UNIQUENESS RESULTS ON A GEOMETRIC PDE IN RIEMANNIAN AND CR GEOMETRY REVISITED

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Abstract. We revisit some uniqueness results for a geometric nonlinear PDE related to the scalar curvature in Riemannian geometry and CR geometry. In the Riemannian case we give a new proof of the uniqueness result assuming only a positive lower bound for Ricci curvature. We apply the same principle in the CR case and reconstruct the Jerison-Lee identity in a more general setting. As a consequence we prove a stronger uniqueness result in the CR case. We also discuss some open problems for further study.

1. Introduction

Let (Σ^n, g) be a Riemannian manifold and $\widetilde{g} = u^{4/(n-2)} g$ another metric conformal to g, where u is a positive smooth function on Σ. The scalar curvatures are related by the following equation

$$-\frac{4(n-1)}{n-2}\Delta_g u + Ru = \widetilde{R} u^{(n+2)/(n-2)}.$$

Let $(\mathbb{S}^n, g_c)$ be the sphere with the standard metric. A conformal metric $\widetilde{g} = u^{4/(n-2)} g_c$ has constant scalar curvature $n(n-1)$ iff

$$-\frac{4}{n(n-2)}\Delta u + u = u^{(n+2)/(n-2)}, \text{ on } \mathbb{S}^n. \tag{1.1}$$

Conformal diffeomorphisms of $\mathbb{S}^n$ give rise to a natural family of solutions to the above equation

$$u_{t,\xi}(x) = (\cosh t + (\sinh t)\, x \cdot \xi)^{-(n-2)/2},$$

where $t \geq 0, \xi \in \mathbb{S}^n$. It is a remarkable theorem that these are all the positive solutions to (1.1). There are now several proofs for this theorem. Analytically, by the stereographic projection (1.1) is equivalent to the following equation

$$-\Delta v = \frac{n(n-2)}{4} v^{(n+2)/(n-2)} \text{ on } \mathbb{R}^n$$

whose positive solutions were classified by Gidas-Ni-Nirenberg [GNN] using the moving plane method. Geometrically, it follows from the following more general theorem of Obata.

Theorem 1. ([O2]) *Suppose $(\Sigma^n, \overline{g})$ is a closed Einstein manifold and $g = \phi^2 \overline{g}$ is a conformal metric with constant scalar curvature, where ϕ is a positive smooth function. Then ϕ must be constant unless $(\Sigma^n, \overline{g})$ is isometric to the standard sphere $(\mathbb{S}^n, g_c)$ up to a scaling and ϕ corresponds to the following function on $\mathbb{S}^n$*

$$\phi(x) = c(\cosh t + \sinh t x \cdot \xi)^{-2}$$

for some $c > 0, t \geq 0$ and $\xi \in \mathbb{S}^n$.

Obata's proof is short and elegant and is based on the following formula

$$\overline{T}=T+(n-2)\,\phi^{-1}\left(D^{2}\phi-\frac{\Delta\phi}{n}g\right),$$

where T and $\overline{T}$ are the traceless Ricci tensor of g and $\overline{g}$, respectively. But this argument is quite subtle as it requires using the unknown metric $g=\phi^{2}\overline{g}$ as the background metric instead of the given Einstein metric $\overline{g}$.

There is parallel story in CR geometry. Let M^{2m+1} be a CR manifold and $\widetilde{\theta}=f^{2/m}\theta$ two pseudohermitian structures. The pseudohermitian scalar curvatures of θ and $\widetilde{\theta}$ are related by the following formula

$$-\frac{2\,(m+1)}{m}\Delta_{b}f+Rf=\widetilde{R}f^{(m+2)/m}.$$

On $\mathbb{S}^{2m+1}=\left\{z\in\mathbb{C}^{m+1}:|z|=1\right\}$ the canonical pseudohermitian structure $\theta_{c}=\left(2\sqrt{-1}\partial\,|z|^{2}\right)|_{\mathbb{S}^{2m+1}}$ satisfies $R_{\alpha\overline{\beta}}=(m+1)\,/2\delta_{\alpha\beta}$ and $R=m\,(m+1)\,/2$. Therefore $\theta=f^{2/m}\theta_{c}$ has scalar curvature $m\,(m+1)\,/2$ iff

$$-\frac{4}{m^{2}}\Delta_{b}f+f=f^{(m+2)/m}\text{ on }\mathbb{S}^{2m+1}. \tag{1.2}$$

Pseudoconformal diffeomorphisms of $\mathbb{S}^{2m+1}$ yield a natural family of solutions to the above equation

$$f_{t,\xi}\,(z)=\left|\cosh t+(\sinh t)\,z\cdot\overline{\xi}\right|^{-1/m}.$$

It is a remarkable result of Jerison and Lee [JL] that these are all the positive solutions of (1.2). The proof is based on a highly nontrivial identity on $\left(\mathbb{S}^{2m+1},\phi\theta_{c}\right)$, with $\phi=f^{2/m}$

(1.3)

$$\begin{aligned}&\operatorname{Re}\left(gD_{\alpha}+\overline{g}E_{\alpha}-3\phi_{0}\sqrt{-1}U_{\alpha}\right)_{,\overline{\alpha}}\\&=\left(\frac{1}{2}+\frac{1}{2}\phi\right)\left(|D_{\alpha\beta}|^{2}+\left|E_{\alpha\overline{\beta}}\right|^{2}\right)\\&+\phi\left[|D_{\alpha}-U_{\alpha}|^{2}+|U_{\alpha}+E_{\alpha}-D_{\alpha}|^{2}+|U_{\alpha}+E_{\alpha}|^{2}+\left|\phi^{-1}\phi_{\overline{\gamma}}D_{\alpha\beta}+\phi^{-1}\phi_{\beta}E_{\alpha\overline{\gamma}}\right|^{2}\right].\end{aligned}$$

where

$$\begin{aligned}D_{\alpha\beta}&=\phi^{-1}\phi_{\alpha,\beta},D_{\alpha}=\phi^{-1}\phi_{\overline{\beta}}D_{\alpha\beta},E_{\alpha}=\phi^{-1}\phi_{\gamma}E_{\alpha\overline{\gamma}},\\E_{\alpha\overline{\beta}}&=\phi^{-1}\phi_{\alpha,\overline{\beta}}-\phi^{-2}\phi_{\alpha}\phi_{\overline{\beta}}-\frac{1}{2}\phi^{-1}\,(g-\phi)\,\delta_{\alpha\overline{\beta}},\\U_{\alpha}&=\frac{2}{m+2}D_{\alpha\beta,\overline{\beta}},g=\frac{1}{2}+\frac{1}{2}\phi+\phi^{-1}\,|\partial\phi|^{2}+i\phi_{0}.\end{aligned}$$

Here and throughout this paper we always work with a local unitary frame $\{T_{\alpha}:\alpha=1,\cdots,m\}$ for $T^{1,0}M$ and $T_{0}=T$ is the Reeb vector field. It should be emphasized that in all these formulas covariant derivatives are calculated w.r.t. the unknown pseudoconformal structure $\phi\theta_{c}$.

The Jerison-Lee identity is in fact valid on any closed Einstein pseudohermitian manifold. Here by Einstein we mean $R_{\alpha\overline{\beta}}=\rho\delta_{\alpha\beta}$ and $A_{\alpha\beta}=0$ (torsion-free). The following more general uniqueness result, which is the analogue of the Obata theorem, was proved in [W].

Theorem 2. *([W]) Let $\left(M^{2m+1},\overline{\theta}\right)$ be a closed Einstein pseudohermitian manifold. Suppose $\theta=\phi\overline{\theta}$ is another pseudohermitian structure with constant pseudohermitian scalar curvature. Then ϕ must be constant unless $\left(M^{2m+1},\overline{\theta}\right)$ is CR isometric to $\left(\mathbb{S}^{2m+1},\theta_c\right)$ up to a scaling and ϕ corresponds to the following function on $\mathbb{S}^{2m+1}$*

$$\phi\left(z\right)=c_m\left|\cosh t+\left(\sinh t\right)z\cdot\overline{\xi}\right|^{-2}$$

for some $t\geq 0$ and $\xi\in\mathbb{S}^{2m+1}$.

