UNIQUENESS RESULTS FOR POSITIVE HARMONIC FUNCTIONS ON $\overline{\mathbb{B}^n}$ SATISFYING A NONLINEAR BOUNDARY CONDITION

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1. Introduction

From the Hardy-Littlewood-Sobolev inequality on $\mathbb{S}^{n-1}$ with sharp constant Beckner [B] derived the following family of inequalities on $\mathbb{B}^n$

Theorem 1. *Let $u \in C^\infty\left(\overline{\mathbb{B}^n}\right)$. Then*
(1.1)
$$c_n{}^{(q-1)/(q+1)}\left(\int_{\mathbb{S}^{n-1}}|u(\xi)|^{q+1}d\sigma(\xi)\right)^{2/(q+1)} \leq (q-1)\int_{\mathbb{B}^n}|\nabla u(x)|^2\,dx+\int_{\mathbb{S}^{n-1}}|u(\xi)|^2\,d\sigma(\xi),$$
for $1<q<\infty$ if $n=2$ and $1<q\leq n/(n-2)$ if $n\geq 3$, where $c_n=2\pi^{n/2}/\Gamma(n/2)=\left|\mathbb{S}^{n-1}\right|$ and $d\sigma$ is the standard volume form on $\mathbb{S}^{n-1}$.

The critical case $q=n/(n-2)$ was also proved by Escobar [E1, E2] by a different method. By Lions [L] the following
$$\inf\frac{\int_{\mathbb{B}^n}|\nabla u|^2+\frac{n-2}{2}\int_{\mathbb{S}^{n-1}}u^2}{\left(\int_{\mathbb{S}^{n-1}}|u|^{2(n-1)/(n-2)}\right)^{(n-2)/(n-1)}}$$
is achieved by a **positive** function u satisfying the following
$$\begin{aligned}\Delta u&=0 &&\text{on}\quad \mathbb{B}^n,\\ \frac{\partial u}{\partial\nu}+\frac{n-2}{2}u&=u^{n/(n-2)} &&\text{on}\quad \mathbb{S}^{n-1}.\end{aligned}$$
Escobar [E2] then classified all positive solutions of the above equation using an integral method and hence proved the inequality (1.1) for $q=n/(n-2)$. The inequality for $1<q<n/(n-2)$ would also follow in the same way from the following

Conjecture 1. *If $u\in C^\infty\left(\overline{\mathbb{B}^n}\right)$ is positive and satisfies the following equation*
$$\begin{aligned}\Delta u&=0 &&\text{on}\quad \mathbb{B}^n,\\ \frac{\partial u}{\partial\nu}+au&=u^q &&\text{on}\quad \mathbb{S}^{n-1},\end{aligned}\tag{1.2}$$
then u is constant, provided $1<q<n/(n-2)$ and $0<a\leq 1/(q-1)$.

In fact, when $1<q<n/(n-2)$ the trace operator $H^1(\mathbb{B}^n)\to L^{q+1}\left(\mathbb{S}^{n-1}\right)$ is compact. It follows trivially that
$$\inf\frac{\int_{\mathbb{B}^n}|\nabla u|^2+a\int_{\mathbb{S}^{n-1}}u^2}{\left(\int_{\mathbb{S}^{n-1}}|u|^{q+1}\right)^{2/(q+1)}}$$

is achieved by a positive function u satisfying (1.2). The conjecture, if true, then implies that u is constant if $a \le 1/(q-1)$ and hence the inequality (1.1). By taking limit $q \nearrow n/(n-2)$ one would also obtain the critical case $q = n/(n-2)$. In [W2] the conjecture is even formulated in a more general context, namely the same uniqueness result should be true on any compact Riemannian manifold with nonnegative Ricci curvature and with principal curvature ≥ 1 on the boundary. In its precise form, the conjecture states

Conjecture 2. *([W2]) Let (M^n, g) be a smooth compact Riemannian manifold with Ric ≥ 0 and $\Pi \ge 1$ on its boundary Σ. Let $u \in C^\infty(M)$ be a positive solution to the following equation*

$$\begin{aligned} \Delta u &= 0 \quad \text{on} \quad M, \\ \frac{\partial u}{\partial \nu} + au &= u^q \quad \text{on} \quad \Sigma, \end{aligned} \tag{1.3}$$

where the parameters a and q are always assumed to satisfy $a > 0$ and $1 < q \le \frac{n}{n-2}$. If $a \le \frac{1}{q-1}$, then u must be constant unless $q = \frac{n}{n-2}, a = \frac{n-2}{2}$, M is isometric to $\overline{\mathbb{B}^n} \subset \mathbb{R}^n$ and u corresponds to

$$u(x) = \left[\frac{2}{n-2}\frac{1-|\xi|^2}{1+|\xi|^2|x|^2-2x\cdot\xi}\right]^{(n-2)/2}$$

for some $\xi \in \mathbb{B}^n$.

