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by

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Abstract. – In this paper, we study the stability of maps in $W^{1,p}$ that are close to the conformal set $K_1 = \mathbb{R}^+ \cdot SO(n)$ in an averaged sense as described in Definition 1.1. We prove that $K_1$ is $W^{1,p}$-compact for all $p \geq n$ but is not $W^{1,p}$-stable for any $1 \leq p < n/2$ when $n \geq 3$. We also prove a coercivity estimate for the integral functional $\int_{\mathbb{R}^n} d_{K_1}^p(\nabla \phi(x))\,dx$ on $W^{1,p}(\mathbb{R}^n; \mathbb{R}^n)$ for certain values of $p$ lower than $n$ using some new estimates for weak solutions of $p$-harmonic equations.

Key words: Weak stability, conformal set.

Résumé. – Dans cet article, nous étudions la stabilité des applications dans $W^{1,p}$ qui sont proches de l’ensemble conforme $K_1 = \mathbb{R}^+ \cdot SO(n)$ dans un sens moyenné décrit dans la Définition 1.1. Nous prouvons que $K_1$ est $W^{1,p}$-compact pour $p \geq n$ mais n’est pas $W^{1,p}$-stable pour tout $1 \leq p < n/2$ si $n \geq 3$. Nous prouvons aussi une estimée de coercivité pour la fonctionnelle $\int_{\mathbb{R}^n} d_{K_1}^p(\nabla \phi(x))\,dx$ on $W^{1,p}(\mathbb{R}^n; \mathbb{R}^n)$ pour certaines valeurs de $p$ inférieures à $n$ en utilisant des estimées nouvelles pour des solutions faibles d’équations $p$-harmoniques.
1. INTRODUCTION

Let \( n \geq 2 \) and \( \mathcal{M}^{n \times n} \) denote the set of all real \( n \times n \) matrices. For each \( l \geq 1 \), we consider the following subset \( K_l \) of \( \mathcal{M}^{n \times n} \) in connection with the theory of \( l \)-quasiregular mappings in \( \mathbb{R}^n \) (see Reshetnyak [21] and Rickman [22]),

\[
K_l = \{ A \in \mathcal{M}^{n \times n} \mid \| A \|^n \leq l \det A \},
\]

(1.1)

where \( \| A \| \) is the norm of \( A \in \mathcal{M}^{n \times n} \) viewed as a linear operator on \( \mathbb{R}^n \), i.e.,

\[
\| A \| = \max_{|h|=1} |Ah| = \max_{|h|=1} \sqrt{h^T A^T A h}.
\]

(1.2)

When \( l = 1 \), set \( K_1 \) is the set of all conformal matrices, which will be called the conformal set in this paper. Note that \( K_1 = \mathbb{R}^+ \cdot SO(n) \). We also consider the set \( R(n) \) of all general orthogonal matrices in \( \mathbb{R}^n \), i.e.,

\[
R(n) = \{ A \in \mathcal{M}^{n \times n} \mid A^T A = \lambda I \text{ for some } \lambda \geq 0 \}.
\]

(1.3)

Let \( \Omega \) be a domain in \( \mathbb{R}^n \), which is assumed throughout this paper to be bounded and smooth. We recall that a map \( u \in W^{1,p}(\Omega; \mathbb{R}^n) \) is said to be (weakly, if \( p < n \)) \( l \)-quasiregular if \( \nabla u(x) \in K_l \) for a.e. \( x \in \Omega \), see [13], [14], [21] and [22]. The Liouville theorem asserts that every \( 1 \)-quasiregular in \( W^{1,1}(\Omega; \mathbb{R}^n) \) is conformal and thus is the restriction of a Möbius map if \( n \geq 3 \).

An important result proved in Iwaniec [13, Theorem 3] is that for each \( n \geq 3 \) and \( l \geq 1 \) there exists a \( p_\ast = p(n,l) < n \) such that every weakly \( l \)-quasiregular map belonging to \( W^{1,p_\ast}(\Omega; \mathbb{R}^n) \) belongs actually to \( W^{1,n}(\Omega; \mathbb{R}^n) \) and is thus an \( l \)-quasiregular map as usually defined in [21] or [22]. Such higher integrability results depend on some new estimates for weak solutions of \( p \)-harmonic equations in Iwaniec [13], and Iwaniec and Sbordone [15].

In this paper, we shall study some properties pertaining to the stability of weakly quasiregular maps. We shall consider the stability of maps in \( W^{1,p}(\Omega; \mathbb{R}^n) \) when their gradients are converging to the conformal set \( K_1 = \mathbb{R}^+ \cdot SO(n) \) in the averaged sense described by (1.4) in Definition 1.1 below. The study is originated from a study of the structures of Young measures whose supports are unbounded. For references in this direction, we refer to [2], [3], [16], [17], [19], [23], [25], [26], [29], [30] and references therein.
We need some notation to proceed. For a function $f$ defined on $\mathcal{M}^{n \times n}$ we use $Z(f)$ and $f^\#$ to denote the zero set and the quasiconvexification of $f$, respectively. For a given set $\mathcal{K} \subset \mathcal{M}^{n \times n}$, denote by $d_\mathcal{K}(A)$ the distance from $A$ to $\mathcal{K}$ for all $A \in \mathcal{M}^{n \times n}$ (in any equivalent Euclidean norm), and let $\mathcal{K}^\#$ be the quasiconvex hull of $\mathcal{K}$. See Dacorogna [8], Yan [26] and Šverák [23] for the relevant definitions.

