ON COMPACT RIEMANNIAN MANIFOLDS WITH CONVEX BOUNDARY AND RICCI CURVATURE BOUNDED FROM BELOW

XIAODONG WANG

ABSTRACT. We propose to study positive harmoninc functions satisfying a nonlinear Neuman condition on a compact Riemannian manifold with non-negative Ricci curvature and strictly convex boundary. A precise conjecture is formulated. We discuss its implications and present some partial results. Related questions are discussed for compact Riemannian manifolds with positive Ricci curvature and convex boundary.

1. INTRODUCTION

For a compact Riemannian manifold (M^n, g) with nonempty boundary $\Sigma = \partial M$, it is interesting to study connections between the intrinsic geometry $g|_\Sigma$ and the extrinsic geometry (the 2nd fundamental form), under a lower bound for scalar curvature or Ricci curvature. We refer to [ST1, ST2, WY1, MW] and references therein for recent works in this direction. Some of these works are motivated by problems in general relativity, in particular about understanding various definitions of quasi-local mass. The following fundamental result was proved by Shi and Tam [ST1].

Theorem 1. *Let (M^n, g) be a compact Riemannian manifold with scalar curvature $R \geq 0$ and with a connected boundary Σ. Suppose*

- *M is spin,*
- *the mean curvature H of Σ is positive,*
- *there exists an isometric embedding $\iota : \Sigma \to \mathbb{R}^n$ as a strictly convex hypersurface.*

Then

$$(1.1) \qquad \int_\Sigma H \leq \int_\Sigma H_0,$$

where H_0 is the mean curvature of $\iota : \Sigma \to \mathbb{R}^n$. Moreover, if equality holds, then M is isometric to the Euclidean domain enclosed by $\iota : \Sigma \to \mathbb{R}^n$.

The right hand side of (1.1) is determined by the intrinsic geometry of Σ. Therefore by the inequality the extrinsic geometry of Σ is constrained by its intrinsic geometry. But the assumption that there is an isometric embedding of Σ into $\mathbb{R}^n$ as a strictly convex hypersurface imposes severe restriction on the kind of intrinsic geometry of Σ for which the theorem is applicable. In the more recent work [MW] Miao and I proved a slightly different inequality under the stronger condition $Ric \geq 0$, but without any restriction on the intrinsic geometry of the boundary.

Theorem 2. *Let (M^n, g) be a compact Riemannian manifold with $Ric \geq 0$ and with a connected boundary Σ that has positive mean curvature H. Let $\iota : \Sigma \to \mathbb{R}^m$ be an isometric embedding. Then*

$$(1.2) \qquad \int_\Sigma H \leq \int_\Sigma \frac{\left|\overrightarrow{H}_0\right|^2}{H},$$

where $\overrightarrow{H}_0$ is the mean curvature vector of $\iota : \Sigma \to \mathbb{R}^m$. Moreover, if equality holds, then $\iota(\Sigma)$ is contained in an n-dimensional plane of $\mathbb{R}^m$ and M is isometric to the Euclidean domain enclosed by $\iota(\Sigma)$ in that n-dimensional plane.

Notice that an isometric embedding $\iota : \Sigma \to \mathbb{R}^m$ always exists by Nash's famous theorem.

When the scalar curvature has a negative lower bound, results similar to Theorem 1 were proved by [WY1] and [ST2]. The counterexample to the Min-Oo conjecture by Brendle, Marques and Neves[BMN] shows that no such result holds when the scalar curvature has a positive lower bound. Results similar to Theorem 2 are also established when the Ricci curvature has a positive or negative lower bound. But in these two cases the inequalities obtained are not sharp. Some rigidity results under stronger assumptions on the boundary were proved in [MW].

In all of these studies the result is basically an estimate on an integral involving the mean curvature. It is natural to ask if one can bound the area of the boundary, the volume of the interior and other more direct geometric or analytic quantities. It is easy to see that for such results to hold a lower bound for the mean curvature is not enough. For example, for any closed $\left(\Sigma^{n-2}, h\right)$ with nonnegative Ricci curvature, $M := \overline{B^2} \times \Sigma$ with the product metric $dx^2 + h$ has nonnegative Ricci curvature and mean curvature $H \geq 1$ while the area of ∂M can be arbitrarily large. Therefore we will in this paper mostly consider compact Riemannian manifolds (M^n, g) with nonnegative Ricci curvature and with a connected boundary Σ whose 2nd fundamental form has a positive lower bound. Motivated by a uniqueness theorem in [BVV], we study positive harmonic functions on M that satisfy a semilinear Neumann condition on the boundary. We formulate a conjecture which has important geometric implications. We will prove some partial results that support this conjecture.[1] Another case we consider is when M has positive Ricci curvature, by scaling we can always assume $Ric \geq n - 1$ and the boundary Σ is convex in the sense that its 2nd fundamental form is nonnegative. There is similarly a natural conjecture on the area of the boundary.