We note that like the Obata argument all calculations have to be carried out with respect to the unknown pseudehermitian structure $\theta=\phi\overline{\theta}$. Complicated formulas relating the curvature tensors of θ and $\overline{\theta}$ as well as various Bianchi identities are also heavily used in the proof.

The Jerison-Lee identity is truly remarkable and a better understanding is highly desirable. In this paper, we propose a different approach to reconstruct the formula. The basic idea is to study the model case carefully and then come up with the right quantities to apply the maximum principle. We first revisit the Riemannian case and give a new(?) proof of the uniqueness results. In fact, this new proof does not require the Einstein condition. A positive lower bound for Ricci curvature suffices. Suppose (M^n,g) is a compact Riemannian manifold with $Ric\geq n-1$ and $u\in C^{\infty}\left(M\right)$ is **positive** and satisfies the following equation

$$-\Delta u+\frac{n\left(n-2\right)}{4}u=\frac{n\left(n-2\right)}{4}u^{(n+2)/(n-2)}.$$

If we write $u=v^{-(n-2)/2}$, then v satisfies

$$\Delta v=\frac{n}{2}v^{-1}\left(\left|\nabla v\right|^2+1-v^2\right).$$

By the study of the model case, we consider $\phi=v^{-1}\left(\left|\nabla v\right|^2+v^2+1\right)$. A simple calculation shows that

$$\Delta\phi\geq\left(n-2\right)\left\langle\nabla\log v,\nabla\phi\right\rangle$$

and therefore the maximum principle comes into play. This simple argument yields the following result which is more general than Obata's theorem.

Theorem 3. *Let (M^n,g) be a smooth compact Riemannian manifold with a (possibly empty) convex boundary. Suppose $u\in C^{\infty}\left(M\right)$ is a positive solution of the following equation*

$$\begin{aligned}-\Delta u+\lambda u&=u^{(n+2)/(n-2)} &&\text{on}\quad M,\\ \tfrac{\partial u}{\partial\nu}&=0 &&\text{on}\quad \partial M,\end{aligned}$$

where $\lambda>0$ is a constant. If $Ric\geq\left(n-1\right)g$ and $\lambda\leq n\left(n-2\right)/4$, then u must be constant unless $\lambda=n\left(n-2\right)/4$ and (M,g) is isometric to $\left(\mathbb{S}^n,g_c\right)$ or $\left(\mathbb{S}^n_+,g_c\right)$. In the latter case u is given on $\mathbb{S}^n$ or $\mathbb{S}^n_+$ by the following formula

$$u(x)=c_n\left(\cosh t+\left(\sinh t\right)x\cdot\xi\right)^{-(n-2)/2}.$$

for some $t\geq 0$ and $\xi\in\mathbb{S}^n$.

The above theorem is actually not new. It is a special case of a theorem by Bidaut-Véron and Véron [BVV] and Ilias [I]. Their method is based on a sophisticated integration by parts which can handle the subcritical case as well. We will say more about their result in the last section.

We apply the same principle to the CR case. Here the first difficulty is that there is no natural first order quantity and therefore we have to go to the 2nd order. There are three natural tensors of order 2 to consider and we must take a suitable contraction and combination to apply the maximum principle. As our argument in the Riemannian case, this approach has the advantage that the calculations are done on a fixed pseudohermitian manifold $\left(\Sigma^{2m+1},\theta\right)$ which does not have to be Einstein. The unknown pseudohermitian structure $\theta=\phi\widetilde{\theta}$ and its curvature tensor do not enter the discussion at all. All it takes is to do covariant derivatives. Of course we are using a lot of hindsight from Jerison-Lee. Besides the identity (1.3) Jerison and Lee [JL] gave three additional divergence formulas on the Heisenberg group. The formula we obtain can be viewed as the generalization of their first formula ((4.2) in [JL]) to any pseudohermitian manifold with torision zero. (One can even drop this condition, but the additional terms involving the torsion $A_{\alpha\beta}$ and its divergence seem too complicated). The calculations are still formidable. But we hope that this approach sheds more light on the Jerison-Lee work. We do get a more general identity, see Theorem 6. As a result we prove a stronger uniqueness theorem.

Theorem 4. *Let $\left(M^{2m+1},\theta\right)$ be a closed pseudohermitian manifold with $A_{\alpha\beta}=0$ and $R_{\alpha\overline{\beta}}\geq\frac{m+1}{2}$. Suppose $f>0$ satisfies the following equation on M*

$$-\Delta_b f+\lambda f=f^{(m+2)/m},$$

where $\lambda>0$ is a constant. If $\lambda\leq m^2/4$, then f is constant unless $\lambda=m^2/4$ and (M,θ) is isometric to $\left(\mathbb{S}^{2m+1},\theta_c\right)$ and in this case

$$f=c_m\left|\cosh t+(\sinh t)\,z\cdot\overline{\xi}\right|^{-1/m}$$

for some $t>0,\xi\in\mathbb{S}^{2m+1}$.

The paper is organized as follows. In the 2nd section we discuss the Riemannian case. In Section 3 we study the model case in CR geometry as a guide for finding the right quantities. In Section 4 we present our reconstruction of the Jerison-Lee identity and prove the above uniqueness result. We discuss some open problems in the last section.

2. The Riemannian case

On $(\mathbb{S}^n,g_c)$ we consider the equation

$$-\frac{4}{n\,(n-2)}\Delta u+u=u^{(n+2)/(n-2)}. \tag{2.1}$$

If u is positive, the equation simply means that $u^{4/(n-2)}g_c$ has the same scalar curvature $n\,(n-1)$. For $t\geq 0,\xi\in\mathbb{S}^n$ the map $\Phi_{t,\xi}:\mathbb{S}^n\rightarrow\mathbb{S}^n$ defined by

$$\Phi_{t,\xi}\left(x\right)=\frac{1}{\cosh t+\sinh t\left(x\cdot\xi\right)}\left(x-\left(x\cdot\xi\right)\xi\right)+\frac{\sinh t+\cosh t\left(x\cdot\xi\right)}{\cosh t+\sinh t\left(x\cdot\xi\right)}\xi$$

is a conformal diffeomorphism with $\Phi_{t,\xi}^{*}g_c=u_{t,\xi}^{4/(n-2)}g_c$ with

$$u_{t,\xi}\left(x\right)=\left(\cosh t+\left(\sinh t\right)x\cdot\xi\right)^{-(n-2)/2}.$$

Therefore these are solutions of the equation (2.1).

If we write $u = v^{-(n-2)/2}$, then $v = \cosh t + (\sinh t)\, x \cdot \xi$. We compute

$$\begin{aligned}|\nabla v|^2 &= \sinh^2 t\, |\nabla (x\cdot\xi)|^2\\ &= \sinh^2 t \left(1-(x\cdot\xi)^2\right)\\ &= \sinh^2 t - (v-\cosh t)^2\\ &= -1 - v^2 + 2v\cosh t.\end{aligned}$$

It follows that $v^{-1}\left(|\nabla v|^2 + v^2 + 1\right) = 2\cosh t$ is a constant.