In this paper, we use the following definition, see also Zhang [30]. We refer to Ball [2], Kinderlehrer and Pedregal [17] and Tartar [25] for more connections of this definition with the Young measures theory.

**Definition 1.1.** We say $\mathcal{K}$ is $W^{1,p}$-stable if for every sequence $u_j \rightharpoonup u_0$ in $W^{1,p}(\Omega; \mathbb{R}^n)$ satisfying

$$\lim_{j \to \infty} \int_\Omega d_\mathcal{K}^p(\nabla u_j(x)) \, dx = 0,$$

(1.4)

it follows that $\nabla u_0(x) \in \mathcal{K}$ for a.e. $x \in \Omega$. We say $\mathcal{K}$ is $W^{1,p}$-compact if every weakly convergent sequence $u_j \rightharpoonup u_0$ in $W^{1,p}(\Omega; \mathbb{R}^n)$ satisfying (1.4) converges strongly to $u_0$ in $W^{1,1}(\Omega; \mathbb{R}^n)$. In terms of Young Measures, $\mathcal{K}$ is $W^{1,p}$-compact if and only if every $W^{1,p}$-gradient Young Measure supported on $\mathcal{K}$ is a Dirac Young Measure on $\mathcal{M}^{n \times n}$.

It should be noted that in many cases $d_\mathcal{K}^p$ in (1.4) can be replaced by other functions $f$ that vanish exactly on $\mathcal{K}$ and satisfy $0 \leq f(A) \leq C(|A|^p + 1)$. For example, when $\mathcal{K}$ is homogeneous, then $d_\mathcal{K}^p$ in (1.4) can be replaced by any non-negative homogeneous functions of degree $p$ that vanish exactly on $\mathcal{K}$.

We also note that it follows from the result in Zhang [29]-[30] that if a compact set $\mathcal{K}$ is $W^{1,p}$-compact for some $p > 1$ then it is $W^{1,p}$-compact for all $p > 1$. One of the main purposes of this paper is to show that this result fails to hold for unbounded sets $\mathcal{K}$. Our counter-example is provided by the conformal set $K_1 = \mathbb{R}^+ \cdot SO(n)$ defined above. More precisely, we shall prove the following result.

**Theorem 1.2.** Suppose $n \geq 3$. Then set $K_1 = \mathbb{R}^+ \cdot SO(n)$ is $W^{1,p}$-compact for all $p \geq n$, but not $W^{1,p}$-stable for any $1 \leq p < n/2$.

The $W^{1,n}$-compactness of $K_1$ follows from a stronger theorem (Theorem 3.1) proved by using the result of Evans and Gariepy [10] (see also Evans [9]) and the theory of polyconvex functions. Note that the $W^{1,p}$-compactness of $K_1$ for $p > n$ has been proved in Ball [3] using the Young measures and polyconvex functions; see also Kinderlehrer [16]. Using the similar techniques of biting Young measures, one can also prove the $W^{1,n}$-compactness of $K_1$ without using the result of [10]; but we do
not pursue such a method in the present paper. For more on biting Young measures, we only refer to [6], [17] and [28].

It is also noted that $K_1 = \mathbb{R}^+ \cdot SO(n)$ is unbounded and contains no rank-one connections, but our theorem says that it may or may not support nontrivial gradient Young measures. This phenomenon also makes the conjecture in Tartar [25] more interesting for Young measures with unbounded supports; of course, this conjecture (in the case of compact supports) has been very well understood and resolved in Bhattacharya et al. [7], see also Šverák [24].

It has been proved in Yan [26] (also Zhang [29]) that if $\mathcal{K}$ is compact then $\mathcal{K}^\# = Z(d^\mathcal{K}_{#})$. For $\mathcal{K} = K_1$, the conformal set, if $n \geq 3$ it is easily seen from the proof of Theorem 3.3 that the set $R(n)$ is contained in $Z(d^\mathcal{K}_{#})$.

More recently, using this observation and the rank-one convex hulls, we have proved in Yan [27] that $d^\mathcal{K}_{K_1}$ actually must be identically zero. For more on the growth condition for conformal energy functions, we refer to the forthcoming paper Yan [27]. Therefore in general the previous result $\mathcal{K}^\# = Z(d^\mathcal{K}_{#})$ does not hold for unbounded sets $\mathcal{K}$ (an example when $n = 2$ was given in Yan [26]).