The paper is organized as follows. In section 2 we discuss some natural PDEs on a compact manifold with boundary. We formulate a uniqueness conjecture on a semilinear Neumann problem in the nonnegative Ricci case and discuss its geometric implications. In Section 3 we prove some topological results. In Section 4 we present some partial results and several other conjectures.

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[1]After this paper was posted to the arXiv, new progress has been made in the following two papers: 1. Q. Guo and X. Wang, Uniqueness results for positive harmonic functions on $\overline{\mathbb{B}^n}$ satisfying a nonlinear boundary condition, arXiv:1912.05568. 2. Q. Guo, F. Hang and X. Wang, Liouville type theorems on manifolds with nonnegative curvature and strictly convex boundary, arXiv:1912.05574

2. FROM PDE TO GEOMETRY: A CONJECTURE

We first recall a theorem proved by Bidaut-Veron and Veron [BVV] .

Theorem 3. *([BVV] and [I]) Let (M^n, g) be a compact Riemannian manifold with a (possibly empty) convex boundary. Suppose $u \in C^\infty(M)$ is a positive solution of the following equation*

$$-\Delta u + \lambda u = u^q \quad on \quad M,$$
$$\frac{\partial u}{\partial \nu} = 0 \quad on \quad \partial M,$$

where $\lambda > 0$ is a constant and $1 < q \leq (n+2)/(n-2)$. If $Ric \geq \frac{(n-1)(q-1)\lambda}{n} g$, then u must be constant unless $q = (n+2)/(n-2)$ and (M^n, g) is isometric to $\left(\mathbb{S}^n, \frac{4\lambda}{n(n-2)} g_0\right)$ or $\left(\mathbb{S}^n_+, \frac{4\lambda}{n(n-2)} g_0\right)$. In the latter case u is given on $\mathbb{S}^n$ or $\mathbb{S}^n_+$ by the following formula

$$u = \frac{1}{(a + x \cdot \xi)^{(n-2)/2}}.$$

for some $\xi \in \mathbb{R}^{n+1}$ and some constant $a > |\xi|$.

This theorem was proved by Bidaut-Veron and Veron [BVV] when $\partial M = \varnothing$ and by Ilias [I] when $\partial M \neq \varnothing$ using the same method. It has some important corollaries. We focus on the case $\partial M \neq \varnothing$. We recall the Yamabe problem on a compact Riemannian manifold (M^n, g) with boundary. The conformal Laplacian is defined to be $L_g = -c_n \Delta_g + R_g$, with $c_n = \frac{4(n-1)}{n-2}$. If $\widetilde{g} = \phi^{4/(n-2)} g$, then

$$L_{\widetilde{g}} u = \phi^{-(n+2)/(n-2)} L_g(u\phi).$$

Under the conformal deformation the mean curvature of the boundary transforms according to the following formula

$$\frac{2(n-1)}{n-2} \frac{\partial \phi}{\partial \nu} + H\phi = \widetilde{H} \phi^{n/(n-2)}.$$

We consider the following functional

$$E_g(u) = \int_M c_n |\nabla u|^2 + Ru^2 + 2\int_{\partial M} Hu^2$$
$$= \int_M u L_g u \, dv_g + \int_{\partial M} \left(c_n \frac{\partial u}{\partial \nu} + 2Hu\right) u \, d\sigma_g$$

This functional is conformally invariant: $E_{\widetilde{g}}(u) = E_g(u\phi)$. If u is positive then

$$E_g(u) = \int_M R_{\widetilde{g}} \, dv_{\widetilde{g}} + \int_{\partial M} 2H_{\widetilde{g}} \, d\sigma_{\widetilde{g}},$$

where $\widetilde{g} = u^{4/(n-2)} g$.

We define

$$\lambda(M, g) = \inf \frac{E_g(u)}{\int_M |u|^2}.$$

The sign of λ is conformally invariant. The Yamabe invariant is defined to be

$$Y(M, g) = \inf \frac{E_g(u)}{\left(\int_M |u|^{2n/(n-2)}\right)^{(n-2)/n}}.$$

Aubin [A] showed that $Y(M, g) \le Y(\mathbb{S}^n) = n(n-1)(|\mathbb{S}^n|)^{2/n}$ when $\partial M = \varnothing$ while Escobar [E1] and Cherrier [C] proved that $Y(M, g) \le Y(\mathbb{S}^n_+) = n(n-1)(|\mathbb{S}^n|/2)^{2/n}$ when $\partial M \ne \varnothing$.

Let (M^n, g) be a compact Riemannian manifold with convex boundary and $Ric \ge (n-1)$. From Theorem 3 one can derive the following

- (Sharp Sobolev inequalities) For $2 < q \le (n+2)/(n-2)$

$$\left(\frac{1}{V} \int_M |u|^{q+1} \right)^{2/(q+1)} \le \frac{q-1}{n} \frac{1}{V} \int_M |\nabla u|^2 + \frac{1}{V} \int_M u^2.$$

- $Y(M, g) \ge n(n-1)V^{2/n}$. Moreover, equality holds iff g is Einstein with totally geodesic boundary.