Suppose now (M^n, g) is a compact Riemannian manifold with $Ric \geq n-1$ and $u \in C^\infty(M)$ is **positive** and satisfies the following equation

$$-\Delta u + \frac{n(n-2)}{4}u = \frac{n(n-2)}{4}u^{(n+2)/(n-2)}. \tag{2.2}$$

If $Ric = n-1$ as in Obata's theorem, the above equation simply means that the scalar curvature of $\widetilde{g} := u^{4/(n-2)}$ equals $n(n-1)$. In the following discussion, this geometric interpretation plays no role. We write $u = v^{-(n-2)/2}$. By direct calculation, $v > 0$ satisfies the following equation

$$\Delta v = \frac{n}{2} v^{-1}\left(|\nabla v|^2 + 1 - v^2\right).$$

In view of the model case, we set $\phi = v^{-1}\left(|\nabla v|^2 + v^2 + 1\right)$. The above equation becomes $\Delta v + nv = \frac{n}{2}\phi$. As $v\phi = |\nabla v|^2 + v^2 + 1$, we compute using the Bochner formula

$$\begin{aligned}&\frac{1}{2}v\Delta\phi + \langle\nabla v, \nabla\phi\rangle + \frac{1}{2}\phi\Delta v\\ =&\left|D^2 v\right|^2 + \langle\nabla v, \nabla\Delta v\rangle + Ric(\nabla v, \nabla v) + v\Delta v + |\nabla v|^2\\ \geq&\frac{(\Delta v)^2}{n} + \langle\nabla v, \nabla\Delta v\rangle + n|\nabla v|^2 + v\Delta v\\ =&\frac{\Delta v}{n}(\Delta v + nv) + \langle\nabla v, \nabla(\Delta v + nv)\rangle\\ =&\frac{1}{2}\phi\Delta v + \frac{n}{2}\langle\nabla v, \nabla\phi\rangle.\end{aligned}$$

Therefore we obtain

$$\Delta\phi \geq (n-2)\langle\nabla\log v, \nabla\phi\rangle.$$

If $\partial M \neq \emptyset$, direct calculation yields the following formula for the normal derivative along ∂M

$$\frac{1}{2}\frac{\partial\phi}{\partial\nu} = f^{-1}\left[\chi\left(D^2 v(\nu,\nu) + f - \frac{1}{2}\phi\right) + \langle\nabla f, \nabla\chi\rangle - \Pi(\nabla f, \nabla f)\right],$$

where $f = v|_{\partial M}, \chi = \frac{\partial v}{\partial\nu}$ and Π is the 2nd fundamental form. By these calculations, we can now deduce the following uniqueness result.

Theorem 5. *Let (M^n, g) be a smooth compact Riemannian manifold with a (possibly empty) convex boundary. Suppose $u \in C^\infty(M)$ is a positive solution of the following equation*

$$\begin{aligned}-\Delta u + \lambda u &= u^{(n+2)/(n-2)} && \text{on} \quad M,\\ \frac{\partial u}{\partial\nu} &= 0 && \text{on} \quad \partial M,\end{aligned}$$

where $\lambda > 0$ is a constant. If $Ric \geq (n-1)\, g$ and $\lambda \leq n\,(n-2)\,/4$, then u must be constant unless $\lambda = n\,(n-2)\,/4$ and (M, g) is isometric to $(\mathbb{S}^n, g_c)$ or $\left(\mathbb{S}^n_+, g_c\right)$. In the latter case u is given on $\mathbb{S}^n$ or $\mathbb{S}^n_+$ by the following formula

$$u(x) = c_n \left(\cosh t + (\sinh t)\, x \cdot \xi\right)^{-(n-2)/2}.$$

for some $t \geq 0$ and $\xi \in \mathbb{S}^n$.

Proof. We first take $\lambda = n\,(n-2)\,/4$. By scaling u we can consider the equivalent equation (2.2). Then the above calculations for the associated v and ϕ yield

$$\Delta\phi \geq (n-2)\,\langle \nabla \log v, \nabla \phi\rangle \text{ on } M; \frac{\partial \phi}{\partial \nu} \leq 0 \text{ on } \partial M$$

under our assumptions. By the maximum principle and Hopf lemma, ϕ must be a constant. Inspecting the proof shows that we must have $D^2 v = \frac{1}{n}\Delta v g$. If v is not constant, it is easy to deduce from this over-determined system that (M, g) is isometric to $(\mathbb{S}^n, g_c)$ or $\left(\mathbb{S}^n_+, g_c\right)$ and v is up to a constant a linear function. This finishes the proof when $\lambda = n\,(n-2)\,/4$.

When $\lambda < n\,(n-2)\,/4$, we consider the scaled metric $\widetilde{g} = cg$. Then u satisfies

$$-\widetilde{\Delta} u + c^{-1}\lambda u = c^{-1} u^{(n+2)/(n-2)}.$$

We choose $c = \frac{4\lambda}{n(n-2)} < 1$. As $Ric\,(\widetilde{g}) \geq \frac{n-1}{c}\widetilde{g} > (n-1)\,\widetilde{g}$, we can apply the result for $\lambda = n\,(n-2)\,/4$ on $(M^n, \widetilde{g})$. ∎

3. The CR sphere

Consider the unit sphere $\mathbb{S}^{2m+1} = \left\{z \in \mathbb{C}^{m+1} : |z| = 1\right\}$ with the canonical pseudohermitian structure

$$\theta_c = \left(2\sqrt{-1}\partial\, |z|^2\right)|_{\mathbb{S}^{2m+1}} = 2\sum_{i=1}^{m+1}\left(x_i dy_i - y_i dx_i\right),$$

i.e. $\theta_c\,(X) = 2\,\langle J\xi, X\rangle$ at $\xi \in \mathbb{S}^{2m+1}$. Then

$$d\theta_c\,(X, Y) = 4\,\langle JX, Y\rangle\,.$$

The Reeb vector field is

$$\begin{aligned} T &= \frac{\sqrt{-1}}{2}\sum_{i=1}^{m+1}\left(z_i\frac{\partial}{\partial z_i} - \overline{z}_i\frac{\partial}{\partial \overline{z}_i}\right) \\ &= \frac{1}{2}\sum_{i=1}^{m+1}\left(x_i\frac{\partial}{\partial y_i} - y_i\frac{\partial}{\partial x_i}\right) \\ &= \frac{1}{2}J\xi. \end{aligned}$$

Therefore the adapted metric $g_c = 4g_0$, where g_0 is the standard metric on $\mathbb{S}^{2m+1}$. We have $R_{\alpha\overline{\beta}} = (m+1)\,/2\delta_{\alpha\beta}, R = m\,(m+1)\,/2$. We now consider the following equation

$$-\frac{4}{m^2}\Delta_b f + f = f^{(m+2)/m} \text{ on } \mathbb{S}^{2m+1}. \tag{3.1}$$

If $f > 0$, it simply means the pseudohermitian structure $f^{2/m}\theta_c$ has the same constant scalar curvature $m\,(m+1)\,/2$.

For $t \geq 0, \xi \in \mathbb{S}^n$ the map $\Phi_{t,\xi} : \mathbb{S}^{2m+1} \to \mathbb{S}^{2m+1}$ defined by

$$\Phi_{t,\xi}(z) = \frac{1}{\cosh t + \sinh t\left(z \cdot \overline{\xi}\right)}\left(z - \left(z \cdot \overline{\xi}\right)\xi\right) + \frac{\sinh t + \cosh t\left(z \cdot \overline{\xi}\right)}{\cosh t + \sinh t\left(z \cdot \overline{\xi}\right)}\xi$$

is a pseudoconformal diffeomorphism with $\Phi^*_{t,\xi}\theta_c = f^{2/m}_{t,\xi}\theta_c$, where

$$f_{t,\xi}(z) = \left|\cosh t + (\sinh t)\, z \cdot \overline{\xi}\right|^{-1/m}.$$

