AN INTEGRAL FORMULA IN KAHLER GEOMETRY WITH APPLICATIONS

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Abstract. We establish an integral formula on a smooth, precompact domain in a Kahler manifold. We apply this formula to study holomorphic extension of CR functions. Using this formula we prove an isoperimetric inequality in terms of a positive lower bound for the Hermitian curvature of the boundary. Combining with a Minkowski type formula on the complex hyperbolic space we prove that any closed, embedded hypersurface of constant mean curvature must be a geodesic sphere, provided the hypersurface is Hopf. A similar result is established on the complex projective space.

1. Introduction

Let M be a compact Riemannian manifold with boundary. Given a real $u \in C^{\infty}(M)$ the following formula was proved by Reilly [R]

$$\begin{aligned} &\int_M \left[(\Delta u)^2 - \left| D^2 u \right|^2 - Ric\left(\nabla u, \nabla u\right) \right] dv \\ = &\int_{\partial M} \left[2\chi \Delta f + H\chi^2 + \Pi\left(\nabla f, \nabla f\right) \right] d\sigma, \end{aligned}$$

where $f = u|_{\partial M}$ and $\chi = \frac{\partial u}{\partial \nu}$. It was used by Reilly [R] to give a simple and elegant proof of the Alexandrov theorem that a closed, embedded hypersurface of constant mean curvature in $\mathbb{R}^n$ must be a round sphere. Since then Reilly's formula has become a very useful tool in geometric analysis and is especially effective on Riemannian manifolds with nonnegative Ricci curvature. For other applications, see [Ro] and the more recent [MW].

In this paper we derive a similar formula on a Kahler manifold. Let M be a Kahler manifold of complex dimension $m + 1 \geq 2$. Suppose $\Omega \subset M$ is a precompact domain with smooth boundary Σ. Then Σ is a real hypersurface in M. Let ν be the outer unit normal along Σ and $T = J\nu$. Let $Av = \nabla_v \nu$ be the shape operator, $\Pi(v_1, v_2) = \langle Av_1, v_2 \rangle$ the second fundamental form and H the mean curvature (the trace of Π). The function $H_b := H - \Pi(T, T)$ will play a fundamental role in our discussion. We call H_b the Hermitian mean curvature of Σ. Set $Z = \left(\nu - \sqrt{-1}T\right)/\sqrt{2}$, a $(1,0)$-vector field along Σ. For a smooth function $F : \Omega \to \mathbb{C}$ we prove the following integral formula

$$\begin{aligned}
&\sqrt{2}\int_\Omega |\Box F|^2 - \left|D^{1,1}F\right|^2 \\
= &\int_\Sigma \left[Z\overline{F}\left(\Box_b f - \sqrt{-1}\Pi\left(T, X_\alpha\right) f_{\overline{\alpha}}\right) + ZF\overline{\left(\Box_b f - \sqrt{-1}\Pi\left(T, X_\alpha\right) f_{\overline{\alpha}}\right)}\right] \\
&+ \sqrt{2}\int_\Sigma \Pi\left(\overline{X_\alpha}, \overline{X_\beta}\right) f_\alpha \overline{f}_\beta + \frac{1}{\sqrt{2}}\int_\Sigma H_b\left|\overline{Z}F\right|^2,
\end{aligned}$$

where $f = F|_\Sigma$, $\Box_b$ is the Kohn Laplacian and $\{X_\alpha\}$ a local unitary frame for $T^{1,0}\Sigma$.

A salient feature of the above identity is that the curvature of M does not appear in it. This makes it effective on a general Kahler manifold. Indeed, we have found several applications for this formula.

The first application is about when one can extend a CR function on Σ to a holomorphic function on Ω. We are able to prove the following

Theorem A. *Let $\Omega \subset M$ be a connected precompact domain with smooth boundary Σ. Suppose $H_b > 0$ on Σ. Then for any $f \in C^\infty(\Sigma)$ which is a CR function, there exists $F \in C^\infty\left(\overline{\Omega}\right) \cap \mathcal{O}(\Omega)$ s.t. $F|_\Sigma = f$.*

Though this result is weaker that a theorem of Kohn-Rossi, our method has the merit of being elementary.

Another application is the following geometric inequality.

Theorem B. *Let $\Omega \subset M$ be a connected precompact domain with smooth boundary Σ. If $H_b > 0$ on Σ, then*

$$\int_\Sigma \frac{1}{H_b} \geq \frac{m+1}{m}|\Omega|.$$

As a corollary we immediately obtain the following isoperimetric inequality in terms of a positive lower bound for the Hermitian mean curvature.

Theorem C. *Let $\Omega \subset M$ be a connected precompact domain with smooth boundary Σ. Let $c = \inf_\Sigma H_b$. If $c > 0$, then*

$$m|\Sigma| \geq c(m+1)|\Omega|. \tag{1.1}$$

In the complex hyperbolic space $\mathbb{CH}^{m+1}$ or in the complex projective space $\mathbb{CP}^{m+1}$ the above inequality is sharp as equality holds if Ω is a geodesic ball. Conversely, we are able to prove

Theorem D. *Let $\Omega \subset \mathbb{CH}^{m+1}$ be a connected precompact domain with smooth boundary Σ. If equality holds in (1.1), then Ω is a geodesic ball.*

The same result holds in $\mathbb{CP}^{m+1}$.

In view of the Alexandrov theorem mentioned before it is natural to study closed hypersurfaces of constant mean curvature in other fundamental Riemannian manifolds. Alexandrov proved the same uniqueness result for $\mathbb{H}^n$ and $\mathbb{S}^n_+$ as well in his groundbreaking paper [A]. More recently, there have been some new spectacular developments. Brendle [Br1] proved a uniqueness result for hypersurfaces of constant mean curvature in certain warped product manifolds of which the deSitter-Schwarzschild spaces are important examples. In another paper [Br2] he proved that the Clifford torus is the only embedded minimal torus in $\mathbb{S}^3$ up to congruence (the Lawson conjecture). Andrews and Li [AL] proved that all embedded tori of constant mean curvature in $\mathbb{S}^3$ are rotationally symmetric and therefore completely classified. Brendle [Br3] further extended these results by proving that a certain class of embedded Weingarten tori in $\mathbb{S}^3$, which include constant mean curvature tori as a special case, must be rotationally symmetric.

After $\mathbb{R}^n, \mathbb{H}^n$ or $\mathbb{S}^n$, the most important Riemannian manifolds are arguably the symmetric spaces. In a symmetric space of rank ≥ 2, geodesic spheres do not have constant mean curvature and it is not clear what to expect. In symmetric spaces of rank one geodesic spheres do have constant mean curvature (and even constant principal curvatures). It is natural to ask if they are the only closed, embedded hypersurfaces of constant mean curvature. The above theorems yield some partial results on this open problem in $\mathbb{CH}^{m+1}$ and $\mathbb{CP}^{m+1}$. Recall that a hypersurface Σ in a Kahler manifold M is called Hopf if T is an eigenvector of the shape operator A at every point of Σ, i.e. $AT = \alpha T$ with $\alpha = \Pi(T, T)$. It is a well known fact that for a Hopf hypersurface in $\mathbb{CH}^{m+1}$ or in $\mathbb{CP}^{m+1}$ the function α is constant. We can prove the following results.