This discussion also yields an analytic proof of the classic result that $V \le |\mathbb{S}^n|$ when $\partial M = \varnothing$ and $V \le |\mathbb{S}^n|/2$ when $\partial M \ne \varnothing$.

Given a compact Riemannian problem (M^n, g) with nonempty boundary, the type II Yamabe problem studied by Escobar [E2] is whether one can find a conformal metric $\widetilde{g} = \phi^{4/(n-2)} g$ with zero scalar curvature and constant mean curvature on the boundary. This leads to the following equation

$$L_g \phi = 0 \text{ on } M,$$

$$\frac{2(n-1)}{n-2} \frac{\partial \phi}{\partial \nu} + H\phi = c\phi^{n/(n-2)} \text{ on } \partial M.$$

Assuming $\lambda(M, g) > 0$ Escobar introduced the following minimization

$$Q(M, \partial M, g) = \inf \frac{E_g(u)}{\left(\int_{\partial M} |u|^{2(n-1)/(n-2)} \right)^{(n-2)/(n-1)}}.$$

Motivated by Theorem 3 we propose to study **positive** solutions of the following equation

$$(2.1) \qquad \begin{array}{rcl} \Delta u = 0 & \text{on} & M, \\ \frac{\partial u}{\partial \nu} + \lambda u = u^q & \text{on} & \partial M, \end{array}$$

where $\lambda > 0$ and $1 < q \le n/(n-2)$, and make the following conjecture.

Conjecture 1. *Let (M^n, g) be a compact Riemannian manifold with $Ric \ge 0$ and $\Pi \ge 1$ on ∂M. If $0 < \lambda \le 1/(q-1)$, then any positive solution u of the above equation must be constant unless $q = n/(n-2)$, M is isometric to $\overline{\mathbb{B}^n} \subset \mathbb{R}^n$ and u corresponds to*

$$u_a(x) = \left[\frac{2}{n-2} \frac{1 - |a|^2}{1 + |a|^2 |x|^2 - 2x \cdot a} \right]^{(n-2)/2}$$

for some $a \in \mathbb{B}^n$.

At the moment this conjecture is completely open. But in dimension 2 an analogous problem was studied by the author [W2] in which the following result was proved.

Theorem 4. *Let (Σ, g) be a compact surface with Gaussian curvature $K \ge 0$ and on the boundary the geodesic curvature $\kappa \ge 1$. Consider the following equation*

$$\begin{array}{rcl} \Delta u = 0 & \text{on} & \Sigma, \\ \frac{\partial u}{\partial \nu} + \lambda = e^u & \text{on} & \partial \Sigma, \end{array}$$

where λ is a positive constant. If $\lambda < 1$ then u is constant; if $\lambda = 1$ and u is not constant, then Σ is isometric to the unit disc $\overline{\mathbb{B}^2}$ and u is given by

$$u\left(z\right) = \log \frac{1 - |a|^2}{1 + |a|^2 |z|^2 - 2\operatorname{Re}\left(z\bar{a}\right)},$$

for some $a \in \mathbb{B}^2$.

Next we discuss a geometric implication of Conjecture 1. For $1 < q < n/\left(n-2\right)$ the following minimization problem

$$\inf \frac{\left(q-1\right)\int_M |\nabla u|^2 + \int_{\partial M} u^2}{\left(\int_{\partial M} |u|^{q+1}\right)^{2/(q+1)}}$$

is achieved by smooth positive function satisfying (2.1) with $\lambda = 1/(q-1)$. If the conjecture is true then the minimizer is constant and therefore the following inequality holds

$$(2.2) \qquad |\partial M|^{(q-1)/(q+1)} \left(\int_{\partial M} |u|^{q+1}\right)^{2/(q+1)} \leq \left(q-1\right)\int_M |\nabla u|^2 + \int_{\partial M} u^2.$$

Letting $q \nearrow n/\left(n-2\right)$ yields

$$|\partial M|^{1/(n-1)} \left(\int_{\partial M} |u|^{2(n-1)/(n-2)}\right)^{(n-2)/(n-1)} \leq \int_M \frac{2}{n-2} |\nabla u|^2 + \int_{\partial M} u^2.$$

Then

$$\begin{aligned} E_g\left(u\right) &= \int_M \frac{4\left(n-1\right)}{n-2} |\nabla u|^2 + Ru^2 + 2\int_{\partial M} Hu^2 \\ &\geq \int_M \frac{4\left(n-1\right)}{n-2} |\nabla u|^2 + 2\left(n-1\right)\int_{\partial M} u^2 \\ &\geq 2\left(n-1\right) |\partial M|^{1/(n-1)} \left(\int_{\partial M} |u|^{2(n-1)/(n-2)}\right)^{(n-2)/(n-1)}. \end{aligned}$$

Therefore

$$Q\left(M, \partial M, g\right) \geq 2\left(n-1\right) |\partial M|^{1/(n-1)}.$$

As $Q\left(M, \partial M, g\right) \leq Q\left(\mathbb{B}^n, \partial\mathbb{B}^n\right) = 2\left(n-1\right)\left|\mathbb{S}^{n-1}\right|^{1/(n-1)}$ we obtain

$$|\partial M| \leq \left|\mathbb{S}^{n-1}\right|.$$

In summary Conjecture 1 implies the following conjecture.