Therefore these are solutions to the equation (3.1). We write such a solution as $f = \phi^{-m/2}$. Then

$$\phi(z) = \left|\cosh t + (\sinh t)\, z \cdot \overline{\xi}\right|^2.$$

We want to see what identities ϕ satisfy. In the following, we always take θ_c and its adapted metric $g_c = 4g_0$ as a background metric. With $f(z) = z \cdot \overline{\xi}$, we have

$$\phi = \cosh^2 t + \sinh^2 t\,|f|^2 + \sinh t \cosh t\left(f + \overline{f}\right).$$

As $f_{\overline{\alpha}} = 0$,

$$\phi_\alpha = \sinh t\left(\cosh t + \sinh t\overline{f}\right) f_\alpha.$$

We also observe, as $T(z) = \frac{1}{2}Jz$,

$$f_0 = \frac{1}{2}\frac{d}{dt}|_{t=0} f\left(e^{ir}z\right) = \frac{\sqrt{-1}}{2}f.$$

If we write $f = u + \sqrt{-1}v$, then $u_0 = -\frac{1}{2}v$. Moreover, $4|\nabla u|^2 = 1 - u^2$ (1st eigenfunction). As a first eigenfunction, we have $D^2 f = -\frac{1}{4}f g_\theta$. It follows

$$\begin{aligned}
f_{\alpha,\beta} &= 0,\\
f_{\alpha,\overline{\beta}} &= D^2 f\left(T_\alpha, T_{\overline{\beta}}\right) + \frac{\sqrt{-1}}{2} f_0 \delta_{\alpha\beta}\\
&= \left(-\frac{1}{4}f + \frac{\sqrt{-1}}{2}f_0\right)\delta_{\alpha\beta}\\
&= -\frac{1}{2}f\delta_{\alpha\beta}.
\end{aligned}$$

Lemma 1. *We have*

$$\phi^{-1}|\partial\phi|^2 = \frac{1}{2}\sinh t\left(1 - |f|^2\right).$$

Proof. As $f_{\overline{\alpha}} = 0$, we have $f_\alpha = 2u_\alpha$. We compute

$$\begin{aligned}
|\partial\phi|^2 &= \phi \sinh^2 t\,|\partial f|^2\\
&= 4\phi\sinh^2 t\,|\partial u|^2\\
&= 2\phi\sinh^2 t\left(|\nabla u|^2 - u_0^2\right)\\
&= \frac{1}{2}\phi\sinh^2 t\left(1 - u^2 - v^2\right)\\
&= \frac{1}{2}\phi\sinh^2 t\left(1 - |f|^2\right).
\end{aligned}$$

□

Let $g = \frac{1}{2} + \frac{1}{2}\phi + \phi^{-1}|\partial\phi|^2 + \sqrt{-1}\phi_0$.

Lemma 2. *The function ϕ satisfies the following three tensor equations*

$$\phi_{\alpha,\beta}=0,$$

$$\phi_{\alpha,\overline{\beta}}-\phi^{-1}\phi_{\alpha}\phi_{\overline{\beta}}=\frac{1}{2}\left(g-\phi\right)\delta_{\alpha\beta}.$$

$$\phi_{0,\alpha}=\frac{\sqrt{-1}}{2}\phi^{-1}\overline{g}\phi_{\alpha}$$

Proof. The 1st identity is obvious. We have $\phi_0=\frac{i}{2}\sinh t\cosh t\left(f-\overline{f}\right)$. Thus

$$\begin{aligned} g&=\frac{1}{2}+\frac{1}{2}\phi+\frac{1}{2}\sinh^2 t\left(1-|f|^2\right)-\frac{1}{2}\sinh t\cosh t\left(f-\overline{f}\right)\\ &=\cosh^2 t+\sinh t\cosh t\overline{f}\\ &=\cosh t\left(\cosh t+\sinh t\overline{f}\right).\end{aligned}$$

We compute

$$\begin{aligned}\phi_{\alpha,\overline{\beta}}&=\sinh^2 tf_{\alpha}\overline{f_{\beta}}-\frac{1}{2}f\sinh t\left(\cosh t+\sinh t\overline{f}\right)\delta_{\alpha\beta}\\ &=\phi^{-1}\phi_{\alpha}\phi_{\overline{\beta}}-\frac{1}{2}\sinh t\left(\cosh tf+\sinh t\,|f|^2\right)\delta_{\alpha\beta}\end{aligned}$$

To finish the proof of the 2nd identity, we check

$$\begin{aligned} g-\phi&=-\sinh^2 t\,|f|^2-\sinh t\cosh tf\\ &=-\sinh t\left(\cosh tf+\sinh t\,|f|^2\right).\end{aligned}$$

The last identity follows from

$$\phi_{0,\alpha}=\frac{i}{2}\sinh t\cosh tf_{\alpha}.$$

□

Remark 1. *There is an additional identity*

$$\phi_{0,0}=\frac{1}{2}\phi^{-1}\left(\frac{1-\phi^2}{4}+\phi^{-1}|\partial\phi|^2+\phi^{-2}|\partial\phi|^4+\phi_0^2\right)$$

which is of higher order as it involves the 2nd order derivative in the direction of the Reeb vector field.

4. Reconstructing the Jerison-Lee identity

We now consider the general case. Let $\left(M^{2m+1},\theta\right)$ be a closed pseudohermitian manifold with torsion $A_{\alpha\beta}=0$. Suppose $f\in C^{\infty}\left(M\right)$ is a positive solution of the following equation

$$-\frac{4}{m^2}\Delta_b f+f=f^{(m+2)/m}.$$

If θ has scalar curvature $m\left(m+1\right)/2$, the equation simply means that $\widetilde{\theta}=f^{2/m}\theta$ has also scalar curvature $m\left(m+1\right)/2$. But this interpretation does not play any role in our discussion. Let $\phi=f^{-2/m}$. The above equation becomes

$$\phi_{\alpha,\overline{\alpha}}=\frac{m}{2}i\phi_0+\frac{m}{4}\left(1-\phi\right)+\frac{m+2}{2}\phi^{-1}|\partial\phi|^2. \tag{4.1}$$

Motivated by the study in the model case, we introduce

$$B_{\alpha\overline{\beta}}=\phi_{\alpha,\overline{\beta}}-\phi^{-1}\phi_\alpha\phi_{\overline{\beta}}-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}\phi+\phi^{-1}\left|\partial\phi\right|^2+i\phi_0\right)\delta_{\alpha\overline{\beta}},$$
$$C_\alpha=i\phi_{0,\alpha}+\frac{1}{2}\phi^{-1}\left(\frac{1}{2}+\frac{1}{2}\phi+\phi^{-1}\left|\partial\phi\right|^2-i\phi_0\right)\phi_\alpha$$

Note that the equation (4.1) simply means $B_{\alpha\overline{\alpha}}=0$. As the study of the model case suggests, to prove the uniqueness result we must prove $\phi_{\alpha\beta}=0, B_{\alpha\overline{\beta}}$ and $C_\alpha=0$. We set

$$g=\frac{1}{2}+\frac{1}{2}\phi+\phi^{-1}\left|\partial\phi\right|^2+i\phi_0.$$

Then we can rewrite these equations as

$$B_{\alpha\overline{\beta}}=\phi_{\alpha,\overline{\beta}}-\phi^{-1}\phi_\alpha\phi_{\overline{\beta}}-\frac{1}{2}\left(g-\phi\right)\delta_{\alpha\beta},$$
$$C_\alpha=i\phi_{0,\alpha}+\frac{1}{2}\phi^{-1}\overline{g}\phi_\alpha,$$
$$\phi_{\alpha\overline{\alpha}}=\frac{m}{2}\left(g-\phi\right)+\phi^{-1}\left|\partial\phi\right|^2.$$

We further introduce the contractions

$$A_\alpha=\phi_{\alpha\beta}\phi_{\overline{\beta}}, B_\alpha=B_{\alpha\overline{\beta}}\phi_\beta.$$

Therefore we have three complex $(1,0)$ vector fields A_α, B_α and C_α. Their conjugates will be denoted by $A_{\overline{\alpha}}, B_{\overline{\alpha}}$ and $C_{\overline{\alpha}}$. Our first goal is to calculate the divergence for these vector fields. We need some preliminary formulas.