The proof of Theorem 3.1 uses the polyconvex function $G(A)$ defined by (2.1) which vanishes exactly on the conformal set $K_1$ and is uniformly strictly quasiconvex in the term used by Evans [9] and Evans and Gariepy [10]. The proof using biting Young measures as in Ball [3] also uses such polyconvex functions. However for $p < n$ both proofs break down since there is no counterpart of polyconvex function $G(A)$ that vanishes exactly on $K_1$ and grows like $|A|^p$ when $p < n$, see Yan [27].

To study the case for $p < n$, we make use of some new estimates for $p$-harmonic equations obtained recently by Iwaniec [13, Theorem 1] (see also [15]). We shall prove the following coercivity result for the functional

$$\int_{\Omega} d_{K_1}^p(|\nabla \phi(x)|) \, dx$$

on $W^{1,p}_0(\Omega; \mathbb{R}^n)$ for certain $p < n$, which follows obviously from Theorem 4.1.

**Theorem 1.3.** - Let $n \geq 3$ and $K_1 = \mathbb{R}^+ \cdot SO(n)$ be the conformal set. Then there exist constants $\alpha(n) < n < \beta(n)$ and $c_0(n) > 0$ such that for all $p \in [\alpha(n), \beta(n)]$

$$c_0(n) \int_{\Omega} |\nabla \phi(x)|^p \, dx \leq \int_{\Omega} d_{K_1}^p(|\nabla \phi(x)|) \, dx \leq \int_{\Omega} |\nabla \phi(x)|^p \, dx \quad (1.5)$$

for all $\phi \in W^{1,p}_0(\Omega; \mathbb{R}^n)$.

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This theorem implies that for certain values of $p$ lower than $n$, any weakly convergent sequence $\{u_j\}$ in $W_0^{1,p}(\Omega; \mathbb{R}^n)$ that satisfies (1.4) must converge to 0 in $W^{1,p}(\Omega; \mathbb{R}^n)$. For functions $\phi$ with the conformal linear boundary conditions, we do not know whether a similar estimate like (1.5) can be obtained; see the remarks in the end of the paper.

Finally, we point out that the estimate like the second one of (1.5) cannot be expected to hold for a constant $\alpha(n) < n/2$.

**Theorem 1.4.** – Let $\alpha(n) < n$ be any constant determined in the previous theorem. Then it follows that $\alpha(n) \geq n/2$.

We now give the plan of the paper. In section 2, we review some notation and preliminaries that are needed to prove our main theorems. In section 3, we prove the $W^{1,p}$-compactness of the conformal set $K_1 = \mathbb{R}^+ \cdot SO(n)$ for $p \geq n$ and study the $W^{1,p}$-stability of $K_1$ for $p < n/2$. In section 4, we prove the coercivity property (1.5) of the integral functional $\int_{\mathbb{R}^n} d_K^p(\nabla \phi(x)) \, dx$ on $W^{1,p}(\mathbb{R}^n; \mathbb{R}^n)$ for certain values of $p$ lower than $n$. We also prove that such a coercivity estimate is not true for $p < n/2$. Finally, in section 5, we make some remarks regarding the $W^{1,p}$-compactness of set $K_1$ for certain lower values of $p < n$.

## 2. NOTATION AND PRELIMINARIES

For $n \geq 2$, let us define

$$G(A) = n^{-n/2}|A|^n - \det A$$

where $|A|^2 = \text{tr} (A^T A)$. It is easily seen that $G(A) \geq 0$ is polyconvex and vanishes exactly on $K_1 = \mathbb{R}^+ \cdot SO(n)$ and is uniformly strictly quasiconvex in the sense defined by Evans and Gariepy in [10], also [9] and [11].

**Lemma 2.1.** – Let $G(A)$ be defined by (2.1). Then for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that for all $A \in \mathcal{M}^{n \times n}$

$$G(A) \leq C_\epsilon d_{K_1}^n(A) + \epsilon |A|^n.$$  \hfill (2.2)

**Proof.** – This follows easily from the homogeneity of $G(A)$ and $d_{K_1}^n(A)$. □

In order to use the estimates for $p$-harmonic tensors, we need some notation on exterior algebras and differential forms on $\mathbb{R}^n$. We follow the notation in Iwaniec and Martin [14].

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Let $e_1, e_2, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$. For $l = 0, 1, \ldots, n$ we denote by $\Lambda^l = \Lambda^l(\mathbb{R}^n)$ the linear space of all $l$-tensors spanned by $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}\}$ for all ordered $l$-tuples $I = (i_1, i_2, \ldots, i_l)$ with $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. Define $\Lambda^l = \{0\}$ if $l < 0$ or $l > n$. The Grassmann algebra $\Lambda = \bigoplus \Lambda^l$ is a graded algebra with respect to the exterior multiplication.

For $\alpha = \sum \alpha^I e_I$ and $\beta = \sum \beta^I e_I$ in $\Lambda$ the inner product is defined by

$$\langle \alpha, \beta \rangle = \sum_I \alpha^I \beta^I,$$

where the summation is taken over all $l$-tuples $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$.