Theorem E. *Let Σ be a closed, embedded hypersurface in the complex hyperbolic space $\mathbb{CH}^{m+1}$ with constant mean curvature. If Σ is Hopf, then it is a geodesic sphere.*

Theorem F. *The same result holds for $\mathbb{CP}^{m+1}$ provided that Σ is disjoint from a hyperplane in $\mathbb{CP}^{m+1}$.*

These results were established by Miquel [M] under an extra condition. More precisely, he needs to assume $\alpha \geq 2\coth\left(2\coth^{-1}(H_b/2m)\right)$

in the case of $\mathbb{CH}^{m+1}$ and $\alpha \geq 2\cot\left(2\cot^{-1}\left(H_b/2m\right)\right)$ in the case of $\mathbb{CP}^{m+1}$.

The paper is organized as follows. In Section 2 we prove the integral identity. In Section 3 we discuss holomorphic extension of CR functions and prove Theorem A. In Section 4 we present some geometric applications, including Theorem B and Theorem C. In Section 5 we discuss real hypersurfaces in $\mathbb{CH}^{m+1}$ and $\mathbb{CP}^{m+1}$. Finally Theorems D, E and F are proved in Section 6.

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2. An integral formula on a domain in a Kahler manifold

Let M be a Kahler manifold of complex dimension $m+1 \geq 2$. We denote the metric (extended as a complex bilinear form on the complexified tangent bundle) by $\langle \cdot, \cdot \rangle$ and the Levi-Civita connection by ∇. Let $\Omega \subset M$ be a precompact domain with smooth boundary. The boundary Σ is endowed with the induced metric and its outer unit normal is denoted by ν. The Levi-Civita connection on Σ will be denoted by ∇^Σ. The shape operator $A : T\Sigma \to T\Sigma$ and the second fundamental form Π of Σ are defined in the usual way: for $u, v \in T\Sigma$,

$$Au = \nabla_u \nu,$$
$$\Pi(u, v) = \langle Au, v \rangle .$$

Note that $T := J\nu$ is a unit tangent vector field on Σ. Denote $Z = \left(\nu - \sqrt{-1}T\right)/\sqrt{2}$.

There is a canonical CR structure on Σ: the distribution $\mathcal{H} = \{u \in T\Sigma : \langle u, T \rangle = 0\}$ is invariant under the complex structure J and therefore $\mathcal{H} \otimes \mathbb{C} = T^{1,0}\Sigma \oplus T^{0,1}\Sigma$, where

$$T^{1,0}\Sigma = \{u - \sqrt{-1}Ju : u \in \mathcal{H}\}, T^{0,1}\Sigma = \overline{T^{1,0}\Sigma}.$$

We consider on Σ the 1-form $\theta = \langle T, \cdot \rangle$. Its Levi form L is the Hermitian symmetric form on $T^{1,0}\Sigma$ defined by $L(X, Y) = -\sqrt{-1}d\theta\left(X, \overline{Y}\right)$. A simple calculation yields

$$L(X, Y) = 2\Pi\left(X, \overline{Y}\right).$$

Recall that Σ is strictly pseudoconvex if the Levi form is positive definite on $T^{1,0}\Sigma$. We denote by H_b the trace of L with respect to the Hermitian metric on $T^{1,0}\Sigma$. A simple calculation shows that

$$H_b = H - \Pi(T, T),$$

where H is the mean curvature of Σ. In other words, H_b is the trace of Π on the contact distribution $\mathcal{H}$. (The notation H_b is in analogy with the sub-Laplacian Δ_b in CR geometry, which equals the trace of a Riemannian Hessian on the contact distribution $\mathcal{H}$.)

For $F \in C^\infty\left(\overline{\Omega}\right)$ we denote by $D^{1,1}F$ its complex Hessian and $\Box F = -\overline{\partial}^*\overline{\partial} F$ its complex Laplacian (which is half of the real Laplacian). Let $\overline{\partial}_b : C^\infty(\Sigma) \to \mathcal{A}^{0,1}(\Sigma)$ be the tangential Cauchy-Riemann operator, $\overline{\partial}_b^* : \mathcal{A}^{0,1}(M) \to C^\infty(\Sigma)$ its dual and $\Box_b = -\overline{\partial}_b^*\overline{\partial}_b$ the Kohn Laplacian. In doing computations we always work with a local unitary frame $\{X_i : 0 \le i \le m\}$ for $T^{1,0}M$. We write

$$F_i = X_i F, F_{\overline{i}} = \overline{X_i} F, F_{i,\overline{j}} = X_i \overline{X_j} f - \nabla_{X_i}\overline{X_j} F$$

etc. We will implicitly use the following rules

$$\begin{aligned} F_{i,\overline{j}} &= F_{\overline{j},i}, F_{i,\overline{jk}} = F_{i,\overline{kj}}, \\ F_{i,\overline{j}k} &= F_{i,k\overline{j}} + R_{k\overline{j}i\overline{l}} F_l. \end{aligned}$$

Along Σ we may and will assume that $X_0 = Z$. Then $\{X_\alpha : 1 \le \alpha \le m\}$ is a local unitary frame for $T^{1,0}\Sigma$. In the following Greek indices range from 1 to m while Latin letters run form 0 to m.

Lemma 1. *For $f \in C^\infty(\Sigma)$ we have*

$$\Box_b f = X_\alpha \overline{X_\alpha} f - \left\langle \nabla^\Sigma_{X_\alpha} \overline{X_\alpha}, X_\beta \right\rangle \overline{X_\beta} f + \sqrt{-1}\Pi(T, X_\alpha) \overline{X_\alpha} f.$$

Proof. For $f, g \in C^\infty(\Sigma)$ we have

$$\left\langle \overline{\partial}_b f, \overline{\overline{\partial}_b g} \right\rangle = f_{\overline{\alpha}} \overline{g}_\alpha = X_\alpha \left(f_{\overline{\alpha}} \overline{g}\right) - \left(X_\alpha f_{\overline{\alpha}}\right) \overline{g}.$$

Consider the vector field $\Xi = \phi_{\overline{\alpha}} X_\alpha$ with $\phi_{\overline{\alpha}} = f_{\overline{\alpha}} \overline{g}$. We compute

$$\begin{aligned} \mathrm{div}\Xi &= \left\langle \nabla^\Sigma_{X_\beta}\Xi, \overline{X_\beta} \right\rangle + \left\langle \nabla^\Sigma_T \Xi, T \right\rangle \\ &= X_\alpha \phi_{\overline{\alpha}} - \left\langle \Xi, \nabla^\Sigma_{X_\beta}\overline{X_\beta} \right\rangle - \left\langle \Xi, \nabla_T T \right\rangle \\ &= X_\alpha \phi_{\overline{\alpha}} - \left\langle \nabla_{X_\beta}\overline{X_\beta}, X_\alpha \right\rangle \phi_{\overline{\alpha}} - \left\langle J\nabla_T \nu, X_\alpha \right\rangle \phi_{\overline{\alpha}} \\ &= X_\alpha \phi_{\overline{\alpha}} - \left\langle \nabla_{X_\beta}\overline{X_\beta}, X_\alpha \right\rangle \phi_{\overline{\alpha}} + \sqrt{-1}\Pi(T, X_\alpha) \phi_{\overline{\alpha}}. \end{aligned}$$