Conjecture 2. *Let (M^n, g) be a compact Riemannian manifold with $Ric \geq 0$ and $\Pi \geq 1$ on ∂M. Then*

$$|\partial M| \leq \left|\mathbb{S}^{n-1}\right|.$$

We remark that the inequality (2.2) for $1 \leq q \leq n/\left(n-2\right)$ on a compact Riemannian manifold (M^n, g) with $Ric \geq 0$ and $\Pi \geq 1$ on ∂M that would follow from Conjecture 1 is known to be true on $\overline{\mathbb{B}^n}$. This was proved by Beckner [B] as a corollary of the Hardy-Littlewood-Sobolev inequality with sharp constant on the sphere. Here is the precise statement

Theorem 5. *(Beckner [B]) For $1 \le q \le n/(n-2)$*

$$c_n^{(q-1)/(q+1)} \left(\int_{\mathbb{S}^{n-1}} |F(\xi)|^{q+1} d\xi \right)^{2/(q+1)} \le (q-1) \int_{\mathbb{B}^n} |\nabla u(x)|^2 \, dx + \int_{\mathbb{S}^n} |F(\xi)|^2 \, d\xi,$$

where u is the harmonic extension of F and $c_n = 2\pi^{n/2}/\Gamma(n/2) = \left| \mathbb{S}^{n-1} \right|.$

3. Boundary effect on topology

In this section we prove some topological results on compact Riemannian manifolds with a lower bound for Ricci curvature and a corresponding lower bound for the 2nd fundamental form on the boundary. Besides their independent interest, these topological results will be used to prove some geometric results in the next section.

Proposition 1. *Let (M^n, g) be a compact Riemannian manifold with boundary Σ. Suppose $Ric \ge 0$.*

- *If the boundary has positive mean curvature, then $H^1(M, \Sigma) = 0$.*
- *If the boundary is strictly convex, then $H^1(M) = 0$.*

This result should be well known. A proof using minimal surfaces for the 2nd part was given by Fraser and Li [FL]. We explain the standard argument with harmonic forms. By the Hodge theory for compact Riemannian manifolds with boundary

$$H^1(M, \Sigma) \cong \mathcal{H}_R^1(M),$$

where $\mathcal{H}_R^1(M)$ is the space of harmonic 1-forms satisfying the relative boundary condition, i.e. $\alpha \in \mathcal{H}_R^1(M)$ iff $d\alpha = 0, d^*\alpha = 0$ and $\alpha \wedge \nu^* = 0$ on the boundary. Note that the boundary condition simply means $\alpha(e_i) = 0, i = 1, \cdots, n-1$. Thus $\langle \nabla_\nu \alpha, \alpha \rangle = \alpha(\nu) \nabla_\nu \alpha(\nu)$ on the boundary. We compute

$$\nabla_\nu \alpha(\nu) = -\sum_{i=1}^{n-1} \nabla_{e_i} \alpha(e_i)$$

$$= \sum_{i=1}^{n-1} (-e_i(\alpha(e_i)) + \alpha(\nabla_{e_i} e_i))$$

$$= -H\alpha(\nu).$$

By the Bochner formula we have

$$\int_M |\nabla \alpha|^2 + Ric(\alpha, \alpha) = \int_\Sigma \langle \nabla_\nu \alpha, \alpha \rangle$$

$$= -\int_\Sigma H[\alpha(\nu)]^2.$$

Clearly $\alpha = 0$ if $Ric \ge 0$ and $H > 0$. Therefore $H^1(M, \Sigma) = 0$.

For the second part we recall

$$H^1(M) \cong \mathcal{H}_A^1(M),$$

where $\mathcal{H}_A^1(M)$ is the space of harmonic 1-forms satisfying the absolute boundary condition, i.e. $\alpha \in \mathcal{H}_A^1(M)$ iff $d\alpha = 0, d^*\alpha = 0$ and $\alpha(\nu) = 0$ on the boundary.