The Hodge star operator $\ast : \Lambda \to \Lambda$ is then defined by the rule that

$$\ast 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

and

$$\alpha \wedge (\ast \beta) = \beta \wedge (\ast \alpha) = \langle \alpha, \beta \rangle (\ast 1)$$

for all $\alpha, \beta \in \Lambda$. It is straightforward to see that $\ast : \Lambda^l \to \Lambda^{n-l}$ and the norm of $\alpha \in \Lambda$ is then given by the formula

$$|\alpha|^2 = \langle \alpha, \alpha \rangle = \ast (\alpha \wedge \ast \alpha) \in \Lambda^0.$$

For each $l = 0, 1, \ldots, n$, a differential form $\alpha$ of degree $l$ defined on $\Omega$

$$\alpha = \sum \alpha^I(x) \, dx_I = \sum \alpha^{i_1i_2\cdots i_l}(x) \, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$$

can be identified with a function $\alpha : \Omega \to \Lambda^l(\mathbb{R}^n)$ with the same coefficients $\{\alpha^I\}$. It is appropriate to introduce the space

$$\mathcal{D}'(\Omega; \Lambda) = \bigoplus \mathcal{D}'(\Omega; \Lambda^l)$$

of all differential forms whose coefficients are Schwartz distributions on $\Omega$. We can also define $L^p(\Omega; \Lambda), W^{1,p}(\Omega; \Lambda)$ or other spaces by requiring all the coefficients belong to the suitable function spaces.

We shall make use of the exterior derivative

$$d : \mathcal{D}'(\Omega; \Lambda^l) \to \mathcal{D}'(\Omega; \Lambda^{l+1}), \quad l = 0, 1, \ldots, n,$$

and its formal adjoint operator, commonly called the Hodge codifferential,

$$d^* : \mathcal{D}'(\Omega; \Lambda^{l+1}) \to \mathcal{D}'(\Omega; \Lambda^l),$$

defined by $d^* = (-1)^{nl+1} \ast d \ast$ on $(l+1)$-forms.
The following observation will be useful in proof of Theorem 1.3.

**Lemma 2.2.** Suppose $F \in K_1 = \mathbb{R}^+ \cdot SO(n)$. Let $f_j$ be the $j$-th column (or row) vector of $F$ for $j = 1, \ldots, n$, each being considered in $\Lambda^1$. Then

$$|f_{i_1} \wedge \cdots \wedge f_{i_l}| = |f_1|^l, \quad 1 \leq i_1 < \cdots < i_l \leq n, \quad l = 1, 2, \ldots, n;$$

and

$$(-1)^{n-1}|f_1 \wedge \cdots \wedge f_{n-1}|^{\frac{2-n}{n-1}}(f_1 \wedge \cdots \wedge f_{n-1}) = *f_n.$$

Finally we need the following estimate on the weak solutions of nonhomogeneous $p$-harmonic equation in $\mathbb{R}^n$ proved in Iwaniec [13] and [12]. We refer to the recent paper of Iwaniec and Sbordone [15] for more discussions.

**Theorem 2.3.** For each $p > 1$, there exists $\nu = \nu(n, p) \in (1, p)$ such that for every $s > \nu$ every weak solution $u$ with $du \in L^s(\mathbb{R}^n; \Lambda)$ to the $p$-harmonic equation

$$d^* [ |g + du|^{p-2}(g + du) ] = d^* h \quad \text{in } \mathbb{R}^n$$

satisfies for a constant $C(n, p, s) > 0$

$$\int_{\mathbb{R}^n} |du|^s \leq C(n, p, s) \int_{\mathbb{R}^n} \left( |g|^s + |h|^\frac{s}{p-1} \right).$$

Moreover, the constant $C(n, p, s)$ can be chosen independent of $s$ for $\nu \leq s \leq n$.

**Proof.** This is Theorem 1 in Iwaniec [13].

3. $W^{1, p}$-Compactness of the Conformal Set $K_1$

Let $K_1 = \mathbb{R}^+ \cdot SO(n)$ be the conformal set defined before. In what follows, we assume $n \geq 3$. We first prove the $W^{1, n}$-compactness of the set $K_1$.