Therefore

$$\begin{aligned} \left\langle \overline{\partial}_b f, \overline{\overline{\partial}_b g} \right\rangle &= \mathrm{div}\Xi - \left[\left(X_\alpha f_{\overline{\alpha}}\right) \overline{g} - \left\langle \nabla_{X_\beta}\overline{X_\beta}, X_\alpha \right\rangle \phi_{\overline{\alpha}} + \sqrt{-1}\Pi(T, X_\alpha) \phi_{\overline{\alpha}} \right] \\ &= \mathrm{div}\Xi - \left[\left(X_\alpha f_{\overline{\alpha}}\right) - \left\langle \nabla_{X_\beta}\overline{X_\beta}, X_\alpha \right\rangle f_{\overline{\alpha}} + \sqrt{-1}\Pi(T, X_\alpha) f_{\overline{\alpha}} \right] \overline{g} \end{aligned}$$

Integrating by parts yields

$$\int_\Sigma \left\langle \overline{\partial}_b f, \overline{\overline{\partial}_b g} \right\rangle = -\int_\Sigma \left[X_\alpha f_{\overline{\alpha}} - \left\langle \nabla_{X_\beta}\overline{X_\beta}, X_\alpha \right\rangle f_{\overline{\alpha}} + \sqrt{-1}\Pi(T, X_\alpha) f_{\overline{\alpha}} \right] \overline{g}.$$

□

Lemma 2. *We have* $\operatorname{div} T = 0$.

Proof. Since T is of unit length, $\left\langle \nabla^{\Sigma}_{T} T, T \right\rangle = 0$. We compute

$$\begin{aligned}
\operatorname{div} T &= \left\langle \nabla^{\Sigma}_{X_\alpha} T, \overline{X_\alpha} \right\rangle + \left\langle \nabla^{\Sigma}_{\overline{X_\alpha}} T, X_\alpha \right\rangle + \left\langle \nabla^{\Sigma}_{T} T, T \right\rangle \\
&= \left\langle \nabla_{X_\alpha} T, \overline{X_\alpha} \right\rangle + \left\langle \nabla_{\overline{X_\alpha}} T, X_\alpha \right\rangle \\
&= \left\langle J \nabla_{X_\alpha} \nu, \overline{X_\alpha} \right\rangle + \left\langle J \nabla_{\overline{X_\alpha}} \nu, X_\alpha \right\rangle \\
&= - \left\langle \nabla_{X_\alpha} \nu, J \overline{X_\alpha} \right\rangle - \left\langle \nabla_{\overline{X_\alpha}} \nu, J X_\alpha \right\rangle \\
&= \sqrt{-1} \left(\left\langle \nabla_{X_\alpha} \nu, \overline{X_\alpha} \right\rangle - \left\langle \nabla_{\overline{X_\alpha}} \nu, X_\alpha \right\rangle \right) \\
&= \sqrt{-1} \left(\Pi \left(X_\alpha, \overline{X_\alpha} \right) - \Pi \left(\overline{X_\alpha}, X_\alpha \right) \right) \\
&= 0.
\end{aligned}$$

□

For later purposes we compare $\Box_b f$ with $\Delta_\Sigma f$.

Proposition 1. *For* $f \in C^\infty(\Sigma)$ *we have*

$$2 \Box_b f = \Delta_\Sigma f - D^2 f (T, T) + \sqrt{-1} \left[2 \Pi (T, X_\alpha) \overline{X_\alpha} f - H_b T f \right]. \tag{2.1}$$

Proof. From the previous lemma we have

$$\begin{aligned}
\Box_b f &= X_\alpha \overline{X_\alpha} f - \nabla^{\Sigma}_{X_\alpha} \overline{X_\alpha} f + \left\langle \nabla^{\Sigma}_{X_\alpha} \overline{X_\alpha}, T \right\rangle T f + \sqrt{-1} \Pi (T, X_\alpha) \overline{X_\alpha} f \\
&= D^2 f \left(X_\alpha, \overline{X_\alpha} \right) - \left\langle \nabla^{\Sigma}_{X_\alpha} T, \overline{X_\alpha} \right\rangle T f + \sqrt{-1} \Pi (T, X_\alpha) \overline{X_\alpha} f \\
&= D^2 f \left(X_\alpha, \overline{X_\alpha} \right) - \sqrt{-1} \Pi \left(X_\alpha, \overline{X_\alpha} \right) T f + \sqrt{-1} \Pi (T, X_\alpha) \overline{X_\alpha} f \\
&= \frac{1}{2} \left(\Delta_\Sigma f - D^2 f (T, T) \right) - \frac{\sqrt{-1}}{2} H_b T f + \sqrt{-1} \Pi (T, X_\alpha) \overline{X_\alpha} f.
\end{aligned}$$

This yields the desired identity. □

We now state again our integral formula in a Kahler manifold.

Theorem 1. *For* $F \in C^\infty \left(\overline{\Omega} \right)$ *denote* $f = F|_\Sigma$. *Then*

$$\begin{aligned}
&\sqrt{2} \int_\Omega^2 |\Box F|^2 - \left| D^{1,1} F \right|^2 \\
= &\int_\Sigma \left[Z \overline{F} \left(\Box_b f - \sqrt{-1} \Pi (T, X_\alpha) f_{\overline{\alpha}} \right) + Z F \overline{\left(\Box_b f - \sqrt{-1} \Pi (T, X_\alpha) f_{\overline{\alpha}} \right)} \right] \\
&+ \sqrt{2} \int_\Sigma \Pi \left(\overline{X_\alpha}, \overline{X_\beta} \right) f_\alpha \overline{f}_\beta + \frac{1}{\sqrt{2}} \int_\Sigma H_b \left| \overline{Z} F \right|^2.
\end{aligned}$$

Proof. Working with a local unitary frame, we have

$$
\begin{aligned}
\left|D^{1,1}F\right|^2 - |\Box F|^2 &= F_{i\bar{j}}\overline{F}_{\bar{i}j} - \Box F\overline{F}_{j\bar{j}} \\
&= \left(F_{i\bar{j}}\overline{F}_j\right)_{\bar{i}} - F_{i\bar{j},\bar{i}}\overline{F}_j - \Box F\overline{F}_{j\bar{j}} \\
&= \left(F_{i\bar{j}}\overline{F}_j\right)_{\bar{i}} - (\Box F)_{\bar{j}}\overline{F}_j - \Box F\overline{F}_{j\bar{j}} \\
&= \left(F_{i\bar{j}}\overline{F}_j\right)_{\bar{i}} - \left(\Box F\overline{F}_j\right)_{\bar{j}}.
\end{aligned}
$$

Integrating by parts we obtain

$$
\begin{aligned}
\sqrt{2}\int_\Omega \left|D^{1,1}F\right|^2 - |\Box F|^2 &= \int_\Sigma D^2F\left(Z, \overline{X_j}\right)\overline{F}_j - \Box F\left(Z\overline{F}\right) \\
&= \int_\Sigma D^2F\left(Z, \overline{X_\alpha}\right)\overline{f}_\alpha - \left(\Box F - D^2F\left(Z, \overline{Z}\right)\right) Z\overline{F}.
\end{aligned}
$$