Working with a local orthonormal frame $\{e_0 = \nu, e_1, \cdots, e_{n-1}\}$ on Σ we have

$$
\begin{aligned}
\langle \nabla_\nu \alpha, \alpha \rangle &= \sum_{i=1}^{n-1} \alpha(e_i) \nabla_\nu \alpha(e_i) \\
&= \sum_{i=1}^{n-1} \alpha(e_i) \nabla_{e_i} \alpha(\nu) \\
&= \sum_{i=1}^{n-1} \alpha(e_i) \left[e_i(\alpha(\nu)) - \alpha(\nabla_{e_i} \nu) \right] \\
&= - \sum_{i,j=1}^{n-1} \Pi_{ij} \alpha(e_i) \alpha(e_j).
\end{aligned}
$$

Therefore

$$
\int_M |\nabla \alpha|^2 + Ric(\alpha, \alpha) = - \int_\Sigma \sum_{i,j=1}^{n-1} \Pi_{ij} \alpha(e_i) \alpha(e_j).
$$

Since $Ric \geq 0$ and $\Pi > 0$, we must have $\alpha = 0$. Therefore $H^1(M) = 0$.

Remark 1. *In the second part if we only assume $\Pi \geq 0$, then the same argument proves that a harmonic form $\alpha \in \mathcal{H}_A^1(M)$ must be parallel. As $\alpha(\nu) = 0$ on the boundary Σ we can write $\alpha = \langle X, \cdot \rangle$ on Σ, where X is a vector field on Σ. As α is parallel it is easy to see that X is a parallel vector field on Σ. Therefore we conclude that either $H^1(M) = 0$ or there exists a nonzero parallel vector field on Σ.*

In dimension 3 we have the following consequence.

Corollary 1. *Let $\left(M^3, g\right)$ be a compact Riemannian 3-manifold with boundary Σ. Suppose $Ric \geq 0$ and the boundary is strictly convex. Then the boundary Σ is topologically a sphere.*

Proof. We have the long exact sequence

$$
\cdots \to H^1(M, \Sigma) \to H^1(M) \to H^1(\Sigma) \to H^2(M, \Sigma) \to \cdots
$$

By Poincare duality $H^2(M, \Sigma) \approx H^1(M)$. Since $H^1(M) = 0$ we must have $H^1(\Sigma) = 0$, i.e. Σ is topologically a sphere. $\qquad\square$

In fact the same argument combined with Remark 1 yields

Proposition 2. *Let $\left(M^3, g\right)$ be a compact Riemannian 3-manifold with boundary Σ. Suppose $Ric \geq 0$ and the boundary is convex. Then Σ is either a topological sphere or a flat torus.*

Therefore the boundary cannot be a Riemann surface of higher genus.

The above two results in dimension 3 may be deduced from the work of Meeks-Simon-Yau [MSY], but the argument here is much more elementary.

The same argument works for the following situation.

Proposition 3. *Let (M^n, g) be a compact Riemannian manifold with positive Ricci curvature and convex boundary. If the boundary is convex, then both $H^1(M, \Sigma)$ and $H^1(M)$ vanish.*

When the Ricci curvature has a negative lower bound, we can also prove the vanishing of $H^1(M, \Sigma)$ if the boundary has sufficiently large mean curvature.

Proposition 4. *Let (M^n, g) be a compact Riemannian manifold with boundary Σ. Suppose $Ric \geq -(n-1)$. If $H \geq (n-1)$, then $H^1(M, \Sigma) = 0$.*

The proof is more complicated. We first recall the following result which can be proved by classic methods.

Proposition 5. *Let (M^n, g) be a compact Riemannian manifold with $Ric \geq -(n-1)$. Let ρ be the distance function to the boundary. Suppose the mean curvature of the boundary satisfies $H \geq (n-1)$. Then in the support sense*

$$\Delta \rho \leq -(n-1).$$

We compute

$$
\begin{aligned}
\Delta e^{c\rho} &= e^{c\rho} \left[c\Delta\rho + c^2 |\nabla\rho|^2 \right] \\
&\leq e^{c\rho} \left[-(n-1)c + c^2 \right] \\
&= e^{c\rho} \left[\left(c - \frac{n-1}{2} \right)^2 - \frac{(n-1)^2}{4} \right].
\end{aligned}
$$

Let $\phi = e^{(n-1)\rho/2}$. We have

$$(3.1) \qquad\qquad \Delta\phi \leq -\frac{(n-1)^2}{4}\phi.$$

It is well know that this implies that the first Dirichlet eigenvalue $\lambda_1 \geq \frac{(n-1)^2}{4}$.

