Math 847 / Homework # 2 – ANSWER KEY *Due: Monday 9/28/2015*

Name: ____________________ **ID:** ______________

NOTICE: *Include all necessary steps to receive full or partial credits. Write clearly on the given papers only.* **No scratch papers are accepted**. *Use backsides if necessary.*

(1) (10 points) Show for $n \geq 3$ that

$$u(0) = ⨍_{\partial B(0,r)} g(y) dS_y + \frac{1}{n(n-2)\alpha_n} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f(x)\, dx,$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B(0,r) \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

Proof. Set

$$\phi(\rho) = ⨍_{\partial B(0,\rho)} u(y) dS_y, \quad 0 < \rho \leq r.$$

Then $\phi(r) = ⨍_{\partial B(0,r)} g dS$ and $\phi(0^+) = u(0)$. We can compute that

$$(*) \qquad \phi'(\rho) = \frac{\rho}{n} ⨍_{B(0,\rho)} \Delta u(x)\, dx = -\frac{1}{n\alpha_n \rho^{n-1}} \int_{B(0,\rho)} f(x)\, dx \stackrel{\text{def}}{=} -\frac{1}{n\alpha_n \rho^{n-1}} F(\rho),$$

where

$$F(\rho) = \int_{B(0,\rho)} f(y)\, dy = \int_0^\rho \left(\int_{\partial B(0,s)} f(y) dS_y \right) ds.$$

So

$$(**) \qquad F'(\rho) = \int_{\partial B(0,\rho)} f(y) dS_y, \quad \lim_{\epsilon \to 0^+} F(\epsilon) = 0, \quad \lim_{\epsilon \to 0^+} F(\epsilon)\epsilon^{-n} = \alpha_n f(0).$$

By $(*)$, since $n \geq 3$, we have for any given $0 < \epsilon < r$,

$$\begin{aligned}
\phi(r) - \phi(\epsilon) &= -\frac{1}{n\alpha_n} \int_\epsilon^r \rho^{1-n} F(\rho) d\rho = \frac{1}{n(n-2)\alpha_n} \int_\epsilon^r F(\rho)\, d(\rho^{2-n} - r^{2-n}) \\
&= \frac{1}{n(n-2)\alpha_n} \left[F(\epsilon)(r^{2-n} - \epsilon^{2-n}) - \int_\epsilon^r (\rho^{2-n} - r^{2-n}) F'(\rho)\, d\rho \right] \\
&= \frac{1}{n(n-2)\alpha_n} \left[F(\epsilon)(r^{2-n} - \epsilon^{2-n}) - \int_\epsilon^r (\rho^{2-n} - r^{2-n}) \int_{\partial B(0,\rho)} f dS\, d\rho \right] \\
&= \frac{1}{n(n-2)\alpha_n} \left[F(\epsilon)(r^{2-n} - \epsilon^{2-n}) - \int_\epsilon^r \int_{\partial B(0,\rho)} (|x|^{2-n} - r^{2-n}) f(x) dS_x\, d\rho \right] \\
&= \frac{1}{n(n-2)\alpha_n} \left[F(\epsilon)(r^{2-n} - \epsilon^{2-n}) - \int_{B(0,r)\setminus B(0,\epsilon)} (|x|^{2-n} - r^{2-n}) f(x)\, dx \right].
\end{aligned}$$

Letting $\epsilon \to 0^+$, using $(**)$ and $\phi(0^+) = u(0)$, since $|x|^{2-n}$ is integrable in $B(0,r)$, we derive the identity wanted.

(Another proof is to directly use Green's function of $B(0,r)$, but you neeed to write down this Green's function.)

(2) (10 points) Prove that there exists a constant C, depending only on n, such that

$$\max_{B(0,1)} |u| \le \max_{\partial B(0,1)} |g| + C \max_{B(0,1)} |f|$$

where u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } B(0,1) \\ u = g & \text{on } \partial B(0,1). \end{cases}$$

Proof. Let $G(x,y)$ be Green's function for ball $B(0,1)$ and $K(x,y)$ be the Poisson kernel of $B(0,1)$. Then any smooth solution $u \in C^2(\bar{B}(0,1))$ can be represented by:

$$(*) \qquad u(x) = \int_{\partial B(0,1)} K(x,y)u(y)\, dS_y + \int_{B(0,1)} G(x,y)(-\Delta u(y))\, dy.$$

Take $u \equiv 1$ and $u = \frac{1}{2n}(1-|x|^2)$ in $(*)$ respectively, and we obtain that

$$\int_{\partial B(0,1)} K(x,y)\, dS_y = 1, \qquad \int_{B(0,1)} G(x,y)\, dy = \frac{1-|x|^2}{2n}$$

for all $x \in B(0,1)$. From the formula of $K(x,y)$ we see

$$K(x,y) > 0 \quad \forall\, x \in B(0,1),\ y \in \partial B(0,1).$$

From the formula $G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x}))$ for $x \neq 0$, since

$$|y-x|^2 < |x|^2|y-\tilde{x}|^2 \quad \forall\, 0 < |x| < 1,\ |y| < 1,$$

we easily see $G(x,y) > 0$ for all $x \neq 0$, $y \in B(0,1)$ with $x \neq y$. Now that since $G(0,y) = G(y,0)$ we see that

$$G(x,y) \ge 0 \text{ for all } x, y \in B(0,1) \text{ with } x \neq y.$$

(*This is true for Green's function of any bounded domain. Explain why, not now but later in the course.*)