We now analyze the boundary terms carefully. We compute on Σ using Lemma 1

$$
\begin{aligned}
\Box F - D^2F\left(Z, \overline{Z}\right) &= D^2F\left(X_\alpha, \overline{X_\alpha}\right) \\
&= X_\alpha\overline{X_\alpha}F - \nabla_{X_\alpha}\overline{X_\alpha}F \\
&= X_\alpha\overline{X_\alpha}f - \left\langle\nabla_{X_\alpha}\overline{X_\alpha}, X_\beta\right\rangle\overline{X_\beta}f - \left\langle\nabla_{X_\alpha}\overline{X_\alpha}, Z\right\rangle\overline{Z}F \\
&= \Box_b f - \sqrt{-1}\Pi\left(T, X_\alpha\right)\overline{X_\alpha}f + \left\langle\nabla_{X_\alpha}Z, \overline{X_\alpha}\right\rangle\overline{Z}F \\
&= \Box_b f - \sqrt{-1}\Pi\left(T, X_\alpha\right)\overline{X_\alpha}f + \sqrt{2}\Pi\left(X_\alpha, \overline{X_\alpha}\right)\overline{Z}F \\
&= \Box_b f - \sqrt{-1}\Pi\left(T, X_\alpha\right)\overline{X_\alpha}f + \frac{H_b}{\sqrt{2}}\overline{Z}F
\end{aligned}
$$

We compute on Σ

$$
\begin{aligned}
D^2F\left(Z, \overline{X_\alpha}\right) &= \overline{X_\alpha}ZF - \nabla_{\overline{X_\alpha}}ZF \\
&= \overline{X_\alpha}ZF - \left\langle\nabla_{\overline{X_\alpha}}Z, \overline{Z}\right\rangle ZF - \left\langle\nabla_{\overline{X_\alpha}}Z, \overline{X_\beta}\right\rangle X_\beta f \\
&= \overline{X_\alpha}ZF - \sqrt{-1}\Pi\left(T, \overline{X_\alpha}\right)ZF - \sqrt{2}\Pi\left(\overline{X_\alpha}, \overline{X_\beta}\right)X_\beta f.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\sqrt{2}\int_{\Omega}\left|D^{1,1}F\right|^{2}-\left|\Box F\right|^{2}\\
=&\int_{\Sigma}\left(\overline{X_{\alpha}}ZF-\sqrt{-1}\Pi\left(T,\overline{X_{\alpha}}\right)ZF-\sqrt{2}\Pi\left(\overline{X_{\alpha}},\overline{X_{\beta}}\right)f_{\beta}\right)\overline{f}_{\alpha}\\
&+\int_{\Sigma}\left(-\Box_{b}f+\sqrt{-1}\Pi\left(T,X_{\alpha}\right)\overline{X_{\alpha}}f-\frac{H_{b}}{\sqrt{2}}\right)Z\overline{F}\\
=&\int_{\Sigma}\left(-Z\overline{F}\Box_{b}f-\overline{\Box_{b}f}ZF+\sqrt{-1}\Pi\left(T,X_{\alpha}\right)f_{\overline{\alpha}}Z\overline{F}-\sqrt{-1}\Pi\left(T,\overline{X_{\alpha}}\right)\overline{f}_{\alpha}ZF\right)\\
&-\sqrt{2}\int_{\Sigma}\Pi\left(\overline{X_{\alpha}},\overline{X_{\beta}}\right)\overline{f}_{\alpha}f_{\beta}-\frac{1}{\sqrt{2}}\int_{\Sigma}H_{b}\left|\overline{Z}F\right|^{2},
\end{aligned}
$$

where in the process we did integration by part on Σ. Reorganizing the terms yields (2.2). $\Box$

3. Holomorphic extension of CR functions

Let M be a complex manifold of complex dimension $m+1\geq 2$ and $\Sigma\subset M$ a real hypersurface. A function f on Σ is called CR if it satisfies the tangential Cauchy-Riemann equations, i.e. $\overline{X}f=0$ for all $X\in T^{1,0}\Sigma$. Obviously, a holomorphic function on a neighborhood of Σ restricts to a CR function on Σ. Conversely, it is an interesting question if all CR functions arise this way. If Σ encloses a domain Ω, one can also ask the global question: does a CR function on Σ extends to a holomorphic function on Ω?

On this problem, we have the following classic result (Theorem 2.3.2' in Hormander [H])

Theorem 2. *Let Ω be a smooth, bounded open set in $\mathbb{C}^{m+1}, m\geq 1$, s.t. $\mathbb{C}^{m+1}\backslash\overline{\Omega}$ is connected. If $u\in C^{\infty}\left(\partial\Omega\right)$ is a CR function, one can find a holomorphic function $U\in C^{\infty}\left(\overline{\Omega}\right)$ s.t. $U=u$ on $\partial\Omega$.*

In a general complex manifold, Kohn and Rossi [KR] proved the following

Theorem 3. *Let Ω be a precompact domain with smooth boundary in a complex manifold M^{m+1}. Suppose the boundary is connected and the Levi form on the boundary has one positive eigenvalue everywhere, then every CR function on $\partial\Omega$ has a holomorphic extension to Ω.*

Their approach is via the solution of the $\overline{\partial}$-Neumann problem, and particularly the regularity of solutions at the boundary.

Using the formula (2.2), we present an elementary new approach to this problem. The basic idea is very simple: given a CR function f on Σ one should try to prove its harmonic extension on Ω is holomorphic. Our approach works in any Kahler manifold with a mild pointwise condition on Σ. The result we can prove is weaker than the Kohn-Rossi theorem, but the method has the merit of being elementary. In particular we avoid the analytically sophisticated $\overline{\partial}$-Neumann problem.

Let M be a Kahler manifold of complex dimension $m+1 \geq 2$. Let $\Omega \subset M$ be a (connected) precompact domain with smooth boundary Σ. For simplicity we assume everything is smooth. But the optimal regularity required for the method to work should be obvious.

Theorem 4. *Suppose that Σ satisfies the following positivity condition*

$$H_b > 0. \tag{3.1}$$

The for any $f \in C^\infty(\Sigma)$ which is a CR function, there exists $F \in C^\infty\left(\overline{\Omega}\right) \cap \mathcal{O}(\Omega)$ s.t. $F|_\Sigma = f$.

Remark 1. *Condition (3.1) is much weaker than strict pseudoconvexity.*

Let $F \in C^\infty\left(\overline{\Omega}\right)$ be the harmonic extension of f. By the integral identity

$$\int_\Omega \left|D^{1,1}F\right|^2 = -\frac{1}{4}\int_\Sigma H_b\left|\overline{Z}F\right|^2.$$

Under the boundary condition we must have $D^{1,1}F = 0$ on Ω and $\overline{Z}F = 0$ on Σ. Integrating by parts we have

$$\begin{aligned}\int_\Omega \left|\overline{\partial}F\right|^2 &= \int_\Omega F_{\overline{j}}\overline{F}_j \\ &= \int_\Omega \left(F_{\overline{j}}\overline{F}\right)_j \\ &= \int_\Sigma \left(\overline{Z}F\right)\overline{f} \\ &= 0.\end{aligned}$$

Therefore F is holomorphic.

Corollary 1. *Under the same assumption, any holomorphic function on $M\backslash\overline{\Omega}$ extends to a holomorphic function on M.*

If we only assume that $H_b \geq 0$, then the argument above shows that the harmonic extension F is pluriharmoinc: $F_{i\overline{j}} = 0$. Is it possible to prove that F is holomorphic? Or equivalently, is Theorem 4 valid under the condition $H_b \geq 0$?