We now prove the first part of Proposition 3. Let $\alpha \in \mathcal{H}_R^1(M)$. By a computation due to Yau we have

$$|\nabla\alpha|^2 \geq \frac{n}{n-1} |\nabla|\alpha||^2.$$

By the Bochner formula we have

$$
\begin{aligned}
\frac{1}{2}\Delta|\alpha|^2 &= |\nabla\alpha|^2 + Ric(\alpha, \alpha) \\
&\geq \frac{n}{n-1} |\nabla|\alpha||^2 - (n-1)|\alpha|^2.
\end{aligned}
$$

Therefore

$$|\alpha|\Delta|\alpha| \geq \frac{1}{n-1} |\nabla|\alpha||^2 - (n-1)|\alpha|^2.$$

Let $f = |\alpha|^{(n-2)/(n-1)}$. Direct calculation yields

$$\Delta f \geq -(n-2)f.$$

Let $u = f/\phi$. Direct calculation yields

$$
\begin{aligned}
\Delta u &\geq \left[\frac{(n-1)^2}{4} - (n-2) \right] u - 2\phi^{-1} \langle \nabla u, \nabla\phi \rangle \\
&= \frac{(n-3)^2}{4} u - 2\phi^{-1} \langle \nabla u, \nabla\phi \rangle
\end{aligned}
$$

Suppose that f is not identically zero. By the maximum principle u must achieve its positive maximum somewhere on the boundary and furthermore at this point we must have

$$\frac{\partial u}{\partial \nu} \geq 0.$$

On the other hand on the boundary, as $H \geq n - 1$

$$\frac{\partial u}{\partial \nu} = u \left[\frac{n-2}{n-1} \frac{\langle \nabla_\nu \alpha, \alpha \rangle}{|\alpha|^2} + \frac{n-1}{2} \right]$$

$$= u \left[-\frac{n-2}{n-1} H + \frac{n-1}{2} \right]$$

$$\leq u \left[-(n-2) + \frac{n-1}{2} \right]$$

$$= -u(n-3)/2.$$

This is strictly negative when $n \geq 4$ and hence a contradiction. Therefore α is identically zero. When $n = 3$ and if α is not identically zero, then by the Hopf lemma u must be a positive constant. By scaling α we can assume that $u \equiv 1$ or $\phi = f$. Therefore $-\Delta\phi = \phi$. By elliptic regularity ϕ is smooth. From the proof of (3.1) it follows that ρ is smooth everywhere and $|\nabla\rho| \equiv 1$. But this is impossible as ρ is not smooth at a cut point, e.g. at a point where it achieves its maximum. Therefore we must have $\alpha = 0$ too when $n = 3$.

The same argument can be used to prove the following: Let (M^n, g) be a compact Riemannian manifold with boundary Σ. Suppose $Ric \geq -(n-1)$. If the second fundamental form of Σ satisfies $\Pi > (n-1)/\sqrt{2(n-2)}$, then $H^1(M) = 0$. The only difference is that at the end we have for $\alpha \in \mathcal{H}_A^1(M)$

$$\frac{\partial u}{\partial \nu} = u \left[\frac{n-2}{n-1} \frac{\langle \nabla_\nu \alpha, \alpha \rangle}{|\alpha|^2} + \frac{n-1}{2} \frac{\sinh R}{\cosh R} \right]$$

$$= u \left[-\frac{n-2}{n-1} \frac{\Pi_{ij} \alpha(e_i) \alpha(e_i)}{|\alpha|^2} + \frac{n-1}{2} \frac{\sinh R}{\cosh R} \right]$$

$$\leq u \left[-\frac{n-2}{n-1} \frac{\cosh R}{\sinh R} + \frac{n-1}{2} \frac{\sinh R}{\cosh R} \right].$$

Is the constant sharp? It seems reasonable to expect $H^1(M) = 0$ if $\Pi > 1$.

4. On the size of the boundary

In this section we prove some estimates on the size of the boundary, in particular we show that Conjecture 2 is true in dimension 3. First we recall a result in Xia [X].

Proposition 6. *Let (M, g) be a compact Riemannian manifold with boundary Σ. Suppose $Ric(g) \geq 0$ and $\Pi \geq 1$. Then $\lambda_1(\Sigma) \geq n - 1$ and the equality holds iff (M, g) is isometric to the unit ball in Euclidean space $\mathbb{R}^n$.*

The proof is based on Reilly's formula [Re]. For completeness and comparison later, we present the proof. Let u be the solution of the following equation

$$\begin{cases} \Delta u = 0 & \text{on } M, \\ \quad u|_\Sigma = f \end{cases}$$

where f is a first eigenfunction on Σ, i.e. $-\triangle_\Sigma f = \lambda_1 f$. Let $\chi = \frac{\partial u}{\partial \nu}$ with ν being the outer unit normal. By Reilly's formula

$$\int_M (\Delta u)^2 - \left| D^2 u \right|^2 - Ric\,(\nabla u, \nabla u)$$
$$= \int_\Sigma \left[2\chi \Delta_\Sigma f + H\chi^2 + \Pi\,(\nabla f, \nabla f) \right] d\sigma$$
$$\geq \int_\Sigma -2\lambda_1 \chi f + (n-1)\,\chi^2 + |\nabla f|^2$$

As $\Delta u = 0$ and $\int_\Sigma |\nabla f|^2 = \lambda_1 \int_\Sigma f^2$

$$\int_\Sigma -2\lambda_1 \chi f + (n-1)\,\chi^2 + \lambda_1 f^2 \leq 0,$$

whence

$$\frac{\lambda_1(\lambda_1 - n + 1)}{n-1} \int_{\partial M} f^2 \geq \int_{\partial M} (\chi - \frac{\lambda_1}{n-1} f)^2 \geq 0.$$

Therefore $\lambda_1 \geq n - 1$.