**Theorem 3.1.** Suppose $u_j \rightharpoonup u_0$ in $W^{1, n}(\Omega; \mathbb{R}^n)$ and satisfies

$$\lim_{j \to \infty} \int_{\Omega} d^n_{K_1}(\nabla u_j(x)) \, dx = 0. \quad (3.1)$$

Then $u_0$ is a conformal map and moreover $u_j \to u_0$ in $W^{1, n}(\Omega; \mathbb{R}^n)$. Consequently, $K_1 = \mathbb{R}^+ \cdot SO(n)$ is $W^{1, n}$-compact.
Proof. – Let $G(A)$ is defined by (2.1). By Lemma 2.1 and (3.1), since 
\[ \{ \| \nabla u_j \|_{L^n(\Omega)} \} \] is bounded, we easily obtain
\[
\lim_{j \to \infty} I(u_j) \equiv \lim_{j \to \infty} \int_{\Omega} G(\nabla u_j(x)) \, dx = 0. \tag{3.2}
\]
Since $G(A)$ is polyconvex and satisfies $0 \leq G(A) \leq C |A|^n$, therefore by
the theorem of Acerbi-Fusco [1], the functional
\[
I(u) = \int_{\Omega} G(\nabla u(x)) \, dx
\]
is weakly lower semicontinuous on $W^{1,n}(\Omega; \mathbb{R}^n)$ (see also Ball and
Murat [4] and Morrey [18]) thus it follows that
\[
0 = I(u_0) \leq \liminf_{j \to \infty} I(u_j) = 0
\]
which implies $u_0$ is a conformal map and $u_j \to u_0$ in $W^{1,n}(\Omega; \mathbb{R}^n)$ by
the result of Evans and Gariepy [10] since $G(A)$ is uniformly strictly
quasiconvex. Finally by definition it follows that $K_1$ is $W^{1,n}$-compact. \hfill \Box

The $W^{1,n}$-compactness of $K_1$ can also be proved by using biting Young
measures as in Ball [2] using Young measures for $p > n$. However, both
methods do not work anymore for $p < n$ mainly because in this case there
is no counterpart of the polyconvex function $G(A)$ vanishing exactly on
$K_1$ and with growth like $|A|^p$; see Yan [27].

Before considering the $W^{1,p}$-compactness of set $K_1$ for $1 < p < n$,
we make some remark about the non-$W^{1,p}$-compactness for a general set
$K \subset M^{n \times n}$ and $1 < p < \infty$.

Let $A \in M^{n \times n}$, we consider the following system of equations or
differential relations,
\[
\begin{align*}
&u \in W^{1,p}(\Omega; \mathbb{R}^n), \\
&\nabla u(x) \in K, \text{ for a.e. } x \in \Omega, \\
&u(x) = A x, \text{ for } x \in \partial \Omega. \tag{3.3}
\end{align*}
\]
Generally, the solvability of (3.3) relies heavily on the structure of $K$. It
is expected that nontrivial solutions of (3.3) (if exist) should be highly
oscillatory if set $K$ does not have certain nice structures.

When $K$ is the compatible two-well in two dimensions, Müller and Šverák [19] recently proved that for certain matrices $A \notin K$, Problem (3.3)
has Lipschitz solutions.

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We now prove the following result. The argument is closely related to that in Ball and Murat [4].

**Theorem 3.2.** Suppose \( u, K \) and \( A \) solve system (3.3). Then \( A \in Z(d^d_\mathcal{K}) \), and \( K \) is not \( W^{1,p} \)-stable if \( A \notin \mathcal{K} \).

**Proof.** First we remark that without loss of generality we can assume \( \Omega \) to be the unit cell \( Q_0 \) centered at origin, since otherwise, using the Vitali covering and the affine boundary condition of \( u(x) \), one can construct a solution \( v \in W^{1,p}(Q_0; \mathbb{R}^n) \) to a system similar to (3.3) only with \( \Omega \) being replaced by \( Q_0 \).

For each \( k = 1, 2, \ldots \), we divide \( Q_0 \) into \( 2^{nk} \) sub-cells with side \( 2^{-k} \), and denote these sub-cells by \( \{Q^k_j\} \) with \( 1 \leq j \leq 2^{nk} \). Suppose

\[
Q^k_j \equiv a^k_j + 2^{-k}Q_0, \quad j = 1, 2, \ldots, 2^{nk}. \tag{3.4}
\]

We now define a map \( u^k : Q_0 \to \mathbb{R}^n \) as follows,

\[
u^k(x) = \begin{cases} \frac{A a^k_j + 2^{-k}u(2^k(x - a^k_j))}{Ax}, & \text{if } x \in Q^k_j \text{ for some } j, \\ \frac{A x}{\text{for other } x \in Q_0}. \end{cases} \tag{3.5}
\]

It is easily seen that \( \nabla u^k(x) \in K \) for a.e. \( x \in Q_0 \) and \( u^k \in W^{1,p}(Q_0; \mathbb{R}^n) \). It is also easy to see for all functions \( W(A) \) defined on \( \mathcal{M}^{n \times n} \) that

\[
\int_{Q_0} W(\nabla u^k(x)) \, dx \to \int_{Q_0} W(\nabla u(x)) \, dx. \tag{3.6}
\]

A calculation also shows that (see e.g., Ball and Murat [4, Corollary A. 2])

\[
u^k \to u_0 \text{ in } W^{1,p}(Q_0; \mathbb{R}^n) \quad \text{as } k \to \infty, \tag{3.7}
\]

where \( u_0(x) \equiv Ax \). Since \( d_\mathcal{K}(\nabla u^k(x)) = 0 \), therefore, it follows from theorem on weak lower semicontinuity (see [1], [4], [8] and [18]) that \( \nabla u_0(x) \equiv A \in Z(d^d_\mathcal{K}) \). If \( A \notin \mathcal{K} \), then \( \nabla u_0(x) \notin \mathcal{K} \), thus by definition and (3.7), this shows that \( \mathcal{K} \) is not \( W^{1,p} \)-stable. We thus complete the proof. \( \square \)