4. Geometric inequalities

As a first application of the integral formula (2.2), we prove the following

Proposition 2. *Let Ω be a connected precompact domain with smooth boundary Σ in a Kahler manifold. If Σ satisfies $H_b > 0$, then Σ is connected.*

Proof. Suppose Σ is not connected and let Σ_1 be a connected component and $\Sigma_2 = \Sigma \backslash \Sigma_1$. Let $f \in C^\infty(\Sigma)$ be the function that is 1 on Σ_1 and 0 on Σ_2. Let $u \in C^\infty\left(\overline{\Omega}\right)$ be the harmonic extension of f and $\chi = \frac{\partial u}{\partial \nu}$. Note that u is real. By the maximum principle and Hopf Lemma,

$$0 < u < 1 \text{ on } \Omega; \ \chi > 0 \text{ on } \Sigma_1; \ \chi < 0 \text{ on } \Sigma_2.$$

By (2.2) we have

$$\int_\Omega \left|D^{1,1}u\right|^2 = -\frac{1}{4}\int_\Sigma H_b \chi^2 < 0,$$

a contradiction. □

Remark 2. *It is interesting to compare our Proposition with the following classic fact in Riemannian geometry: a compact connected Riemannian manifold with mean convex boundary and nonnegative Ricci curvature has at most two boundary components; moreover if ∂M has two components, then M is isometric to a cylinder $N \times [0, a]$ over some connected closed Riemannian manifold N with nonnegative Ricci curvature (cf.* [I]*). Note that in the Kahler case we do not impose any curvature assumption on Ω. The proof here is similar to the proof in* [HW] *of the aforementioned fact in Riemannian geometry using Reilly's formula.*

We now consider $F \in C^\infty\left(\overline{\Omega}\right)$ which is the solution of the following boundary value problem

$$\begin{cases} \Box F = (m+1) & \text{on } \overline{\Omega}, \\ F = 0 & \text{on } \Sigma. \end{cases} \tag{4.1}$$

Note that F is real. Denote $\chi = \frac{\partial F}{\partial \nu}$. By the strong maximum principle and the Hopf Lemma

$$F < 0 \text{ on } \Omega; \ \chi > 0 \text{ on } \Sigma.$$

Theorem 5. *Let $\Omega \subset M$ be a connected precompact domain with smooth boundary Σ. If $H_b > 0$ on Σ, then*

$$\int_\Sigma \frac{1}{H_b} \geq \frac{m+1}{m} |\Omega| . \tag{4.2}$$

Proof. Integrating the equation (4.1) yields

$$\begin{aligned}(m+1)|\Omega| &= \frac{1}{2}\int_\Sigma \chi \\ &\leq \frac{1}{2}\left(\int_\Sigma H_b\chi^2\right)^{1/2}\left(\int_\Sigma \frac{1}{H_b}\right)^{1/2}.\end{aligned}$$

Thus

$$\frac{1}{4}\int_\Sigma H_b\chi^2 \geq (m+1)^2 |\Omega|^2 \Big/ \int_\Sigma \frac{1}{H_b}. \tag{4.3}$$

By the integral identity (2.2) applied to F (noting $\overline{Z}F = \frac{1}{\sqrt{2}}H_b$) we obtain

$$\begin{aligned}\frac{1}{4}\int_\Sigma H_b\chi^2 &= \int_\Omega |\Box F|^2 - \left|D^{1,1}F\right|^2 \\ &\leq \left(1-\frac{1}{m+1}\right)\int_\Omega |\Box F|^2 \\ &= (m+1)\, m\, |\Omega| .\end{aligned}$$

Combined with (4.3) this implies

$$\int_\Sigma \frac{1}{H_b} \geq \frac{m+1}{m} |\Omega| .$$

□

Remark 3. *The theorem and its proof are similar to the following result of Ros* [R]*:*

Let (M^n, g) be a compact Riemannian manifold with boundary. If $Ric \geq 0$ and the mean curvature H of ∂M is positive, then

$$\int_{\partial M} \frac{1}{H} d\sigma \geq \frac{n}{n-1} V.$$

The equality holds iff M is isometric to an Euclidean ball.

But Theorem 5 is valid in any Khaler manifold: there is no curvature assumption on Ω.

Remark 4. *It is clear from the proof that if equality holds in (4.2) we must have $D^{1,1}F = I$ and $H_b\chi = a$ (a constant) on Σ. In fact the first identity implies the second (see below). Therefore on any Kahler manifold if F is a (local) Kahler potential and c a regular value for F s.t. $\{F \leq c\}$ is compact, then for $\Omega = \{F < c\}$ we have equality in (4.2).*

As an immediate corollary, we obtain the following isoperimetric inequality in terms of a positive lower bound for H_b.

Theorem 6. *Let $\Omega \subset M$ be a connected precompact domain with smooth boundary Σ. Let $c = \inf_\Sigma H_b$. If $c > 0$, then*

$$m\,|\Sigma| \geq c\,(m+1)\,|\Omega|\,.$$

We now analyze the equality case in Theorem 6. From the proof we must have

(1) $D^{1,1}F = I$, i.e. for $(1,0)$-vectors $X\,, Y$

$$D^2F\left(X,\overline{Y}\right) = \left\langle X,\overline{Y}\right\rangle. \tag{4.4}$$

(2) $H_b \equiv c$ and χ is a positive constant.

Lemma 3. *On Σ we have $\chi c = 2m$ and for $X\,, Y \in T^{1,0}\Sigma$*

$$\begin{aligned}\Pi\left(X,\overline{Y}\right) &= \frac{c}{2m}\left\langle X,\overline{Y}\right\rangle,\\ \Pi\left(T,X\right) &= 0.\end{aligned}$$

Proof. Working with a unitary frame along Σ we have by (4.4)

$$\begin{aligned}\delta_{\alpha\beta} &= F_{\alpha\overline{\beta}}\\ &= X_\alpha\overline{X_\beta}F - \left(\nabla_{X_\alpha}\overline{X_\beta}\right)F\\ &= \left\langle\nabla_{X_\alpha}\nu,\overline{X_\beta}\right\rangle\chi\\ &= \Pi\left(X_\alpha,\overline{X_\beta}\right)\chi,\end{aligned}$$

as $F|_\Sigma = 0$. Therefore $\Pi\left(X_\alpha, \overline{X_\beta}\right) = \frac{1}{\chi}\delta_{\alpha\beta}$. Taking trace yields $\chi c = 2m$. Similarly, as χ is constant

$$\begin{aligned}
0 &= F_{\alpha\bar{0}} \\
&= X_\alpha \overline{Z} F - \left(\nabla_{X_\alpha}\overline{Z}\right) F \\
&= X_\alpha \chi - \left\langle \nabla_{X_\alpha}\overline{Z}, \nu\right\rangle \chi \\
&= \left\langle \nabla_{X_\alpha}\nu, \overline{Z}\right\rangle \chi \\
&= \frac{\sqrt{-1}}{\sqrt{2}}\Pi\left(X_\alpha, T\right)\chi.
\end{aligned}$$

This implies $\Pi\left(X_\alpha, T\right) = 0$. □

Remark 5. *Since $c > 0$ the 1st identity implies that Σ is strictly pseudoconvex.*

Let A be the shape operator on Σ, i.e. $A : T\Sigma \to T\Sigma$ is the symmetric endomorphism defined by $Av = \nabla_v \nu$. We have $\Pi(u, v) = \langle Au, v\rangle$. In the first identity if we take $X = u - \sqrt{-1}Ju, Y = v - \sqrt{-1}Jv$, we obtain for any $u, v \in \mathcal{H}$

$$\langle Au, v\rangle + \langle AJu, Jv\rangle = \frac{c}{m}\langle u, v\rangle .$$

In other words, restricted on $\mathcal{H}$ we have

$$A - JAJ = \frac{c}{m} I. \tag{4.5}$$

The second identity in the above lemma implies $AT = \alpha T$, where $\alpha = \Pi(T, T)$.