This conjecture, if true, would yield Beckner-type inequalities and interesting geometric results for such manifolds. We refer to [W2] for further discussion.

Conjecture 2 has been completely confirmed in dimension 2 in [GHW].

Theorem 2. *Let (Σ, g) be a compact surface with Gaussian curvature $K \ge 0$ and geodesic curvature $\kappa \ge 1$ on the boundary. The only positive solution to the following equation*

$$\begin{aligned} \Delta u &= 0 \quad \text{on} \quad \Sigma, \\ \frac{\partial u}{\partial \nu} + au &= u^q \quad \text{on} \quad \partial\Sigma, \end{aligned}$$

where $a > 0$ and $q > 1$ are constants such that $a(q-1) \le 1$.

In higher dimensions the following partial result is proved in [GHW].

Theorem 3. *Let (M^n, g) be a smooth compact Riemannian manifold with nonnegative sectional curvature and the second fundamental form of the boundary $\Pi \ge 1$. Then the only positive solution to (1.3) is constant if $a \le \frac{1}{q-1}$, provided $3 \le n \le 8$ and $1 < q \le \frac{4n}{5n-9}$.*

In this short note we study the problem on the model space $\overline{\mathbb{B}^n}, n \ge 3$. The main result is the following partial result in dimensions $n \ge 3$.

Theorem 4. *If $u \in C^\infty\left(\overline{\mathbb{B}^n}\right)$ is positive and satisfies the equation (1.2), then u is constant, provided $1 < q < n/(n-2)$ and $0 < a \le (n-2)/2$.*

The paper is organized as follows. In section 2 we transform the equation (1.2) to a new equation on the upper half space $\overline{\mathbb{R}^n_+} = \{x_n \ge 0\}$. We then study the new equation by the method of moving planes and prove that the solution is axially symmetric with respect to the x_n-axis. In section 3 we go back to the ball and finish the proof of Theorem 4 by an integral identity.

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2. Analysis on $\overline{\mathbb{R}^n_+}$

In this section we prove the following partial result.

Proposition 1. *Suppose $u \in C^\infty\left(\overline{\mathbb{B}^n}\right)$ is positive and satisfies the equation (1.2) with $1 < q < n/(n-2)$ and $0 < a \le (n-2)/2$. If $\xi \in \mathbb{S}^{n-1}$ is a critical point of $u|_{\mathbb{S}^{n-1}}$, then u is axially symmetric w.r.t. the line through the origin and ξ.*

Without loss of generality, we assume that the north pole e_n is a critical point of $u|_{\mathbb{S}^{n-1}}$. The inverse of the stereographic projection $\Psi : \overline{\mathbb{R}^n_+} \to \overline{\mathbb{B}^n} \setminus \{e_n\}$ is given by

$$\Psi(x) = \left(\frac{2x_1}{1+|x|^2+2x_n}, \cdots, \frac{2x_{n-1}}{1+|x|^2+2x_n}, \frac{-1+|x|^2}{1+|x|^2+2x_n}\right).$$

Let $v(x) = u \circ \Psi(x)\left(\frac{2}{1+|x|^2+2x_n}\right)^{(n-2)/2}$. Then v satisfies the following equation on $\overline{\mathbb{R}^n_+}$

$$\begin{aligned} \Delta v &= 0 && \text{on} \quad \mathbb{R}^n_+, \\ -\frac{\partial v}{\partial x_n} &= \alpha\left(\frac{2}{1+|x|^2}\right)^2 v + \left(\frac{2}{1+|x|^2}\right)^\beta v^q && \text{on} \quad \mathbb{R}^{n-1}, \end{aligned} \tag{2.1}$$

where

$$\alpha = \frac{n-2}{2} - a, \beta = \frac{n-q(n-2)}{2}.$$

By our assumption α and β are both nonnegative. As e_n is a critical point of $u|_{\mathbb{S}^{n-1}}$, using the Taylor expansion of u at e_n we have, as $x \to \infty$,