It is proved in [13] that there exists a \( p_* = p(n,l) < n \) for each \( n \geq 3 \) and \( l \geq 1 \) such that every weakly \( l \)-quasiregular map belonging to \( W^{1,p_*}(\Omega; \mathbb{R}^n) \) belongs actually to \( W^{1,n}(\Omega; \mathbb{R}^n) \); thus is an \( l \)-quasiregular map as usually defined in [21] and [22]. The general conjecture is that \( p_* = \frac{nl}{l+1} \); see also [14]. From this it follows that problem (3.8) can not have a solution when \( p \geq p_* = p(n) \) and \( l = 1 \) unless \( A \in K_1 \).

The following results are based on the existence of weakly \( l \)-quasiregular maps that are not \( l \)-quasiregular when \( n \geq 3 \). Recall that \( R(n) \) is the set
of all general orthogonal matrices in $\mathbb{R}^n$ defined by (1.3). See also Iwaniec and Martin [14, section 12].

**THEOREM 3.3.** – Let $1 \leq p < \frac{n l}{l + 1}$ and $A \in R(n)$ with $\det A = -1$. Then the following problem has a solution:

$$
\begin{align*}
 u &\in W^{1,p}(B; \mathbb{R}^n), \\
 \nabla u(x) &\in K_l, \text{ for a.e. } x \in B, \\
u(x) &= A x, \text{ for } x \in \partial B,
\end{align*}
$$

(3.8)

where $B$ is the unit open ball in $\mathbb{R}^n$.

**Proof.** – For a given $l \geq 1$, define a radial map $\Phi_l : B \to \mathbb{R}^n$ as follows:

$$
\Phi_l(x) = \left( \frac{1}{|x|} \right)^{1 + \frac{1}{l}} x \quad \text{for } x \in B.
$$

(3.9)

When $l = 1$, $\Phi_1$ is the inversion with respect to the unit sphere.

It is easily seen that $\Phi_l(x) = x$ for $x \in \partial B$ and that

$$
\nabla \Phi_l(x) = \left( \frac{1}{|x|} \right)^{1 + \frac{1}{l}} \left( I - \frac{l + 1}{l} \frac{x}{|x|} \otimes \frac{x}{|x|} \right).
$$

Thus

$$
\|\nabla \Phi_l(x)\|^n = |x|^{-n(1 + \frac{1}{l})} = -l \det \nabla \Phi_l(x) \quad \text{for } x \in B \setminus \{0\}.
$$

(3.10)

For $A \in R(n)$ with $\det A = -1$, define $u(x) = \Phi_l(Ax)$ for $x \in B$. We claim that $u$ solves (3.8) for any $1 \leq p < \frac{n l}{l + 1}$.

Note that $\nabla u(x) = \nabla \Phi_l(Ax) A$ for $x \in B \setminus \{0\}$ and $|Ax| = |x|$ for any $x \in \mathbb{R}^n$. Therefore by (3.10), it follows that $\|\nabla u(x)\|^n = l \det \nabla u(x)$ for $x \in B \setminus \{0\}$ and $u(x) = A x$ if $x \in \partial B$. What is left to check is $u \in W^{1,p}(B; \mathbb{R}^n)$ for any $1 \leq p < \frac{n l}{l + 1}$. Our calculation shows

$$
\|\nabla u(x)\|^p = |x|^{-p(1 + \frac{1}{l})} \quad \text{for } x \in B \setminus \{0\}.
$$

Thus

$$
\int_B \|\nabla u(x)\|^p \, dx = \frac{l \omega_n}{ln - p(l + 1)} < \infty,
$$

where $\omega_n$ is the area of $\partial B$. Thus $u \in W^{1,p}(B; \mathbb{R}^n)$ for any $1 \leq p < \frac{n l}{l + 1}$. We thus complete the proof. □

Combining Theorems 3.2 and 3.3, we have proved the following corollary.

**COROLLARY 3.4.** – For any $l \geq 1$ and $1 \leq p < \frac{n l}{l + 1}$, the set $K_l$ is not $W^{1,p}$-stable. Moreover, $R(n) \subset Z(d_{K_l}^\#)$.

As mentioned in the introduction, using rank-one convex hulls, we can prove $R(n)^\# \equiv M_n^{n \times n}$ for $n \geq 3$. Therefore the previous corollary actually implies that $d_{K_l}^\#$ must be identically zero. See Yan [27] for more.
4. THE COERCIVITY OF $\int_{\mathbb{R}^n} d_{K_1}^p(\nabla u(x)) \, dx$ ON $W^{1,p}(\mathbb{R}^n; \mathbb{R}^n)$

Let $K = K_1 = \mathbb{R}^+ \cdot SO(n)$ be the conformal set. This section is devoted to proving the following result.