Definition 1. *A hypersurface Σ in a complex manifold is called a Hopf hypersurface if T is an eigenvector of the shape operator at every point of Σ.*

Such hypersurfaces have been studied intensively in $\mathbb{CP}^{m+1}$ and $\mathbb{CH}^{m+1}$, cf. Niebergall-Ryan [NR] for a detailed survey of the subject. We will have further discussion on Hopf hypersurfaces in the next section.

5. Hopf hypersurfaces in $\mathbb{CH}^{m+1}$ and $\mathbb{CP}^{m+1}$

In this Section we discuss hypersurfaces in $\mathbb{CP}^{m+1}$ and $\mathbb{CH}^{m+1}$. For basic facts on $\mathbb{CP}^{m+1}$ and $\mathbb{CH}^{m+1}$ we refer to [KN]. We will take an intrinsic approach. The starting point is that they are the unique simply connected, Kahler manifold of constant holomorphic sectional curvature 4κ, with $\kappa = -1$ for $\mathbb{CH}^{m+1}$ and $\kappa = 1$ for $\mathbb{CP}^{m+1}$. It follows that the sectional curvature of a J-invariant 2-plane is 4κ while the sectional curvature of a totally real 2-plane is κ. The curvature tensor is explicitly given by

(5.1)
$$\begin{aligned} R\left(v_1, v_2, v_3, v_4\right) = \kappa \left[\left\langle v_1, v_3\right\rangle \left\langle v_2, v_4\right\rangle - \left\langle v_1, v_4\right\rangle \left\langle v_2, v_3\right\rangle + \left\langle v_1, Jv_3\right\rangle \left\langle v_2, Jv_4\right\rangle \right. \\ \left. - \left\langle v_1, Jv_4\right\rangle \left\langle v_2, Jv_3\right\rangle + 2\left\langle v_1, Jv_2\right\rangle \left\langle v_3, Jv_4\right\rangle\right]. \end{aligned}$$

Equivalently, for $(1,0)$-vectors X, Y, Z, W

$$R\left(X, \overline{Y}, Z, \overline{W}\right) = -2\kappa\left[\left\langle X, \overline{Y}\right\rangle \left\langle Z, \overline{W}\right\rangle - \left\langle X, \overline{W}\right\rangle \left\langle Z, \overline{Y}\right\rangle\right].$$

We now focus on $\mathbb{CH}^{m+1}$. Fix a point $o \in \mathbb{CH}^{m+1}$ and let r be the distance function to o. The Hessian of r is given by

$$\begin{aligned} \nabla_{\nabla r}\nabla r = 0, \nabla_{J\nabla r}\nabla r = 2\frac{\cosh 2r}{\sinh 2r}J\nabla r, \\ \nabla_v \nabla r = \frac{\cosh r}{\sinh r}v, \text{ if } \left\langle v, \nabla r\right\rangle = \left\langle v, J\nabla r\right\rangle = 0. \end{aligned} \tag{5.2}$$

From (5.2) it is also clear that on a geodesic sphere of radius $a > 0$ the shape operator A has the following form

$$AT = 2\frac{\cosh 2a}{\sinh 2a}T, A|_{\mathcal{H}} = \frac{\cosh a}{\sinh a}I.$$

Let $\Phi = \log\cosh r$. It is a smooth function on $\mathbb{CH}^{m+1}$. A straightforward calculation shows that $D^{(1,1)}\Phi = I$, i.e. for any $(1,0)$-vectors X, Y

$$D^2\Phi\left(X, \overline{Y}\right) = \left\langle X, \overline{Y}\right\rangle.$$

In particular $\Box\Phi = m+1$.

Remark 6. *The existence of F is also clear from the ball model of $\mathbb{CH}^{m+1}$: the unit ball in $\mathbb{C}^{m+1}$ with the Kahler form*

$$\omega = -\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\left(1 - |z|^2\right).$$

As a simple consequence of the existence of F, we prove the following Minkowski type formula (cf. [M], where it is derived in a different way).

Proposition 3. *Suppose Σ is a closed Hopf hypersurface in $\mathbb{CH}^{m+1}$. Then*

$$2m\left|\Sigma\right| = \int_\Sigma H_b\left\langle \nabla\Phi, \nu\right\rangle. \tag{5.3}$$

Proof. Let $\Phi = \Phi|_\Sigma, \chi = \frac{\partial\Phi}{\partial\nu} = \left\langle\nabla\Phi, \nu\right\rangle$. Then

$$2(m+1) = \Delta\Phi|_\Sigma = \Delta_\Sigma\Phi + H\chi + D^2\Phi\left(\nu, \nu\right).$$

For $Z = \left(\nu - \sqrt{-1}T\right)/\sqrt{2}$ we have

$$1 = D^2\Phi\left(Z, \overline{Z}\right) = \frac{1}{2}\left[D^2\Phi\left(\nu,\nu\right) + D^2\Phi\left(T,T\right)\right].$$

On the other hand

$$\begin{aligned} D^2\Phi\left(T,T\right) &= TT\Phi - \left(\nabla_T T\right)\Phi \\ &= TT\Phi - \left(\nabla_T^\Sigma T\right)\Phi + \Pi\left(T,T\right)\chi \\ &= D^2\Phi\left(T,T\right) + \Pi\left(T,T\right)\chi. \end{aligned}$$

Combining these identities yields

$$2m = \Delta_\Sigma\Phi - D^2\Phi\left(T,T\right) + H_b\chi.$$

Integrating over Σ yields

$$2m\left|\Sigma\right| = \int_\Sigma H_b\left\langle\nabla\Phi,\nu\right\rangle - \int_\Sigma D^2\Phi\left(T,T\right).$$

Integrating the identity (2.1) yields

$$\int_\Sigma D^2\Phi\left(T,T\right) = 2\operatorname{Re}\int_\Sigma \sqrt{-1}\Pi\left(T,X_\alpha\right)\overline{X_\alpha}\Phi.$$

Since Σ is Hopf, the right hand side is clearly zero. Thus we obtain (5.3). ☐

Remark 7. *On $\mathbb{C}^{m+1}$ consider $\Phi = \left|z\right|^2/2$. First recall the classic Minkowski formula: If $\Sigma \subset \mathbb{C}^{m+1}$ is a closed hypersurface then*

$$\left(2m+1\right)\left|\Sigma\right| = \int_\Sigma H\left\langle\nabla\Phi,\nu\right\rangle$$

By the same argument used to prove Proposition (3) we have the following formula

$$2m\left|\Sigma\right| = \int_\Sigma H_b\left\langle\nabla\Phi,\nu\right\rangle,$$

provided that Σ is Hopf.

We now discuss the case of $\mathbb{CP}^{m+1}$ endowed with the Fubini-Study metric normalized to have holomorphic sectional curvature 4. The diameter is $\pi/2$. For any $o = [\xi] \in \mathbb{CP}^{m+1}$ ($\xi \in \mathbb{C}^{m+2}\setminus\{0\}$) the set of points whose distance to o equals $\pi/2$ is also its cut-locus

$$C\left(o\right) := \left\{[\zeta] \in \mathbb{CP}^{m+1} : \left\langle\zeta,\overline{\xi}\right\rangle = 0\right\},$$

which is a totally geodesic $\mathbb{CP}^m$, called a hyperplane. On the domain

$$\mathbb{CP}^{m+1}\backslash C\left(o\right) = \left\{p \in \mathbb{CP}^{m+1} : d\left(o,p\right) < \pi/2\right\}$$

the function $\Phi = \log \cos r$ is smooth and satisfies

$$D^2\Phi\left(X, \overline{Y}\right) = \left\langle X, \overline{Y}\right\rangle.$$

for any $(1,0)$-vectors X, Y. Therefore $\Delta\Phi = 2\Box F = 2(m+1)$.