$$\begin{aligned} v(x) &= c_0 |x|^{2-n}\left(1 + c_1 \frac{x_n}{|x|^2} + O\left(|x|^{-2}\right)\right), \\ \frac{\partial v}{\partial x_i} &= c_0\left(-\frac{(n-2)x_i}{|x|^n} - \frac{nc_1 x_n x_i}{|x|^{n+2}} + O\left(\frac{1}{|x|^{n+1}}\right)\right), i = 1, \cdots, n-1, \end{aligned}$$

where c_0 is a positive constant. We will prove that v is axially symmetric with respect to the x_n-axis by the method of moving planes. We will follow the approach in the classic work of Gidas, Ni and Nirenberg [GNN].

Remark 1. *When $\alpha = 0, \beta = 0$ the equation (2.1) reduces to the following*

$$\begin{aligned} \Delta v &= 0 \quad \text{on} \quad \mathbb{R}^n_+, \\ -\frac{\partial v}{\partial x_n} &= v^q \quad \text{on} \quad \mathbb{R}^{n-1}. \end{aligned}$$

This equation is invariant under translations $x \to x + \eta$ when $\eta_n = 0$. When $q = n/(n-2)$ it is further invariant under Mobius transformations and all positive solutions were classified by Li and Zhu [LZ] *using the more powerful moving sphere method. Ou* [Ou] *studied the case $q < n/(n-2)$ by the method of moving planes and proved that there is no positive solution.*

Since the equation (2.1) is invariant under rotation about the x_n-axis, it suffices to show that v is even in x_1. For $\lambda \in \mathbb{R}$ we define $x^\lambda = (2\lambda - x_1, x_{2,} \cdots, x_n)$ and $\Sigma_\lambda = \{x \in \mathbb{R}^n : x_1 \le \lambda\}$. Let $\Lambda = \left\{\lambda > 0 : v(x) \ge v\left(x^\lambda\right) \text{ on } \Sigma_\lambda\right\}$.

Lemma 1. *For any $\lambda_0 > 0$ there exists $R_0 > 0$ s.t. for all $\lambda \ge \lambda_0, x \in \Sigma_\lambda$ with $|x| \ge R_0$*

$$v(x) \ge v\left(x^\lambda\right).$$

By this lemma, it is obvious that Λ contains all sufficiently large λ. To prove the lemma, we assume $c_0 = 1$ without loss of generality in the following proof. First observe that we always have $|x| \le \left|x^\lambda\right|$ when $\lambda > 0$ and $x \in \Sigma_\lambda$. We consider several cases.

Case 1. $\left|x^\lambda\right| \ge 2|x|$.

Then

$$\begin{aligned} v(x) - v\left(x^\lambda\right) &= |x|^{2-n} - \left|x^\lambda\right|^{2-n} + O\left(|x|^{1-n}\right) \\ &= |x|^{2-n}\left(1 - \left(\frac{|x|}{|x^\lambda|}\right)^{n-2} + O\left(|x|^{-1}\right)\right) \\ &\ge |x|^{2-n}\left(1 - 2^{-(n-2)} + O\left(|x|^{-1}\right)\right) \\ &\ge 0, \text{ if } |x| \text{ is sufficiently large.} \end{aligned}$$

Case 2. $\left|x^\lambda\right| \le 2|x|$, but $|x'| \le |x|/\sqrt{2}$. Here we write $x = (x_1, x')$.

We have, with $y_1 = 2\lambda - x_1$

$$\begin{aligned} v(x) - v\left(x^\lambda\right) &= \int_{x_1}^{y_1} -\frac{\partial u}{\partial x_1}(t, x')\,dt \\ &= \int_{x_1}^{y_1}\left[\frac{(n-2)t}{\left(t^2 + |x'|^2\right)^{n/2}} + \frac{nc_1 x_n t}{\left(t^2 + |x'|^2\right)^{(n+2)/2}}\right]dt + O\left(\frac{y_1 - x_1}{|x|^{n+1}}\right) \\ &= \frac{1}{|x|^{n-2}} - \frac{1}{|x^\lambda|^{n-2}} + \widetilde{c}x_n\left(\frac{1}{|x|^n} - \frac{1}{|x^\lambda|^n}\right) + O\left(\frac{y_1 - x_1}{|x|^{n+1}}\right) \\ &= \frac{(n-2)\left(\left|x^\lambda\right| - |x|\right)}{r^{n-1}} + \frac{n\widetilde{c}x_n\left(\left|x^\lambda\right| - |x|\right)}{s^{n+1}} + O\left(\frac{y_1 - x_1}{|x|^{n+1}}\right), \end{aligned}$$

