^-)$ exists. Hence F is bounded above by $G(b^-)$ and so $F(d^-)$ exists and is finite. This shows that f is improperly integrable on $[c,b)$. By the similar argument, we also show that f is improperly integrable on $(a,c]$; thus f is improperly integrable on (a,b). The order property

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

follows easily from the order property of the Riemann integrals of f and g on each subinterval $[c,d]$ of (a,b). □

Example 6.1. Show that $f(x) = (\sin x)/x^{3/2}$ is improperly integrable on $(0,1]$.

Proof. Since $0 \le \sin x \le x$ for all $x \in [0,1]$ (use elementary calculus to prove it!), it follows that

$$0 \le f(x) \le x \cdot x^{-3/2} = x^{-1/2} \quad \forall\, x \in (0,1].$$

Since $x^{-1/2}$ is improperly integrable on $(0,1]$, by the theorem above, f is improperly integrable on $(0,1]$. □

Example 6.2. Show that $f(x) = (\ln x)/x^{5/2}$ is improperly integrable on $[1,+\infty)$.

Proof. Since $0 \le \ln x \le x$ for all $x \ge 1$ (use elementary calculus to prove it!), it follows that

$$0 \le f(x) \le x \cdot x^{-5/2} = x^{-3/2} \quad \forall\, x \ge 1.$$

Since $x^{-3/2}$ is improperly integrable on $[1, +\infty)$, by the theorem above, f is improperly integrable on $[1, +\infty)$. $\square$

Lemma 6.24. *If f is bounded and locally integrable on (a, b) and $|g|$ is improperly integrable on (a, b), then $|fg|$ is improperly integrable on (a, b).*

Proof. Use $0 \le |fg| \le M|g|$ and the **Comparison Theorem** above. $\square$

Definition 6.9. Let $f\colon (a, b) \to \mathbf{R}$. We say that f is **absolutely integrable** on (a, b) if f is locally integrable on (a, b) and $|f|$ is improperly integrable on (a, b).

We say that f is **conditionally integrable** on (a, b) if f is improperly integrable on (a, b) but $|f|$ is not improperly integrable on (a, b).

Theorem 6.25. *If f is absolutely integrable on (a, b), then f is improperly integrable on (a, b) and*

$$\left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx.$$

Proof. Since $0 \le |f| + f \le 2|f|$, by the **Comparison Theorem**, $f + |f|$ is improperly integrable on (a, b). Hence, by the Linear Property, $f = (|f| + f) - |f|$ is also improperly integrable on (a, b). Moreover, for all $c < d$ in (a, b),

$$\left| \int_c^d f(x)\, dx \right| \le \int_c^d |f(x)|\, dx.$$

We then complete the proof by taking the limit as $c \to a^+$ and $d \to b^-$. $\square$

The converse of Theorem 6.25 is false.

Example 6.3. Prove that $f(x) = \dfrac{\sin x}{x}$ is conditionally integrable on $[1, +\infty)$.

Proof. Integrating by parts, we have for all $d > 1$,

$$\int_1^d \frac{\sin x}{x}\, dx = -\frac{\cos x}{x}\Big|_1^d - \int_1^d \frac{\cos x}{x^2}\, dx.$$

Since $1/x^2$ is absolutely integrable on $[1, +\infty)$, we have $(\cos x)/x^2$ is absolutely integrable on $[1, +\infty)$; hence $(\cos x)/x^2$ is improperly integrable on $[1, +\infty)$. Taking the limit as $d \to +\infty$ above, we have

$$\int_1^{+\infty} \frac{\sin x}{x}\, dx = \cos(1) - \int_1^{+\infty} \frac{\cos x}{x^2}\, dx$$

exists and is finite. This proves that $(\sin x)/x$ is improperly integrable on $[1, +\infty)$.

We now show that $|\sin x|/x$ is not improperly integrable on $[1, +\infty)$, which proves that $(\sin x)/x$ is conditionally integrable on $[1, +\infty)$. Note that if $n \in \mathbf{N}$ and $n \ge 2$ then

$$\int_1^{n\pi} \frac{|\sin x|}{x}\, dx \ge \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\, dx \ge \sum_{k=2}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x|\, dx = \frac{2}{\pi} \sum_{k=2}^{n} \frac{1}{k}.$$

Hence

$$\lim_{n\to\infty} \int_1^{n\pi} \frac{|\sin x|}{x}\, dx = +\infty.$$

So $|\sin x|/x$ is not improperly integrable on $[1, +\infty)$. $\square$