Therefore as in the complex hyperbolic case we have

Proposition 4. *Suppose Σ is a closed Hopf hypersurface in $\mathbb{CH}^m$. If there exists $o \in \mathbb{CP}^{m+1}$ s.t. $\Sigma \cap C(o) = \varnothing$, then*

$$2m|\Sigma| = \int_\Sigma H_b \langle \nabla\Phi, \nu\rangle.$$

Again, this formula was first proved by Miquel [M].

Similar to the complex hyperbolic case, on a geodesic sphere of radius $a \in (0, \pi/2)$ in $\mathbb{CP}^{m+1}$ the shape operator A has the following form

$$AT = 2\frac{\cos 2a}{\sin 2a}T, A|_{\mathcal{H}} = \frac{\cos a}{\sin a}I.$$

More generally let Σ be a tube over a totally geodesic $\mathbb{CP}^k$, i.e.

$$\Sigma = \left\{p \in \mathbb{CP}^{m+1} : d\left(p, \mathbb{CP}^k\right) = a\right\}$$

for some $a \in (0, \pi/2)$. Then the shape operator A has three eigenvalues:

- $\lambda_1 = 2\cos 2a/\sin 2a$ of multiplicity 1,
- $\lambda_2 = \cos a/\sin a$ of multiplicity $2(m-k)$,
- $\lambda_3 = -\sin a/\cos a$ of multiplicity $2k$.

6. Uniqueness results in $\mathbb{CH}^{m+1}$ and $\mathbb{CP}^{m+1}$

We now prove the following rigidity result.

Theorem 7. *Let M be a simply connected Kahler manifold of constant holomorphic sectional curvature 4κ with $\dim_{\mathbb{C}} M = m+1 \geq 2$. In other words $M = \mathbb{CP}^{m+1}$ if $\kappa = 1$; $M = \mathbb{CH}^{m+1}$ if $\kappa = -1$. Let $\Omega \subset M$ be a connected precompact domain with smooth boundary Σ. Let $c = \inf_\Sigma H_b$. If equality holds in Theorem 6, i.e. $m|\Sigma| = c(m+1)|\Omega|$, then Ω is a geodesic ball.*

Suppose Σ has constant Hermitian mean curvature c. Let Ω be the domain enclosed by Σ (in the case of $\mathbb{CP}^{m+1}$ we choose Ω to be the one disjoint from the hyperplane in the assumption). By Proposition 3 or Proposition 4, Since $m|\Sigma| = c(m+1)|\Omega|$, we have $c > 0$. By the discussion in Section 4, we know that there exists $F \in C^\infty\left(\overline{\Omega}\right)$ s.t. $F = 0$ on Σ and $D^{1,1}F = I$. Moreover Σ is a Hopf hypersurface,

$$AT = \alpha T \tag{6.1}$$

and its shape operator satisfies on $\mathcal{H}$

$$A - JAJ = \frac{c}{m} I. \tag{6.2}$$

Lemma 4. *The function α is constant.*

Proof. This is a well know fact on Hopf hypersurfaces in complex spaces forms, cf. [NR]. We provide a direct proof. Let X be a tangential vector field on Σ s.t. $\langle X, T\rangle = 0$. Differentiating (6.1) yields

$$\begin{aligned} X\alpha &= \langle \nabla_X \nabla_T \nu, T\rangle \\ &= \langle \nabla_T \nabla_X \nu, T\rangle - \left\langle \nabla_{[T,X]}\nu, T\right\rangle + R\left(T, X, \nu, T\right). \end{aligned}$$

From (6.1) we obtain

$$\begin{aligned} \nabla_T T &= J\nabla_T v \\ &= JAT \\ &= -\alpha\nu. \end{aligned}$$

Thus

$$\begin{aligned} \langle \nabla_T \nabla_X \nu, T\rangle &= T\left\langle \nabla_X \nu, T\right\rangle - \langle \nabla_X \nu, \nabla_T T\rangle \\ &= T\Pi\left(T, X\right) - \alpha\left\langle \nabla_X \nu, \nu\right\rangle \\ &= 0. \end{aligned}$$

On the other hand

$$\begin{aligned} \left\langle \nabla_{[T,X]}\nu, T\right\rangle &= \langle \nabla_T \nu, [T, X]\rangle \\ &= \alpha\left\langle T, [T, X]\right\rangle \\ &= \alpha\left\langle T, \nabla_T X - \nabla_X T\right\rangle \\ &= \alpha\left\langle T, \nabla_T X\right\rangle \\ &= \alpha\left(T\left\langle X, T\right\rangle - \langle \nabla_T T, X\rangle\right) \\ &= 0. \end{aligned}$$

By the formula for curvature, $R\left(T, X, \nu, T\right) = 0$. Therefore $X\alpha = 0$.

Since Σ is strictly pseudoconvex, $T\alpha = 0$ as well. Therefore α is constant. $\square$

For further discussion, we need the following fact on Hopf hypersurfaces in complex space forms (Lemma 2.2. in Niebergall and Ryan [NR]). Let $\varphi : T\Sigma \to T\Sigma$ be the endomorphism s.t. $\varphi T = 0$ and $\varphi|_{\mathcal{H}} = J$.

Lemma 5. *Let Σ be a Hopf hypersurface in a Kahler manifold M of constant holomorphic sectional curvature κ. Then*

$$A\varphi A - \frac{a}{2}\left(A\varphi + \varphi A\right) = \kappa\varphi. \tag{6.3}$$

This is in fact a simple consequence of the Codazzi equation. We sketch the proof here. By direct calculation, for any tangent vector $\left(\nabla_X^\Sigma A\right) T = (aI - A)\,\varphi AX$. Thus

$$\left\langle \left(\nabla_X^\Sigma A\right) Y, T \right\rangle = \left\langle \left(\nabla_X^\Sigma A\right) T, Y \right\rangle = \left\langle (aI - A)\,\varphi AX, Y \right\rangle.$$

By the Codazzi equation

$$\begin{aligned} \left\langle (aI - A)\,\varphi AX, Y \right\rangle - \left\langle (aI - A)\,\varphi AY, X \right\rangle &= \left\langle \left(\nabla_X^\Sigma A\right) Y, T \right\rangle - \left\langle \left(\nabla_Y^\Sigma A\right) X, T \right\rangle \\ &= R\,(X, Y, T, \nu) \\ &= 2\kappa \left\langle X, \varphi Y \right\rangle, \end{aligned}$$

where in the last step we used the curvature formula (5.1). The identity (6.3) follows easily.