where in the last step we used the mean value theorem to get $r, s \in (|x|, 2|x|)$. On the other hand, as $\left|x^\lambda\right|^2 - |x|^2 = y_1^2 - x_1^2 = 2\lambda(y_1 - x_1)$, we have

$$|y_1 - x_1| \le \frac{\left|x^\lambda\right|^2 - |x|^2}{2\lambda_0} \le \frac{2|x|}{\lambda_0}\left(\left|x^\lambda\right| - |x|\right).$$

Therefore

$$\begin{aligned} v(x) - v\left(x^\lambda\right) &\ge \frac{(n-2)\left(\left|x^\lambda\right| - |x|\right)}{2^{n-1}|x|^{n-1}} + O\left(\frac{\left|x^\lambda\right| - |x|}{|x|^n}\right) + O\left(\frac{\left|x^\lambda\right| - |x|}{\lambda_0|x|^n}\right) \\ &= \frac{\left|x^\lambda\right| - |x|}{|x|^{n-1}}\left[\frac{(n-2)}{2^{n-1}} + O\left(\frac{1 + \lambda_0^{-1}}{|x|}\right)\right] \\ &\ge 0, \text{ if } |x| \text{ is sufficiently large, depending on } \lambda_0. \end{aligned}$$

Case 3. $\left|x^{\lambda}\right| \leq 2\left|x\right|, \left|x'\right| \leq \left|x\right|/\sqrt{2}$.

It follows $\left|x_1\right| \geq \left|x\right|/\sqrt{2}$. If $x_1 \geq \left|x\right|/\sqrt{2}$, by the mean value theorem, there is $s \in (x_1, y_1)$

$$\begin{aligned} v(x) - v\left(x^{\lambda}\right) &= \frac{\partial u}{\partial x_1}(s, x')(x_1 - y_1) \\ &= \left[\frac{(n-2)s}{\left(s^2 + \left|x'\right|^2\right)^{n/2}} + O\left(\frac{1}{\left(s^2 + \left|x'\right|^2\right)^{n/2}}\right)\right](y_1 - x_1) \\ &\geq \left[\frac{(n-2)}{\sqrt{2}\left|x\right|^{n-1}} + O\left(\frac{1}{\left|x\right|^n}\right)\right](y_1 - x_1) \\ &\geq 0, \text{ if } \left|x\right| \text{ is sufficiently large.} \end{aligned}$$

If $x_1 \leq -\left|x\right|/\sqrt{2}$, we let $\overline{x} = (-x_1, x')$. By the asymptotic expansion

$$v(x) - v(\overline{x}) = O\left(\frac{1}{\left|x\right|^n}\right).$$

By the same type of analysis as in Case 2, we have

$$\begin{aligned} v(\overline{x}) - v\left(x^{\lambda}\right) &\geq \left[\frac{(n-2)}{\sqrt{2}\left|x\right|^{n-1}} + O\left(\frac{1}{\left|x\right|^n}\right)\right](y_1 + x_1) \\ &= 2\lambda\left[\frac{(n-2)}{\sqrt{2}\left|x\right|^{n-1}} + O\left(\frac{1}{\left|x\right|^n}\right)\right]. \end{aligned}$$

It follows that

$$\begin{aligned} v(x) - v\left(x^{\lambda}\right) &= v(x) - v(\overline{x}) + v(\overline{x}) - v\left(x^{\lambda}\right) \\ &\geq 0, \text{if } \left|x\right| \text{ is sufficiently large,depending on } \lambda_0. \end{aligned}$$

This finishes the proof of Lemma 1.

Lemma 2. *If $\lambda_0 \in \Lambda$, then there exist $\varepsilon < 0$ s.t. $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \subset \Lambda$.*

The function v^* defined by $v^*(x) = v\left(x^{\lambda_0}\right)$ satisfies

$$\begin{array}{rll} \Delta v^* = 0 & \text{on} & \mathbb{R}^n_+, \\ -\frac{\partial v}{\partial x_n} = \alpha\left(\frac{2}{1+\left|x^{\lambda}\right|^2}\right)^2 v + \left(\frac{2}{1+\left|x^{\lambda}\right|^2}\right)^{\beta} v^q & \text{on} & \mathbb{R}^{n-1}, \end{array}$$