Lemma 3. *We have*

$$\left(\phi^{-1}\left|\partial\phi\right|^2\right)_{\overline{\alpha}}=\phi^{-1}\left(A_{\overline{\alpha}}+B_{\overline{\alpha}}\right)+\frac{1}{2}\phi^{-1}\left(g-\phi\right)\phi_{\overline{\alpha}},$$
$$g_{\overline{\alpha}}=\phi^{-1}\left(A_{\overline{\alpha}}+B_{\overline{\alpha}}\right)-C_{\overline{\alpha}}+\phi^{-1}g\phi_{\overline{\alpha}},$$
$$\overline{g}_{\overline{\alpha}}=\phi^{-1}\left(A_{\overline{\alpha}}+B_{\overline{\alpha}}\right)+C_{\overline{\alpha}}$$

Proof. We compute

$$\left(\phi^{-1}\left|\partial\phi\right|^2\right)_{\overline{\alpha}}=\phi^{-1}\left(\phi_\beta\phi_{\overline{\beta},\overline{\alpha}}+\phi_{\beta,\overline{\alpha}}\phi_{\overline{\beta}}\right)-\phi^{-2}\left|\partial\phi\right|^2\phi_{\overline{\alpha}}$$
$$=\phi^{-1}\left(A_{\overline{\alpha}}+\overline{\phi_{\alpha,\overline{\beta}}\phi_\beta}+i\phi_0\phi_{\overline{\alpha}}\right)-\phi^{-2}\left|\partial\phi\right|^2\phi_{\overline{\alpha}}.$$

Eliminating $\phi_{\alpha,\overline{\beta}}$ using the formula for $B_{\alpha\overline{\beta}}$ yields

$$\left(\phi^{-1}\left|\partial\phi\right|^2\right)_{\overline{\alpha}}=\phi^{-1}\left(A_{\overline{\alpha}}+\overline{B_{\alpha\overline{\beta}}\phi_\beta+\phi^{-1}\left|\partial\phi\right|^2\phi_\alpha+\frac{1}{2}\left(g-\phi\right)\phi_\alpha}+i\phi_0\phi_{\overline{\alpha}}\right)$$
$$-\phi^{-2}\left|\partial\phi\right|^2\phi_{\overline{\alpha}}.$$
$$=\phi^{-1}\left(A_{\overline{\alpha}}+B_{\overline{\alpha}}+\frac{1}{2}\left(\overline{g}+2i\phi_0-\phi\right)\phi_{\overline{\alpha}}\right)$$
$$=\phi^{-1}\left(A_{\overline{\alpha}}+B_{\overline{\alpha}}+\frac{1}{2}\left(g-\phi\right)\phi_{\overline{\alpha}}\right).$$

This proves the 1st identity. Differentiating g yields

$$\begin{aligned} g_{\overline{\alpha}} &= \frac{1}{2}\phi_{\overline{\alpha}} + \left(\phi^{-1}\left|\partial\phi\right|^2\right)_{\overline{\alpha}} + i\phi_{0,\overline{\alpha}} \\ &= \phi^{-1}\left(A_{\overline{\alpha}} + B_{\overline{\alpha}}\right) + \frac{1}{2}\phi^{-1} g\phi_{\overline{\alpha}} + i\phi_{0,\overline{\alpha}} \\ &= \phi^{-1}\left(A_{\overline{\alpha}} + B_{\overline{\alpha}}\right) - C_{\overline{\alpha}} + \phi^{-1} g\phi_{\overline{\alpha}} \end{aligned}$$

Differentiating $\overline{g}$ yields

$$\begin{aligned} \overline{g}_{\overline{\alpha}} &= \phi^{-1}\left(A_{\overline{\alpha}} + B_{\overline{\alpha}}\right) + \frac{1}{2}\phi^{-1} g\phi_{\overline{\alpha}} - i\phi_{0,\overline{\alpha}} \\ &= \phi^{-1}\left(A_{\overline{\alpha}} + B_{\overline{\alpha}}\right) + C_{\overline{\alpha}}. \end{aligned}$$

□

Lemma 4. *We have*

$$\begin{aligned} \phi_{\alpha,\beta\overline{\beta}} &= \frac{m+2}{2}\left[\phi^{-1}\left(A_\alpha + B_\alpha\right) + C_\alpha\right] + R_{\alpha\overline{\beta}}\phi_\beta - \frac{m+1}{2}\phi_\alpha, \\ B_{\alpha\overline{\beta},\overline{\alpha}} &= \frac{(m-1)}{2}\phi^{-1}A_{\overline{\beta}} + \frac{m+1}{2}\phi^{-1}B_{\overline{\beta}} - \frac{(m-1)}{2}C_{\overline{\beta}}. \end{aligned}$$

Proof. We compute, using Lemma 3

$$\begin{aligned} \phi_{\alpha,\beta\overline{\beta}} =& \phi_{\beta,\overline{\beta}\alpha} + i\phi_{\alpha,0} + R_{\alpha\overline{\beta}}\phi_\beta \\ =& \frac{m}{2}\left(g_\alpha - \phi_\alpha\right) + \left(\phi^{-1}\left|\partial\phi\right|^2\right)_\alpha + i\phi_{\alpha,0} + R_{\alpha\overline{\beta}}\phi_\beta \\ =& \frac{m}{2}\left[\phi^{-1}\left(A_\alpha + B_\alpha\right) + C_\alpha - \phi_\alpha\right] + \phi^{-1}\left(A_\alpha + B_\alpha\right) + \frac{1}{2}\phi^{-1}\left(\overline{g} - \phi\right)\phi_\alpha \\ & + i\phi_{\alpha,0} + R_{\alpha\overline{\beta}}\phi_\beta \\ =& \frac{m}{2}\left[\phi^{-1}\left(A_\alpha + B_\alpha\right) + C_\alpha\right] + \phi^{-1}\left(A_\alpha + B_\alpha\right) + \frac{1}{2}\phi^{-1}\overline{g}\phi_\alpha + i\phi_{\alpha,0} \\ & + R_{\alpha\overline{\beta}}\phi_\beta - \frac{m+1}{2}\phi_\alpha \\ =& \frac{m+2}{2}\left[\phi^{-1}\left(A_\alpha + B_\alpha\right) + C_\alpha\right] + R_{\alpha\overline{\beta}}\phi_\beta - \frac{m+1}{2}\phi_\alpha. \end{aligned}$$

Similarly, using the equation of ϕ

$$\begin{aligned} B_{\alpha\overline{\beta},\overline{\alpha}} =& \phi_{\alpha,\overline{\beta}\overline{\alpha}} - \phi^{-1}\left(\phi_{\alpha,\overline{\alpha}}\phi_{\overline{\beta}} + \phi_\alpha\phi_{\overline{\beta},\overline{\alpha}}\right) + \phi^{-2}\left|\partial\phi\right|^2\phi_{\overline{\beta}} - \frac{1}{2}\left(g_{\overline{\beta}} - \phi_{\overline{\beta}}\right) \\ =& \phi_{\alpha,\overline{\alpha}\overline{\beta}} - \phi^{-1}\left(\phi_{\alpha,\overline{\alpha}}\phi_{\overline{\beta}} + A_{\overline{\beta}}\right) + \phi^{-2}\left|\partial\phi\right|^2\phi_{\overline{\beta}} - \frac{1}{2}\left(g_{\overline{\beta}} - \phi_{\overline{\beta}}\right) \\ =& \frac{m-1}{2}\left(g_{\overline{\beta}} - \phi_{\overline{\beta}}\right) + \left(\phi^{-1}\left|\partial\phi\right|^2\right)_{\overline{\beta}} - \frac{m}{2}\phi^{-1}\left(g - \phi\right)\phi_{\overline{\beta}} - \phi^{-1}A_{\overline{\beta}}. \end{aligned}$$