Suppose $u \in \mathcal{H}$ is an eigenvector of A, $Au = \lambda u$. From (6.2) we easily obtain $AJu = \left(\frac{c}{m} - \lambda\right) Ju$, i.e. Ju is also an eigenvector with eigenvalue $\frac{c}{m} - \lambda$. Applying (6.3) yields

$$\lambda \left(\frac{c}{m} - \lambda\right) = \frac{\alpha c}{2m} + \kappa. \tag{6.4}$$

This means that besides α the principal curvatures of Σ can only take at most two values, the two roots λ and $\lambda^* = \frac{c}{m} - \lambda$ of the quadratic equation

$$x \left(\frac{c}{m} - x\right) = \frac{\alpha c}{2m} + \kappa.$$

Therefore Σ is a Hopf hypersurface with constant principal curvatures. Such hypersurfaces in $\mathbb{CP}^{m+1}$ and $\mathbb{CH}^{m+1}$ are completely classified (even locally) by Kimura [K] and Berndt [B]. We could finish the proof of Theorem 7 by doing a case by case analysis of the classification list. But there is a more direct approach which avoids using the classification. All we need is a fundamental formula from [B]. (Except this formula our proof is self-contained.)

There are two possibilities:

(1) The two roots coincide $\lambda = \lambda^*$.
  In this case we have $\lambda = c/2m$ and $A = \lambda I$ on $\mathcal{H}$.
(2) The two roots are different $\lambda \neq \lambda^*$.
  In this case we have an orthogonal decomposition $\mathcal{H} = E \oplus JE$, where E is a real subspace of dimension m and with respect to this decomposition A is given by the matrix

$$\begin{bmatrix} \lambda I & 0 \\ 0 & \lambda^* I \end{bmatrix}.$$

By the fundamental formula in Berndt [B, Theorem 2] we must have

$$\lambda\lambda^{*}+\kappa=0. \tag{6.5}$$

For further discussion we discuss the two cases $\mathbb{CH}^{m+1}$ and $\mathbb{CP}^{m+1}$ separately.

- $M=\mathbb{CH}^{m+1}$

Lemma 6. *$\alpha>2$ and $\lambda,\lambda^{*}>1$.*

Proof. This is a simple comparison. Let o be a point enclosed by Σ and ρ the distance function to o on $\mathbb{CH}^{m+1}$. Let $p\in\Sigma$ be a farthest point on Σ to o and $a=d(o,p)$. Then at p we have $\nu=\nabla\rho$ and for any $X\in T_{p}\Sigma$

$$\Pi(X,X)\geq D^{2}\rho(X,X).$$

By (5.2) we have by taking either $X=T$ or $u\in\mathcal{H}$ in the above inequality

$$\begin{aligned}\alpha&\geq 2\frac{\cosh 2a}{\sinh 2a},\\ A&\geq\frac{\cosh a}{\sinh a}I\text{ on }\mathcal{H}.\end{aligned}$$

The second inequality implies that λ and λ^{*} are at least $\frac{\cosh a}{\sinh a}>1$. $\square$

Lemma 7. *Let $r>0$ be the number s.t. $\alpha=2\frac{\cosh 2r}{\sinh 2r}$. Then $A=\frac{\cosh r}{\sinh r}I$ on $\mathcal{H}$.*

Proof. If $\lambda\neq\lambda^{*}$ we would have by (6.5) $\lambda\lambda^{*}=1$. But by the previous Lemma all eigenvalues are greater than 1. Therefore $\lambda=\lambda^{*}=\frac{c}{2m}$ and $A=\frac{c}{2m}I$ on $\mathcal{H}$. Then (6.4) becomes

$$\left(\frac{c}{2m}\right)^{2}=\frac{\alpha c}{2m}-1.$$

If $\alpha=2\frac{\cosh 2r}{\sinh 2r}$, then we can easily obtain from the above equation

$$\frac{c}{2m}=\frac{\cosh r}{\sinh r}$$

as the other root $\sinh r/\cosh r<1$ must be discarded. $\square$

We can now finish the proof. For $a>0$ consider the map $\Phi_{a}:\Sigma\to\mathbb{CH}^{m+1}$ defined by

$$\Phi_{a}(p)=\exp_{p}(-a\nu(p)).$$

By solving the Jacobi equation along the geodesic $\gamma_p(t) = \exp_p(-a\nu(p))$ we have for $u \in T_p\Sigma$

$$(\Phi_a)_* u = \begin{cases} \left(\cosh 2a - \frac{\cosh 2r}{\sinh 2r}\sinh 2a\right) U(a), & \text{if } u = T, \\ \left(\cosh 2a - \frac{\cosh r}{\sinh r}\sinh 2a\right) U(a) & \text{if } u \in \mathcal{H}, \end{cases}$$

where $U(t)$ denotes the parallel vector field along γ_p with $U(0) = u$.

This shows that Φ_r is fully degenerate and hence maps Σ to a point o. Therefore Σ is the geodesic sphere with center o and radius r.

- $M = \mathbb{CP}^{m+1}$

The discussion is parallel. Let $o \in \mathbb{CP}^{m+1}$ s.t. $\Sigma \subset \mathbb{CP}^{m+1} \backslash C(o)$. Let $p \in \Sigma$ be a point on Σ s.t. $a := d(o,p) = \max_{x\in\Sigma} d(o,x) \in (0, \pi/2)$. By a comparison argument similar to the proof of Lemma 6 we see that $A \geq \frac{\cos a}{\sin a} I$ on $\mathcal{H}$ at p. This implies that both λ and λ^* are positive. As $\kappa = 1$ the identity (6.5) is impossible. Therefore $\lambda = \lambda^* = \frac{c}{2m}$ and $A = \frac{c}{2m} I$ on $\mathcal{H}$.

Let $r \in (0, \pi/2)$ be the number s.t. $\lambda = \cos r / \sin r$. From (6.4) we easily obtain $\alpha = 2\cos 2r / \sin 2r$. By a similar argument as in the hyperbolic case we conclude that Σ is a geodesic sphere of radius r.

As a corollary, we obtain the following uniqueness theorem for Hopf hypersurfaces of constant mean curvature.

Theorem 8. *Let M be a simply connected Kahler manifold of constant holomorphic sectional curvature 4κ with* $\dim_{\mathbb{C}} M = m+1 \geq 2$. *In other words $M = \mathbb{CP}^{m+1}$ if $\kappa = 1$; $M = \mathbb{CH}^{m+1}$ if $\kappa = -1$. Let Σ be a closed, embedded hypersurface in M. When $M = \mathbb{CP}^{m+1}$ we further assume that Σ is disjoint from a hyperplane. If Σ has constant mean curvature and is Hopf, then it is a geodesic sphere.*

Remark 8. *A tube over a totally geodesic $\mathbb{CP}^k$ ($0 < k < m$) in $\mathbb{CP}^{m+1}$ discussed above shows that the extra condition that Σ is disjoint from a hyperplane is necessary.*

Proof. Since Σ is Hopf, $\alpha = \Pi(T,T)$ is constant. As $H_b = H - \alpha$ and H is constant, we see that H_b is constant. Suppose $H_b = c$. Let Ω be the domain enclosed by Σ (in the case of $\mathbb{CP}^{m+1}$ we take Ω to be the one disjoint from the same hyperplane). Since Σ is Hopf we have by

Proposition 3 and Proposition 4 (using the same notation there)

$$\begin{aligned} 2m\,|\Sigma| &= \int_\Sigma H_b \langle \nabla\Phi, \nu\rangle \\ &= c\int_\Sigma \langle \nabla\Phi, \nu\rangle \\ &= c\int_\Omega \Delta\Phi \\ &= 2\,(m+1)\,c\,|\Omega|\,, \end{aligned}$$

i.e. $m\,|\Sigma| = (m+1)\,c\,|\Omega|$. Therefore Σ is a geodesic sphere by Theorem 7. □

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