Therefore the function $w = v - v^*$ satisfies, as both α and β are nonnegative

$$\begin{array}{rll} \Delta w = 0 & \text{on} & \mathbb{R}^n_+, \\ -\frac{\partial w}{\partial x_n} \geq qw & \text{on} & \mathbb{R}^{n-1}, \end{array}$$

where

$$q = \alpha\left(\frac{2}{1+\left|x\right|^2}\right)^2 + \left(\frac{2}{1+\left|x\right|^2}\right)^{\beta}\frac{v^q - (v^*)^q}{v - v^*}.$$

By the assumption we have $w \geq 0$ on Σ_{λ_0}. Moreover by the asymptotic expansion of v it is clear that w cannot be identically zero on Σ_{λ_0}. We claim that $\frac{\partial w}{\partial x_1} < 0$ everywhere on the half-plane $\{x : x_1 = \lambda_0, x_n \geq 0\}$. If $x_n > 0$, this follows from the Hopf Lemma. When $x_n = 0$ one can adapt Hopf's argument to get the same conclusion, as observed in Ou [Ou]. We elaborate the idea. First observe that, in view of the boundary condition for w and the Hopf lemma again, we have $w(x) > 0$ on $\{x : x_1 < \lambda_0, x_n = 0\}$. Given ξ with $\xi_1 = \lambda_0, \xi_n = 0$ we consider on the annulus $A = \left\{x \in \overline{\mathbb{R}^n_+} : \rho/2 \leq |x - \xi - \rho e_1| \leq \rho\right\}$ the following function

$$\widetilde{w}(x) = w(x) + \delta\left(e^{-\alpha\rho^2} - e^{-\alpha|x-\xi-\rho e_1|^2}\right).$$

Note that $\widetilde{w} = 0$ on $\partial A \cap \{|x - \xi - \rho e_1| = \rho\}$. For appropriately chosen positive numbers ρ, α, and δ, we can guarantee

$$\begin{aligned} &\Delta\widetilde{w} \leq 0 \text{ on } A; \\ &\widetilde{w} \geq 0 \text{ on } \partial A \cap \{|x - \xi - \rho e_1| = \rho/2\} \end{aligned}$$

while on $\partial A \cap \mathbb{R}^{n-1}$ we have $-\frac{\partial\widetilde{w}}{\partial x_n} = -\frac{\partial w}{\partial x_n} \geq 0$. By the maximum principle $\widetilde{w} \geq 0$ on A. As $\widetilde{w}(\xi) = 0$ we must have $\frac{\partial\widetilde{w}}{\partial x_1}(\xi) \leq 0$. Hence $\frac{\partial w}{\partial x_1}(\xi) = \frac{\partial\widetilde{w}}{\partial x_1}(\xi) - \delta\alpha < 0$.

We now finish the proof of Lemma 2 by contradiction. Suppose there is a sequence of positive $\mu_k \to \lambda_0$ and $x_k \in \Sigma_{\mu_k}$ s.t. $v(x_k) < v(x_k^{\mu_k})$. By Lemma 1 $\{x_k\}$ is bounded. Passing to a subsequence we assume $x_k \to \overline{x}$ as $k \to \infty$. Then $v(\overline{x}) \leq v\left(\overline{x}^{\lambda_0}\right)$. This implies that $\overline{x}$ lies in the half-plane $\{x : x_1 = \lambda_0, x_n \geq 0\}$ and hence $\frac{\partial u}{\partial x_1}(\overline{x}) < 0$. On the other hand by the mean value theorem there exists ξ_k in the segment joining x_k and $x_k^{\mu_k}$ s.t. $\frac{\partial u}{\partial x_1}(\xi_k) > 0$. In the limit we obtain $\frac{\partial u}{\partial x_1}(\overline{x}) \geq 0$, a contradiction.

By Lemma 2 Λ is open in $(0, \infty)$. It is also clearly closed. Therefore $\Lambda = (0, \infty)$ and we have $v(x_1, x') \geq v(-x_1, x')$ when $x_1 \leq 0$. As the equation is symmetric w.r.t. $(x_1, x') \to (-x_1, x')$, we must have $v(x_1, x') = v(-x_1, x')$, i.e. v is even in x_1.