To complete the proof, we use formula $(*)$ and the fact that $G \ge 0$, $K \ge 0$ to deduce

$$\begin{aligned} |u(x)| &\le (\max_{\partial B(0,1)} |g|) \left(\int_{\partial B(0,1)} K(x,y) dS_y \right) + (\max_{B(0,1)} |f|) \left(\int_{B(0,1)} G(x,y) dy \right) \\ &= \max_{\partial B(0,1)} |g| + (\max_{B(0,1)} |f|) \left(\frac{1-|x|^2}{2n} \right) \\ &\le \max_{\partial B(0,1)} |g| + \frac{1}{2n} \max_{B(0,1)} |f|, \quad \forall\, x \in B(0,1). \end{aligned}$$

(3) (10 points) Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \le u(x) \le r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B(0,r)$.

Proof. Poisson's formula says

$$u(x) = \int_{\partial B(0,r)} K(x,y)u(y)\,dS_y,$$

where

$$K(x,y) = \frac{r^2-|x|^2}{n\alpha_n r|x-y|^n}, \quad |x|<r,\ |y|=r.$$

Since for $|x|<r$ and $|y|=r$, we have $r-|x| \le |y-x| \le r+|x|$. Therefore,

$$\frac{r^2-|x|^2}{n\alpha_n r(r+|x|)^n} \le K(x,y) \le \frac{r^2-|x|^2}{n\alpha_n r(r-|x|)^n}$$

for $x \in B(0,r)$ and $y \in \partial B(0,r)$. So, if u is harmonic and $u \ge 0$ in $B(0,r)$, then u is given by the Poisson formula above and hence we have

$$\frac{r^2-|x|^2}{n\alpha_n r(r+|x|)^n}\int_{\partial B(0,r)} u(y)dS_y \le u(x) \le \frac{r^2-|x|^2}{n\alpha_n r(r-|x|)^n}\int_{\partial B(0,r)} u(y)dS_y.$$

By the mean value property,

$$\int_{\partial B(0,r)} u(y)dS_y = n\alpha_n r^{n-1}u(0),$$

which, by simplifying, proves what we needed to prove.

(4) (10 points) Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+, |x| \le 1$. Show Du is **not bounded** near $x = 0$.

Proof. Poisson's formula gives

$$u(x) = u(x', x_n) = \frac{2x_n}{n\alpha_n} \int_{\mathbb{R}^{n-1}} \frac{g(y)\, dy}{(|y - x'|^2 + x_n^2)^{n/2}} \quad (x' \in \mathbb{R}^{n-1},\ x_n > 0).$$

Clearly, $u(0', 0) = g(0') = 0$. We want to show that

$$\lim_{\lambda \to 0^+} \frac{u(0', \lambda) - u(0', 0)}{\lambda - 0} = +\infty,$$

which shows that u_{x_n} is not bounded near $x = 0$. Let $0 < \lambda < 1$. We compute

$$u(0', \lambda) = \frac{2\lambda}{n\alpha_n} \int_{y \in \mathbb{R}^{n-1}} \frac{g(y)}{(|y|^2 + \lambda^2)^{n/2}}\, dy$$

$$= \frac{2\lambda}{n\alpha_n} \left(\int_{|y| \le 1} \frac{|y|}{(|y|^2 + \lambda^2)^{n/2}}\, dy + \int_{|y| \ge 1} \frac{g(y)}{(|y|^2 + \lambda^2)^{n/2}}\, dy \right)$$

$$\equiv \frac{2\lambda}{n\alpha_n} (I_1(\lambda) + I_2(\lambda)).$$

Let $|g(y)| \le M$ for all $y \in \mathbb{R}^{n-1}$. We first estimate $I_2(\lambda)$.

$$|I_2(\lambda)| \le M \int_{|y| \ge 1} |y|^{-n} dy = B < \infty.$$

Note that, since $y \in \mathbb{R}^{n-1}$,

$$I_1(\lambda) = \int_{|y| \le 1} \frac{|y|}{(|y|^2 + \lambda^2)^{n/2}}\, dy = c_n \int_0^1 (r^2 + \lambda^2)^{-n/2} r^{n-1}\, dr$$

$$= c_n \int_0^{\frac{1}{\lambda}} (s^2 + 1)^{-n/2} s^{n-1}\, ds \ge c_n \int_1^{\frac{1}{\lambda}} (s^2 + 1)^{-n/2} s^{n-1}\, ds.$$

Now note that for $s \ge 1$ we have $s^2 + 1 \le 2s^2$, thus

$$(s^2 + 1)^{-n/2} s^{n-1} \ge 2^{-n/2} s^{-1} \quad \text{for } s \ge 1,$$

and hence

$$I_1(\lambda) \ge c_n 2^{-n/2} \int_1^{\frac{1}{\lambda}} s^{-1}\, ds = A|\ln \lambda|$$

for some constant $A > 0$. Finally we have

$$\lim_{\lambda \to 0^+} \frac{u(0', \lambda) - u(0', 0)}{\lambda - 0} = \lim_{\lambda \to 0^+} \frac{2}{n\alpha_n} (I_1(\lambda) + I_2(\lambda))$$

$$\ge \frac{2}{n\alpha_n} \lim_{\lambda \to 0^+} (A|\ln \lambda| - B) = +\infty.$$