**Theorem 4.1.** – For each $n \geq 3$, there exists $\alpha(n) < n$ such that for all $p \geq \alpha(n)$

$$\int_{\mathbb{R}^n} |\nabla \phi(x)|^p \, dx \leq C(n,p) \int_{\mathbb{R}^n} d_{K}^p(\nabla \phi(x)) \, dx$$

(4.1)

for all $\phi \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^n)$. Moreover, $1 \leq C(n,p) \leq C(n) < \infty$ for $\alpha(n) \leq p \leq n$.

**Proof.** – We have only to prove (4.1) for $\phi \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^n)$. Let us assume

$$\nabla \phi(x) = A(x) - B(x), \ A(x) \in K_1, |B(x)| = d_K(\nabla \phi(x)), \ a.e. \quad (4.2)$$

We can also assume $B$ has compact support and is bounded. Let $\phi_i$ be the $i$-th coordinate function of $\phi$, then $d\phi_i$ is a 1-form. Let $\beta_i(x)$ be the $i$-th row vector of $B(x)$ considered as a 1-form. Since $\nabla \phi(x) + B(x) \in K_1$ thus by Lemma 2.2, we have

$$|(d\phi_{i_1} + \beta_{i_1}) \wedge \cdots \wedge (d\phi_{i_l} + \beta_{i_l})|$$

$$= |d\phi_1 + \beta_1|^l, \ 1 \leq i_1 < \cdots < i_l \leq n, \quad l = 1, 2, \ldots, n; \quad (4.3)$$

and

$$|(d\phi_1 + \beta_1) \wedge \cdots \wedge (d\phi_{n-1} + \beta_{n-1})|^{\frac{2-n}{n-1}} (d\phi_1 + \beta_1) \wedge \cdots \wedge (d\phi_{n-1} + \beta_{n-1})$$

$$= (-1)^{n-1} * (d\phi_n + \beta_n). \quad (4.4)$$

Let

$$u = \phi_{n-1} d\phi_1 \wedge \cdots \wedge d\phi_{n-2}, \quad du = (-1)^n d\phi_1 \wedge \cdots \wedge d\phi_{n-1},$$

$$g = (-1)^n [(d\phi_1 + \beta_1) \wedge \cdots \wedge (d\phi_{n-1} + \beta_{n-1})] - (d\phi_1 \wedge \cdots \wedge d\phi_{n-1}),$$

$$h = - * \beta_n.$$ 

Then it follows from (4.4) and $d^* = *d$ on 1-forms that

$$d^*[|g + du|^{p-2}(g + du)] = d^* h \quad \text{in} \ \mathbb{R}^n, \ \text{where} \ p = \frac{n}{n-1}. \quad (4.5)$$
Therefore by Theorem 2.3, there exists $1 < \nu < \frac{m}{n-1}$ such that for all $s \geq \nu$,

$$
\int_{\mathbb{R}^n} |du|^s \leq C(n, s) \int_{\mathbb{R}^n} \left( |g|^s + |h|^{s(n-1)} \right).
$$

(4.6)

By (4.3) and definition of $du$ and $g$, it follows that for each $j = 1, 2, ..., n$,

$$
|d\phi_j|^{(n-1)s} \leq |g + du|^s + |\beta_j|^{(n-1)s} \leq |g|^s + |du|^s + |\beta_j|^{(n-1)s}
$$

and

$$
|g|^s \leq \sum |d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_l}|^s |\beta_{j_1} \wedge \cdots \wedge \beta_{j_m}|^s,
$$

where the summation is over all $l + m = (n - 1)$ and $l > 0, m > 0$ and $i_l \leq (n - 1)$. From this we have

$$
|g|^s \leq \epsilon \sum_{j=1}^{n-1} (|d\phi_j|^{s(n-1)}) + C(\epsilon)|B|^{s(n-1)},
$$

where $\epsilon > 0$ is arbitrary. Combining these pointwise estimates, integrating over $\mathbb{R}^n$ and using (4.6), we obtain for each $j = 1, 2, ..., n$,

$$
\int_{\mathbb{R}^n} |d\phi_j|^{s(n-1)} \leq \epsilon \sum_{j=1}^{n-1} \int_{\mathbb{R}^n} |d\phi_j|^{s(n-1)} + C(n, s, \epsilon) \int_{\mathbb{R}^n} |B|^{s(n-1)},
$$

for a different arbitrary $\epsilon > 0$, to be chosen later. Summing this inequality for $j$ from 1 to $n$ and choosing $\epsilon = 1/(2n)$, it follows that

$$
\sum_{j=1}^{n} \int_{\mathbb{R}^n} |d\phi_j|^{s(n-1)} \leq C(n, s) \int_{\mathbb{R}^n} |B|^{s(n-1)},
$$

(4.7)

for all $s \geq \nu$. Now let $\alpha(n) = \nu(n - 1)$ then $\alpha(n) < n$. For this $\alpha(n)$ it is easy to see (4.1) follows from (4.7). The proof is thus completed. \(\square\)

**Theorem 4.2.** Let $\alpha(n) < n$ be any constant determined in the previous theorem. Then it follows that $\alpha(n) \geq n/2$.