If $\lambda_1 = n-1$, then we must have $D^2 u = 0, \chi = f$ and $\Pi = g|_\Sigma$. As a consequence we have $D^2 f = -fg$ on $\partial \Sigma$. By the well known Obata theorem ∂M is isometric to the standard sphere S^{n-1}. Let $f_1, \cdots, f_n$ be a standard basis of the first eigenspace on $\partial M \cong S^{n-1}$ and $u_1, \cdots, u_n$ the corresponding harmonic extensions on M. We know that $\nabla u_1, \cdots, \nabla u_n$ are parallel vector fields on M. It is then easy to see that $U = (u_1, \cdots, u_n)$ isometrically embeds M into $\mathbb{R}^n$ with the image the unit ball.

This is basically the same argument used by Choi and Wang [CW] to prove that the 1st eigenvalue of an embedded minimal hypersurface $\Sigma^{n-1} \subset \mathbb{S}^n$ is at least $(n-1)/2$. But a conjecture of Yau the 1st eigenvalue should equal to $n-1$. By the same argument we have the following estimate for a general compact Riemannian manifold with $Ric \geq n$ and with a convex boundary.

Proposition 7. *Let (M^n, g) be a compact Riemannian manifold with boundary Σ. Suppose $Ric(g) \geq n - 1$ and $\Pi \geq 0$. Then $\lambda_1(\Sigma) \geq (n-1)/2$.*

Remark 2. *In view of Yau's conjecture, we also conjecture that in this case the best lower bound is $n - 1$.*

Proof. Let u be the solution of the following equation

$$(4.1) \qquad\qquad \begin{cases} \Delta u = 0 & \text{on } M, \\ \quad u|_\Sigma = f \end{cases}$$

where f is a first eigenfunction on Σ, i.e. $-\triangle_\Sigma f = \lambda_1 f$. Let $\chi = \frac{\partial u}{\partial \nu}$ with ν being the outer unit normal. By Reilly's formula

$$\int_M (\Delta u)^2 - \left| D^2 u \right|^2 - Ric\left(\nabla u, \nabla u\right)$$

$$= \int_\Sigma \left[2\chi \Delta_\Sigma f + H\chi^2 + \Pi\left(\nabla f, \nabla f\right) \right] d\sigma$$

$$\geq \int_\Sigma -2\lambda_1 \chi f,$$

as $\Pi \geq 0$. Thus we get

$$2\lambda_1 \int_\Sigma \chi f \geq (n-1) \int_M |\nabla u|^2$$

From the equation of u we have $\int_\Sigma |\nabla u|^2 = \int_\Sigma f\chi$. Thus

$$[2\lambda_1 - (n-1)] \int_M |\nabla u|^2 \geq 0,$$

Therefore $\lambda_1 \geq (n-1)/2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We will also need the following result due to Ros [Ros], which was also proved by Reilly's formula.

Theorem 6. *(Ros) Let (M,g) be a compact Riemannian manifold with boundary. If $Ric \geq 0$ and the mean curvature H of ∂M is positive, then*

$$\int_{\partial M} \frac{1}{H} d\sigma \geq \frac{n}{n-1} V.$$

The equality holds iff M is isometric to an Euclidean ball.

We can now prove the following result in dimension 3.

Theorem 7. *Let (M^3, g) be a compact Riemannian manifold with boundary Σ. Suppose $Ric(g) \geq 0$ and $\Pi \geq g|_\Sigma$. Then*

- $A(\Sigma) \leq 4\pi$;
- $V(M) \leq 4\pi/3$.

Moreover if equality holds in either case, M is isometric to the unit ball $\overline{\mathbb{B}^3} \subset \mathbb{R}^3$.

Proof. By Proposition 6 we have $\lambda_1(\Sigma) \geq 2$. By Corollary 1 Σ is topologically $\mathbb{S}^2$. Then by a theorem of Hersch [H] (see also [SY, page 135]) $A(\Sigma) \leq 8\pi/\lambda_1(\Sigma)$ and moreover equality holds iff Σ is a round sphere. By Proposition 6 we have $\lambda_1(\Sigma) \geq 2$. Therefore $A(\Sigma) \leq 4\pi$. If equality holds, then $\lambda_1(\Sigma) = 2$ and hence M is isometric to $\overline{\mathbb{B}^3}$ by the rigidity part of Proposition 6.

