MTH 930 HOMEWORK ASSIGNMENT 4

DUE OCT. 11 IN CLASS

(1) Let \((M^n, g)\) be a Riemannian manifold. For \(p \in M\), pick \(\delta > 0\) s.t. \(\exp_p : B_\delta \to B(p, \delta)\) is a diffeomorphism, where
\[
B_\delta = \{ v \in T_p M : |v| < \delta \},
B(p, \delta) = \{ q \in M : d(p, q) < \delta \}.
\]
Let \(\{e_i\}\) be an orthonormal basis for \(T_p M\) and \(\phi : \mathbb{R}^n \to T_p M\) the isometry \(\phi (x) = \sum_i x^i e_i\). We can introduce a local chart \(B(p, \delta) \to \mathbb{R}^n\) by \(x = \Phi(q) = \phi^{-1} \circ \exp_p^{-1}(q)\). Write \(g = g_{ij}(x) \, dx_i \otimes dx_j\) in these coordinates. Prove
\[
\begin{align*}
g_{ij}(0) &= \delta_{ij}, \\
g_{ij}(x) \, x_j &= x_i.
\end{align*}
\]
(Hint: \(t \to (tx_1, \cdots, tx_n)\) is a geodesic.)

(2) Let \((M^n, g)\) be a complete Riemannian manifold \(g\) another metric on \(M\) s.t. \(\bar{g} \geq g\), i.e. for any \(X \in TM\), \(\bar{g}(X, X) \geq g(X, X)\). Show that \((M, \bar{g})\) is also complete.

(3) Let \((M^n, g)\) be a complete Riemannian manifold and \(K \subset M\) a closed subset. The distance from \(K\) to \(p \in M\) is defined as
\[
d(p, K) := \inf \{ d(p, q) : q \in K \}.
\]
- Prove the infimum is achieved at some point \(q \in K\).
- Further assume that \(K\) is a submanifold. Let \(\gamma : [0, \ell] \to M\) be a minimizing geodesic from \(p\) to \(q\). Prove that \(\gamma'(t) \perp T_q K\).

(4) Consider the conformal ball model of the hyperbolic space: \(B^n\) with \(g = \frac{4}{(1 - |x|^2)^2} \, dx^2\). Prove
\[
d(0, x) = \log \frac{1 + |x|}{1 - |x|}.
\]

(5) Let \((M^n, g)\) be a Riemannian manifold with \(\sec \leq \kappa\). Let \(\gamma : [0, \ell] \to M\) be a unit-speed geodesic and \(J\) a normal Jacobi field along \(\gamma\) with \(J(0) = 0, \, J(0) = 1\). Suppose \(J(t) \neq 0\) for all \(t \in (0, \ell]\). Prove
\[
|J(t)| \geq \begin{cases} 
\frac{\sin \sqrt{\kappa t}}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\
\frac{\sinh \sqrt{\kappa t}}{\sqrt{\kappa}} & \text{if } \kappa = 0,
\end{cases}
\]
(Hint: inspect the proof of the Cartan-Hadamard theorem where the case \(\kappa = 0\) is proved.)