**Proof.** We suppose $\alpha(n) < n/2$. Let $\Phi_1 : B_1 \to \mathbb{R}^n$ be the inversion with respect to the unit sphere as defined by (3.9). Let $A \in R(n)$ with $\text{det} A = -1$. Define $u(x) = \Phi_1(Ax)$ for $x \in B_1$. Then

$$
\nabla u(x) = \nabla \Phi_1(Ax) A \in K_1 = \mathbb{R}^+ \cdot SO(n), \quad \text{a.e. } x \in B_1.
$$
Now let $\rho \in C^\infty_0(\mathbb{R}^n)$ with $\rho(x) = 1$ for $x \in B_{1/2}$ and $\rho(x) = 0$ for $x \notin B_1$, and

$$0 \leq \rho(x) \leq 1, \quad |\nabla \rho(x)| \leq 2.$$ 

Let $\phi(x) = \rho(x)(u(x) - c)$, where $c$ is a constant to be chosen later. Then $\phi \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^n)$ for all $1 \leq p < n/2$.

For $0 < \epsilon < \frac{n}{2} - \alpha(n)$, applying (4.1) to $\phi \in W^{1,\frac{n}{2} - \epsilon}(\mathbb{R}^n; \mathbb{R}^n)$, we obtain

$$\int_{B_{1/2}} |\nabla u(x)|^{\frac{n}{2} - \epsilon} \, dx \leq C(n) \left( \int_{B_1} d_K^\frac{n}{2} \left( \nabla \rho(x) \otimes (u(x) - c) + \rho(x) \nabla u(x) \right) \right) \, dx,$$

from which and using $d_K(A + B) \leq d_K(A) + |B|$ it follows that

$$\int_{B_{1/2}} |\nabla u|^{\frac{n}{2} - \epsilon} \leq C(n) \int_{B_1} |u - c|^{\frac{n}{2} - \epsilon} \leq C(n) \left( \int_{B_1} |\nabla u|^{\frac{n^2 - 2n + 1}{2n - 2}} \right)^{\frac{3n - 2n + 1}{2n}}, \quad (4.8)$$

where we have chosen $c = \frac{1}{|B_1|} \int_{B_1} u$ and applied the Sobolev inequality.

In (4.8), letting $\epsilon \to 0$ we would have

$$\int_{B_{1/2}} |\nabla u(x)|^{n/2} \, dx \leq C(n) \left( \int_{B_1} |\nabla u(x)|^{n/3} \, dx \right)^{3/2} < \infty,$$

which is a contradiction, since $u \notin W^{1,n/2}(B_{1/2}; \mathbb{R}^n)$ as we showed before. We have thus completed the proof. \(\square\)

### 5. A CONCLUDING REMARK

As we mentioned before, it is proved in Iwaniec [13, Theorem 3] that there exists a minimal $p_\ast = p(n) \in [n/2, n]$ for each $n \geq 3$ such that if a map $u(x)$ belonging to $W^{1,p_\ast}(\Omega; \mathbb{R}^n)$ satisfies $\nabla u(x) \in K_1 = \mathbb{R}_+^n \cdot SO(n)$ a.e. then it belongs actually to $W^{1,n}(\Omega; \mathbb{R}^n)$. Note that $p_\ast = n/2$ when $n$ is even, by the results in [14].

For a weakly convergent unperturbed sequence $\{u_j\}$ in $W^{1,p_\ast}(\Omega; \mathbb{R}^n)$ with $\nabla u_j(x) \in K_1$ for a.e. $x \in \Omega$, the strong convergence follows easily from Theorem 3.1 and this higher integrability result.
Now, if we only assume the distance from $\nabla u_j(x)$ to the conformal set is small and approaches zero as $j \to \infty$, then we do not usually have the higher integrability for $u_j \in W^{1,p}(\Omega; \mathbb{R}^n)$. In the even dimensions, there are some linear structures (see [14] and [27]) among the subdeterminants of half dimension size, that may compensate some loss of the stability due to the weak convergence of $\{u_j\}$. But I have not come up with the definite results in this aspect even in even dimensions. Therefore, it would be interesting to consider the following problem.

**Problem 5.1.** - Determine whether $K_1 = \mathbb{R}^+ \cdot SO(n)$ is $W^{1,p}$-compact for some $p < n$. If it is, whether the minimal value of such $p$ is equal to $p^*$ given above.

**Remark.** - Most recently, in Müller, Šverák and Yan [20], it is proved that for even dimensions $n \geq 4$ the minimal $\alpha(n)$ in Problem 5.1 is $n/2$.

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**REFERENCES**


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