3. Back to $\overline{\mathbb{B}^n}$

Before we finish the proof of Theorem 4, we need to explain an integral identity to be used. Let (M^n, g) be a compact Riemannian manifold with boundary Σ which may be empty. We denote by T the Einstein tensor, i.e. $T = Ric - \frac{R}{n}g$, where R is the scalar curvature. If Ξ is a conformal vector field, then according to [S] the following identity holds

$$\int_M \Xi R dv_g = \frac{2n}{n-2}\int_\Sigma T(\Xi, \nu) d\sigma_g.$$

We further assume that the boundary is umbilic, i.e. the 2nd fundamental form is a proportional to the 1st fundamental form. More precisely for any $X, Y \in T\Sigma$

$$\Pi(X, Y) = \frac{H}{n-1}g(X, Y),$$

where H denotes the mean curvature.

By the Codazzi equation we have for any $X, Y, Z \in T\Sigma$

$$\begin{aligned} R(X, Y, Z, \nu) &= \nabla_X \Pi(Y, Z) - \nabla_Y \Pi(X, Z) \\ &= \frac{XH}{n-1}g(Y, Z) - \frac{YH}{n-1}g(X, Z). \end{aligned}$$

Taking trace yields

$$T(X,\nu) = Ric(X,\nu) = -\frac{n-2}{n-1}XH.$$

Therefore we obtain

Proposition 2. *Suppose (M^n, g) is compact with an umbilic boundary and Ξ a conformal vector field on M s.t. Ξ is tangential on the boundary, then*

$$\int_M \Xi R dv_g = -\frac{2n}{n-1}\int_\Sigma \Xi H d\sigma_g.$$

When $u > 0$ satisfies (1.2) the metric $g = u^{4/(n-2)}dx^2$ on $\overline{\mathbb{B}^n}$ has scalar curvature $R = 0$ and mean curvature

$$H = \left[\left(\frac{n-2}{2} - a\right)u + u^q\right]u^{-n/(n-2)} = \left(\frac{n-2}{2} - a\right)u^{-2/(n-2)} + u^{q-n/(n-2)}.$$

on the boundary $\mathbb{S}^{n-1}$. Being umbilic is conformally invariant. Therefore we can apply Proposition 2 in this situation. For each $i = 1, \cdots, n$

$$\Xi_i(x) = x_i x - \frac{1+|x|^2}{2}e_i$$

is a conformal vector field on $\overline{\mathbb{B}^n}$ and its restriction on the boundary $\mathbb{S}^{n-1}$ is given by

$$\Xi_i(\xi) = \xi_i\xi - e_i = \nabla\xi_i.$$

Therefore by Proposition 2 we have for $i = 1, \cdots, n$

$$\int_{\mathbb{S}^{n-1}} \left\langle \nabla\left[\left(\frac{n-2}{2} - a\right)u^{-2/(n-2)} + u^{q-n/(n-2)}\right], \nabla\xi_i\right\rangle u^{2(n-1)/(n-2)}d\sigma = 0,$$

here the gradient, the pairing and the volume element $d\sigma$ are all with respect to the standard metric on $\mathbb{S}^{n-1}$. Simplifying yields

$$\int_{\mathbb{S}^{n-1}} \left[\left(1 - \frac{2a}{n-2}\right)u + \left(\frac{n}{n-2} - q\right)u^q\right]\langle\nabla u, \nabla\xi_i\rangle d\sigma = 0. \tag{3.1}$$

We note that the 1st factor in the integrand is positive as $a \le \frac{n-2}{2}$ and $q < \frac{n}{n-2}$. By Proposition 1, we know that $u|_{\mathbb{S}^{n-1}}$ is axially symmetric w.r.t. the x_n-axis. Thus we write $u(\xi) = f(\xi_n)$, with f a smooth function on $[-1, 1]$. If f has a critical point $t_0 \in (-1, 1)$, then every point $\xi \in \mathbb{S}^{n-1}$ with $\xi_n = t_0$ is a critical point of $u|_{\mathbb{S}^{n-1}}$. By theorem, then $u|_{\mathbb{S}^{n-1}}$ has is axially symmetric w.r.t. the line passing through 0 and ξ. In other words, $u(x)$ only depends on the distance between x and ξ. It is easy to see that then f must then be constant.

If f has no critical point in $(-1, 1)$, then f' is either everywhere positive or everywhere negative. This implies that $\langle\nabla u, \nabla\xi_n\rangle$ is either everywhere positive or everywhere negative. We then have a contradiction with (3.1) when $i = n$.

Therefore $u|_{\mathbb{S}^{n-1}}$ and hence u itself is constant.

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