Using Lemma 3, we obtain

$$\begin{aligned}
B_{\alpha\overline{\beta},\overline{\alpha}} =& \frac{m-1}{2}\left[\phi^{-1}\left(A_{\overline{\beta}}+B_{\overline{\beta}}\right)-C_{\overline{\beta}}+\phi^{-1}\left(g-\phi\right)\phi_{\overline{\beta}}\right] \\
&+\left[\phi^{-1}\left(A_{\overline{\beta}}+B_{\overline{\beta}}\right)+\frac{1}{2}\phi^{-1}\left(g-\phi\right)\phi_{\overline{\beta}}\right]-\frac{m}{2}\phi^{-1}\left(g-\phi\right)\phi_{\overline{\beta}}-\phi^{-1}A_{\overline{\beta}} \\
=& \frac{m-1}{2}\left[\phi^{-1}\left(A_{\overline{\beta}}+B_{\overline{\beta}}\right)-C_{\overline{\beta}}\right]+\phi^{-1}B_{\overline{\beta}} \\
=& \frac{m-1}{2}\left(\phi^{-1}A_{\overline{\beta}}-C_{\overline{\beta}}\right)+\frac{m+1}{2}\phi^{-1}B_{\overline{\beta}}.
\end{aligned}$$

□

We are ready to calculated the divergence for the three vector fields A_α, B_α and C_α.

Lemma 5. *We have*

$$\begin{aligned}
A_{\alpha,\overline{\alpha}} =& \frac{m+2}{2}\left[C_\alpha+\phi^{-1}\left(A_\alpha+B_\alpha\right)\right]\phi_{\overline{\alpha}}+\left|\phi_{\alpha\beta}\right|^2+Q, \\
B_{\alpha,\overline{\alpha}} =& \left[\frac{(m-1)}{2}\phi^{-1}A_{\overline{\alpha}}+\frac{m+1}{2}\phi^{-1}B_{\overline{\alpha}}-\frac{(m-1)}{2}C_{\overline{\alpha}}\right]\phi_\alpha+\phi^{-1}B_\alpha\phi_{\overline{\alpha}}+\left|B_{\alpha\overline{\beta}}\right|^2 \\
C_{\alpha,\overline{\alpha}} =& \frac{m+2}{2}\phi^{-1}C_\alpha\phi_{\overline{\alpha}}-\frac{m+1}{2}\phi^{-1}C_{\overline{\alpha}}\phi_\alpha+\frac{1}{2}\phi^{-2}\left(A_{\overline{\alpha}}+B_{\overline{\alpha}}\right)\phi_\alpha+S,
\end{aligned}$$

where

$$\begin{aligned}
Q &= R_{\alpha\overline{\beta}}\phi_\beta\phi_{\overline{\alpha}}-\frac{m+1}{2}\left|\partial\phi\right|^2 \\
S &= -\frac{m}{2}\left[\phi_{0,0}-\frac{1}{2}\phi^{-1}\left(\frac{1-\phi^2}{4}+\phi^{-1}\left|\partial\phi\right|^2+\phi^{-2}\left|\partial\phi\right|^4+\phi_0^2\right)\right]
\end{aligned}$$

Proof. The 1st two identities follow directly from Lemma 3 and Lemma 4. To prove the 3rd identity, we first note $\phi_{0,\alpha}=\phi_{\alpha,0}$ and $\phi_{\alpha,0\overline{\beta}}=\phi_{\alpha,\overline{\beta}0}$ as we assume that (M,θ) is torsion-free. We compute

$$\begin{aligned}
\left(\phi^{-1}\left|\partial\phi\right|^2\right)_0 =& \phi^{-1}\left(\phi_\beta\phi_{\overline{\beta},0}+\phi_{\beta,0}\phi_{\overline{\beta}}\right)-\phi^{-2}\left|\partial\phi\right|^2\phi_0 \\
=& \phi^{-1}\phi_\beta\left(iC_{\overline{\beta}}-\frac{i}{2}\phi^{-1}g\phi_{\overline{\beta}}\right)+\phi^{-1}\phi_{\overline{\beta}}\left(-iC_\beta+\frac{i}{2}\phi^{-1}\overline{g}\phi_\beta\right) \\
&-\phi^{-2}\left|\partial\phi\right|^2\phi_0 \\
=& i\phi^{-1}\left(\phi_\beta C_{\overline{\beta}}-\phi_{\overline{\beta}}C_\beta\right)+\frac{i}{2}\phi^{-2}\left|\partial\phi\right|^2\left(\overline{g}-g\right)-\phi^{-2}\left|\partial\phi\right|^2\phi_0 \\
=& i\phi^{-1}\left(\phi_\beta C_{\overline{\beta}}-\phi_{\overline{\beta}}C_\beta\right).
\end{aligned}$$

Using these identities as well as the previous lemmas, we compute

$$
\begin{aligned}
C_{\alpha,\overline{\alpha}} =& i\phi_{\alpha,\overline{\alpha}0}+\frac{1}{2}\phi^{-1}\overline{g}_{\overline{\alpha}}\phi_\alpha+\frac{1}{2}\phi^{-1}\overline{g}\phi_{\alpha\overline{\alpha}}-\frac{1}{2}\phi^{-2}\overline{g}\left|\partial\phi\right|^2\\
=& i\left[\frac{m}{2}\left(i\phi_{00}-\frac{1}{2}\phi_0\right)+\frac{m+2}{2}\left(\phi^{-1}\left|\partial\phi\right|^2\right)_0\right]+\frac{1}{2}\phi^{-1}\phi_\alpha\left[\phi^{-1}\left(A_{\overline{\alpha}}+B_{\overline{\alpha}}\right)+C_{\overline{\alpha}}\right]\\
&+\frac{m}{4}\phi^{-1}\overline{g}\left(g-\phi\right)\\
=&-\frac{m}{4}i\phi_0-\frac{m}{2}\phi_{00}-\frac{m+2}{2}\phi^{-1}\left(\phi_\alpha C_{\overline{\alpha}}-\phi_{\overline{\alpha}}C_\alpha\right)\\
&+\frac{1}{2}\phi^{-1}\phi_\alpha\left[\phi^{-1}\left(A_{\overline{\alpha}}+B_{\overline{\alpha}}\right)+C_{\overline{\alpha}}\right]+\frac{m}{4}\phi^{-1}\overline{g}\left(g-\phi\right)\\
=&\frac{m+2}{2}\phi^{-1}\phi_{\overline{\alpha}}C_\alpha-\frac{m+1}{2}\phi^{-1}\phi_\alpha C_{\overline{\alpha}}+\frac{1}{2}\phi^{-2}\phi_\alpha\left(A_{\overline{\alpha}}+B_{\overline{\alpha}}\right)\\
&-\frac{m}{2}\phi_{00}+\frac{m}{4}\phi^{-1}\left|g\right|^2-\frac{m}{4}\left(\overline{g}+i\phi_0\right)\\
=&\frac{m+2}{2}\phi^{-1}\phi_{\overline{\alpha}}C_\alpha-\frac{m+1}{2}\phi^{-1}\phi_\alpha C_{\overline{\alpha}}+\frac{1}{2}\phi^{-2}\phi_\alpha\left(A_{\overline{\alpha}}+B_{\overline{\alpha}}\right)\\
&-\frac{m}{2}\left[\phi_{00}-\frac{1}{2}\phi^{-1}\left|g\right|^2+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\phi+\phi^{-1}\left|\partial\phi\right|^2\right)\right].
\end{aligned}
$$

Using the formula for g in the last term yields the 3rd identity. □

These three formulas demonstrate that the three vector fields A_α, B_α and C_α intertwine with one another. The basic strategy, following Jerison-Lee, is to come up with a linear combination of the three vectors fields whose divergence is a sum of squares. The divergence of C_α is much more complicated as the last term S involves 2nd order derivatives in the T direction which we don't know how to control. But fortunately this extra term is purely real. Another simple but crucial fact for the final formula is the following.