The 2nd part easily follows from combining the first part and Theorem 6. $\qquad\square$

Example 1. *Let $\left(S^{n-2}, h\right)$ be compact Riemannian manifold with nonnegative Ricci curvature. Then $\overline{B^2} \times S$ has nonnegative Ricci curvature and the boundary has mean curvature $H = 1$. This show that the conjecture is not true if the condition on 2nd fundamental form is weakened to a condition on the mean curvature.*

In the case of positive Ricci curvature we make the following

Conjecture 3. *Let (M^n, g) be a compact Riemannian manifold with $Ric \geq n-1$ and $\Pi \geq 0$ on $\Sigma = \partial M$. Then*

$$|\Sigma| \leq \left|\mathbb{S}^{n-1}\right|.$$

Moreover if equality holds then (M^n, g) is isometric to the hemisphere $\mathbb{S}^n_+ = \{x \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} \geq 0\} \subset \mathbb{R}^{n+1}$.

In [HW] the following rigidity result was established.

Theorem 8. *Let (M^n, g) ($n \geq 2$) be a compact Riemannian manifold with non-empty boundary $\Sigma = \partial M$. Suppose*
- *$Ric \geq (n-1)g$,*
- *$(\Sigma, g|_\Sigma)$ is isometric to the standard sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$,*
- *Σ is convex in M in the sense that its second fundamental form is nonnegative.*

Then (M, g) is isometric to the hemisphere $\mathbb{S}^n_+$.

Therefore the conjecture, if true, is a far-reaching generalization of the above rigidity result. When $n = 2$ the above theorem can be reformulated as follows.

Theorem 9. *Let (M^2, g) be compact surface with boundary and the Gaussian curvature $K \geq 1$. Suppose the geodesic curvature k of the boundary γ satisfies $k \geq 0$. Then $L(\gamma) \leq 2\pi$. Moreover equality holds iff (M, g) is isometric to $\mathbb{S}^2_+$.*

It implies a classic result of Toponogov [T]:

> Let (M^2, g) be a closed surface with Gaussian curvature $K \geq 1$. Then any simple closed geodesic in M has length at most 2π. Moreover if there is one with length 2π, then M is isometric to the standard sphere $\mathbb{S}^2$.

We refer to [HW] for more details. In view of this connection, Conjecture 3 can be viewed as a generalization of Toponogov's theorem in higher dimensions. We note that Marques and Neves [MN] offered a generalization of Toponogov's theorem in dimension 3 in terms of the scalar curvature.

As an evidence for Conjecture 3 we show that it is true under the stronger condition that sectional curvatures are at least one.

Proposition 8. *Let (M^n, g) be a compact Riemannian manifold with $\sec \geq 1$ and $\Pi \geq 0$ on $\Sigma = \partial M$. Then*

$$|\Sigma| \leq \left|\mathbb{S}^{n-1}\right|.$$

Moreover if equality holds then (M^n, g) is isometric to the hemisphere $\mathbb{S}^n_+ = \{x \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} \geq 0\} \subset \mathbb{R}^{n+1}$.

Proof. The proof of the inequality is elementary. By the Gauss equation for any orthonormal pair $X, Y \in T_p\Sigma$

$$R^\Sigma(X, Y, X, Y) = R^\Sigma(X, Y, X, Y) + \Pi(X, X)\Pi(Y, Y) - \Pi(X, Y)^2$$

$$\geq 1 + \Pi(X, X)\Pi(Y, Y) - \Pi(X, Y)^2.$$

Since $\Pi \geq 0$ it is a simple algebraic fact that $\Pi(X, X)\Pi(Y, Y) - \Pi(X, Y)^2 \geq 0$. Therefore $R^\Sigma(X, Y, X, Y) \geq 1$, i.e. $\sec_\Sigma \geq 1$. By the Bishop-Gromov volume comparison we have $|\Sigma| \leq \left|\mathbb{S}^{n-1}\right|$.

Moreover if $|\Sigma| = \left|\mathbb{S}^{n-1}\right|$, then Σ is isometric to $\mathbb{S}^{n-1}$. By Theorem 8 M is isometric to the hemisphere $\mathbb{S}^n_+$. $\qquad\square$

Similarly we have the following parallel result when sectional curvature is non-negative.

Proposition 9. *Let (M^n, g) be a compact Riemannian manifold with $\sec \geq 0$ and $\Pi \geq 1$ on $\Sigma = \partial M$. Then*

$$|\Sigma| \leq \left|\mathbb{S}^{n-1}\right|.$$

Moreover if equality holds then (M^n, g) is isometric to the hemisphere $\overline{\mathbb{B}^n} = \{x \in \mathbb{R}^n : |x| \leq 1\}$.

Using Proposition 7 one can easily prove the following by the same method used to prove Theorem 7.

Proposition 10. *Let (M^3, g) be a compact Riemannian manifold with boundary Σ. Suppose $Ric(g) \geq 2$ and $\Pi \geq 0$. Then $A(\Sigma) \leq 8\pi$*

As stated in Conjecture 3 the optimal upper bound should be 4π.

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48824

E-mail address: xwang@math.msu.edu