Lemma 6. *$B_\alpha\phi_{\overline{\alpha}}$ is real, i.e. $B_\alpha\phi_{\overline{\alpha}}=B_{\overline{\alpha}}\phi_\alpha$.*

Proof. We have

$$
\begin{aligned}
B_{\alpha\overline{\beta}}&=\phi_{\alpha,\overline{\beta}}-\frac{i}{2}\phi_0\delta_{\alpha\beta}-\phi^{-1}\phi_\alpha\phi_{\overline{\beta}}-\frac{1}{2}\left(\operatorname{Re}g-\phi\right)\delta_{\alpha\beta}\\
&=\frac{1}{2}\left(\phi_{\alpha,\overline{\beta}}+\phi_{\overline{\alpha},\beta}\right)-\phi^{-1}\phi_\alpha\phi_{\overline{\beta}}-\frac{1}{2}\left(\operatorname{Re}g-\phi\right)\delta_{\alpha\beta}.
\end{aligned}
$$

It is clearly hermitian. Therefore $B_\alpha\phi_{\overline{\alpha}}=B_{\alpha\overline{\beta}}\phi_\beta\phi_{\overline{\alpha}}$ is real. □

We can now state the final formula, which can be viewed as a generalization of the formula (4.2) on the Heisenberg group in [JL].

Theorem 6. *Let $\left(M^{2m+1},\theta\right)$ is a closed pseudohermitian manifold with torsion $A_{\alpha\beta}=0$. Suppose $\phi\in C^\infty\left(M\right)$ is positive and satisfies the equation (4.1). Then*

$$
\begin{aligned}
&\operatorname{Re}\left[\left(\phi^{-(m+1)}\left(\left(\overline{g}+3i\phi_0\right)\phi^{-1}A_\alpha+\left(\overline{g}-i\phi_0\right)\phi^{-1}B_\alpha-i3\phi_0C_\alpha\right)\right)_{\overline{\alpha}}\right]\\
=&\phi^{-(m+1)}\left[\left(\frac{1}{2}+\frac{1}{2}\phi\right)\left(\left|\phi_{\alpha\beta}\right|^2+\left|B_{\alpha\overline{\beta}}\right|^2\right)+\left(\frac{1}{2}+\frac{1}{2}\phi+\phi^{-1}\left|\partial\phi\right|^2\right)Q\right.\\
&\left.+\left|\phi^{-1}A_\alpha-C_\alpha\right|^2+\left|\phi^{-1}B_\alpha+C_\alpha\right|^2+\left|C_\alpha\right|^2+\phi^{-1}\left|\phi_{\alpha\beta}\phi_{\overline{\gamma}}+B_{\alpha\overline{\gamma}}\phi_\beta\right|^2\right],
\end{aligned}
$$

where

$$Q = R_{\alpha\overline{\beta}}\phi_\beta\phi_{\overline{\alpha}} - \frac{m+1}{2}\left|\partial\phi\right|^2.$$

Proof. By direct calculation, the divergence on the LHS is given by

$$\begin{aligned}&\phi^{-(m+1)}\left[\left(\overline{g}+3i\phi_0\right)\left(\phi^{-1}A_{\alpha,\overline{\alpha}}-(m+2)\,\phi^{-2}A_\alpha\phi_{\overline{\alpha}}\right)+\phi^{-1}\left(\overline{g}_{\overline{\alpha}}+3i\phi_{0,\overline{\alpha}}\right)A_\alpha\right.\\&+\left(\overline{g}-i\phi_0\right)\left(\phi^{-1}B_{\alpha,\overline{\alpha}}-(m+2)\,\phi^{-2}B_\alpha\phi_{\overline{\alpha}}\right)+\phi^{-1}\left(\overline{g}_{\overline{\alpha}}-i\phi_{0,\overline{\alpha}}\right)B_\alpha\\&\left.-i3\left(\phi_0C_{\alpha,\overline{\alpha}}+\phi_{0,\overline{\alpha}}C_\alpha-(m+1)\,\phi^{-1}\phi_0C_\alpha\phi_{\overline{\alpha}}\right)\right].\end{aligned}$$

Using Lemma as well as the formula $-i\phi_{0,\overline{\alpha}} = C_{\overline{\alpha}} - \frac{1}{2}\phi^{-1}g\phi_{\overline{\alpha}}$ the above expression is (ignoring the factor $\phi^{-(m+1)}$)

$$\begin{aligned}&\left(\overline{g}+3i\phi_0\right)\left(\frac{m+2}{2}\phi^{-1}C_\alpha\phi_{\overline{\alpha}}-\frac{m+2}{2}\phi^{-2}A_\alpha\phi_{\overline{\alpha}}+\frac{m+2}{2}\phi^{-2}B_\alpha\phi_{\overline{\alpha}}+\left|\phi_{\alpha\beta}\right|^2+Q\right)\\&+\phi^{-1}\left(\phi^{-1}\left(A_{\overline{\alpha}}+B_{\overline{\alpha}}\right)-2C_{\overline{\alpha}}+\frac{3}{2}\phi^{-1}g\phi_{\overline{\alpha}}\right)A_\alpha\\&+\left(\overline{g}-i\phi_0\right)\left(-\frac{m-1}{2}\phi^{-1}C_{\overline{\alpha}}\phi_\alpha+\frac{m-1}{2}\phi^{-2}A_{\overline{\alpha}}\phi_\alpha-\frac{m+1}{2}\phi^{-2}B_{\overline{\alpha}}\phi_\alpha+\left|B_{\alpha\overline{\beta}}\right|^2\right)\\&+\phi^{-1}\left(\phi^{-1}\left(A_{\overline{\alpha}}+B_{\overline{\alpha}}\right)+2C_{\overline{\alpha}}-\frac{1}{2}\phi^{-1}g\phi_{\overline{\alpha}}\right)B_\alpha\\&-i3\left(\phi_0\left(-\frac{m}{2}\phi^{-1}C_\alpha\phi_{\overline{\alpha}}-\frac{m+1}{2}\phi^{-1}C_{\overline{\alpha}}\phi_\alpha+\frac{1}{2}\phi^{-2}\left(A_{\overline{\alpha}}+B_{\overline{\alpha}}\right)\phi_\alpha\right)\right)\\&+3\left(C_{\overline{\alpha}}-\frac{1}{2}\phi^{-1}g\phi_{\overline{\alpha}}\right)C_\alpha-3i\phi_0S.\end{aligned}$$

After expansions and cancellations, we arrive at

$$\begin{aligned}&\left(\overline{g}+3i\phi_0\right)\left(\left|\phi_{\alpha\beta}\right|^2+Q\right)+\left(\overline{g}-i\phi_0\right)\left|B_{\alpha\overline{\beta}}\right|^2+\phi^{-2}\left|A_\alpha+B_\alpha\right|^2+2\phi^{-1}\left(B_\alpha-A_\alpha\right)C_{\overline{\alpha}}\\&+3\left|C_\alpha\right|^2+\phi^{-2}E_1+\phi^{-2}E_2+\phi^{-1}E_3-3i\phi_0S,\end{aligned}$$

where

$$\begin{aligned}E_1&=\frac{(m-1)}{2}\left(A_{\overline{\alpha}}\phi_\alpha-A_\alpha\phi_{\overline{\alpha}}\right)\operatorname{Re}g-\left(m+\frac{1}{2}\right)i\phi_0\left(A_\alpha\phi_{\overline{\alpha}}+A_{\overline{\alpha}}\phi_\alpha\right),\\E_2&=(2m+1)\,i\phi_0B_\alpha\phi_{\overline{\alpha}}\\E_3&=\frac{(m-1)}{2}\left(C_\alpha\phi_{\overline{\alpha}}-C_{\overline{\alpha}}\phi_\alpha\right)\operatorname{Re}g+\frac{5m+1}{2}i\phi_0\left(C_\alpha\phi_{\overline{\alpha}}+C_{\overline{\alpha}}\phi_\alpha\right).\end{aligned}$$

It is clear that all these three terms as well as the last term (recall S is real) are purely imaginary. Therefore

$$\begin{aligned}
&\operatorname{Re}\left[\left(\phi^{-(m+1)}\left(\left(\overline{g}+3i\phi_0\right)\phi^{-1}A_\alpha+\left(\overline{g}-i\phi_0\right)\phi^{-1}B_\alpha-i3\phi_0C_\alpha\right)\right)_{\overline{\alpha}}\right]\\
=&\phi^{-(m+1)}\operatorname{Re}\left[\left(\overline{g}+3i\phi_0\right)\left(\left|\phi_{\alpha\beta}\right|^2+Q\right)+\left(\overline{g}-i\phi_0\right)\left|B_{\alpha\overline{\beta}}\right|^2+\phi^{-2}\left|A_\alpha+B_\alpha\right|^2\right.\\
&\left.-2\phi^{-1}\left(A_\alpha+B_\alpha\right)C_{\overline{\alpha}}+3\left|C_\alpha\right|^2\right]\\
=&\phi^{-(m+1)}\left[\left(\frac{1}{2}+\frac{1}{2}\phi+\phi^{-1}\left|\partial\phi\right|^2\right)\left(\left|\phi_{\alpha\beta}\right|^2+\left|B_{\alpha\overline{\beta}}\right|^2+Q\right)+\phi^{-2}\left|A_\alpha+B_\alpha\right|^2\right.\\
&\left.+3\left|C_\alpha\right|^2+2\phi^{-1}\operatorname{Re}\left(B_\alpha-A_\alpha\right)C_{\overline{\alpha}}\right].
\end{aligned}$$

It is then elementary to show that this equals the RHS. □

Remark 2. *The divergence formula (4.2) in* [JL] *has been used recently by Ma and Ou* [MO] *to show that the following equation on the Heisenberg group* $\mathbb{H}^m$

$$-\Delta_b u = u^q,$$

where $q<(m+2)/m$, *has no positive solution.*

With this identity, we can now prove the following.

Theorem 7. *Let* $\left(M^{2m+1},\theta\right)$ *be a closed pseudohermitian manifold with* $A_{\alpha\beta}=0$ *and* $R_{\alpha\overline{\beta}}\geq\frac{m+1}{2}$. *Suppose* $f>0$ *satisfies the following equation on* M

$$-\Delta_b f+\lambda f=f^{(m+2)/m},$$

where $\lambda>0$ *is a constant. If* $\lambda\leq m^2/4$, *then* f *is constant unless* $\lambda=m^2/4$ *and* (M,θ) *is isometric to* $\left(\mathbb{S}^{2m+1},\theta_c\right)$ *and in this case*

$$f=c_m\left|\cosh t+\left(\sinh t\right)z\cdot\overline{\xi}\right|^{-1/m}$$

for some $t>0,\xi\in\mathbb{S}^{2m+1}$.

It suffices to prove it for $\lambda=m^2/4$ as the case when $\lambda<m^2/4$ then follows by scaling θ as in the proof of Theorem 5. Therefore in the following we assume $\lambda=m^2/4$. By scaling f we consider the equivalent equation $\frac{4}{m^2}\Delta_b f+f=f^{(m+2)/m}$. By our previous discussion, $\phi:=f^{-m/2}$ satisfies the identity in Theorem 6. Under our assumption, $Q\geq0$. The RHS of the identity is nonnegative while the LHS is a divergence. Therefore, all the terms on the RHS must vanish. In particular,

$$\phi_{\alpha\beta}=0,B_{\alpha\overline{\beta}}=0,R_{\alpha\overline{\beta}}\phi_\beta=\frac{m+1}{2}\phi_\alpha.$$

The arguments in [W] can then be applied to finish the proof.

5. Some open problems

In the introduction, we have alluded to a more general uniqueness result in the Riemannian case, which can be stated as follows.

Theorem 8. *([BVV, I]) Let (M^n, g) be a smooth compact Riemannian manifold with a (possibly empty) convex boundary. Suppose $u \in C^\infty(M)$ is a positive solution of the following equation*

$$\begin{aligned} -\Delta u + \lambda u = u^q \quad & on \quad M, \\ \frac{\partial u}{\partial \nu} = 0 \quad & on \quad \partial M, \end{aligned}$$

where $\lambda > 0$ is a constant and $1 < q \leq (n+2)/(n-2)$. If $Ric \geq (n-1)g$ and $\lambda(q-1) \leq n$, then u must be constant unless $q = (n+2)/(n-2), \lambda = n(n-2)/4$ and (M, g) is isometric to $\left(\mathbb{S}^n, \frac{4\lambda}{n(n-2)} g_{\mathbb{S}^n}\right)$ or $\left(\mathbb{S}^n_+, \frac{4\lambda}{n(n-2)} g_{\mathbb{S}^n}\right)$. In the latter case u is given on $\mathbb{S}^n$ or $\mathbb{S}^n_+$ by the following formula

$$u(x) = c_n (\cosh t + (\sinh t)\, x \cdot \xi)^{-(n-2)/2}.$$

for some $t \geq 0$ and $\xi \in \mathbb{S}^n$.

Our Theorem 5 corresponds to the special case $q = (n+2)/(n-2)$. It would be interesting if the method presented in Section 2 can be sharpened to give a new proof of the above theorem in it full generality.

It is natural to wonder if there is a CR analogue of Theorem 8. We are tempted to make the following conjecture.

Conjecture 1. *Let $\left(M^{2m+1}, \theta\right)$ be a closed pseudohermitian manifold with $A_{\alpha\beta} = 0$ and $R_{\alpha\overline{\beta}} \geq \frac{m+1}{2}$. Suppose $f > 0$ satisfies the following equation on M*

$$-\Delta_b f + \lambda f = f^q,$$

where $\lambda > 0$ is a constant and $1 < q \leq (m+2)/m$. If $\lambda(q-1) \leq m/2$, then f is constant unless $q = (m+2)/m, \lambda = m^2/4$ and (M, θ) is isometric to $\left(\mathbb{S}^{2m+1}, \theta_c\right)$ and in this case

$$f(z) = c_m \left|\cosh t + (\sinh t)\, z \cdot \overline{\xi}\right|^{-1/m}$$

for some $t > 0, \xi \in \mathbb{S}^{2m+1}$.

The case $q = (m+2)/m$ is exactly our Theorem 7.

The Conjecture, if true, would imply the following sharp Sobolev inequality on any closed pseudohermitian manifold $\left(M^{2m+1}, \theta\right)$ with $A_{\alpha\beta} = 0$ and $R_{\alpha\overline{\beta}} \geq \frac{m+1}{2}$: for $1 \leq q \leq (m+2)/m$

$$\left(\frac{1}{V}\int_M |F|^{q+1} dv\right)^{2/(q+1)} \leq \frac{2(q-1)}{m}\frac{1}{V}\int_M |\nabla_b F|^2 dv + \frac{1}{V}\int_M |F|^2 dv.$$

This family of inequalities was proved on $\left(\mathbb{S}^{2m+1}, \theta_c\right)$ by Frank and Lieb [FL] as a corollary of their sharp Hardy-Littlewood integral inequality on the Heisenberg group. It is interesting to know if it holds in a more general setting.

A possible approach to the above conjecture is to modify the Jerison-Lee identity. We hope to report progress on this front in the future.

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