Math. Res. Lett. **18** (2011), *no.* 00, 10001–100NN

LIOUVILLE TYPE THEOREMS ON MANIFOLDS WITH NONNEGATIVE CURVATURE AND STRICTLY CONVEX BOUNDARY

Qianqiao Guo, Fengbo Hang, and Xiaodong Wang

Abstract. We prove some Liouville type theorems on smooth compact Riemannian manifolds with nonnegative sectional curvature and strictly convex boundary. This gives a nonlinear generalization in low dimension of the recent sharp lower bound for the first Steklov eigenvalue by Xia-Xiong and verifies partially a conjecture by the third named author. As a consequence, we derive several sharp Sobolev trace inequalities on such manifolds.

1. Introduction

In [BVV, section 6], a remarkable calculation of Bidaut-Véron and Véron implies the following Liouville type theorem (see also [I] for the case of Neumann boundary condition):

Theorem 1. *(*[BVV, I] *) Let (M^n, g) be a smooth compact Riemannian manifold with a (possibly empty) convex boundary. Suppose $u \in C^\infty(M)$ is a positive solution of the following equation*

$$\begin{aligned} -\Delta u + \lambda u &= u^q \quad \text{on} \quad M, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \partial M, \end{aligned}$$

where $\lambda > 0$ is a constant and $1 < q \le (n+2)/(n-2)$. If $Ric \ge \frac{(n-1)(q-1)\lambda}{n} g$, then u must be constant unless $q = (n+2)/(n-2)$ and (M, g) is isometric to $\left(\mathbb{S}^n, \frac{4\lambda}{n(n-2)} g_{\mathbb{S}^n}\right)$ or $\left(\mathbb{S}^n_+, \frac{4\lambda}{n(n-2)} g_{\mathbb{S}^n}\right)$. In the latter case u is given on $\mathbb{S}^n$ or $\mathbb{S}^n_+$ by the following formula

$$u(x) = \frac{1}{(a + x \cdot \xi)^{(n-2)/2}}.$$

for some $\xi \in \mathbb{R}^{n+1}$ and some constant $a > |\xi|$.

By convex boundary we mean that the 2nd fundamental form Π is nonnegative. To be precise, throughout this paper ν denotes the outer unit normal on the boundary and the second fundamental form is defined as

$$\Pi(X, Y) = \langle \nabla_X \nu, Y \rangle.$$

for $X, Y \in T_p(\partial M)$.

This theorem has some very interesting corollaries. In particular it yields a sharp lower bound for type I Yamabe invariant (see [BVV, section 6] and [W2]). It is proposed in [W2] that a similar result should hold for type II Yamabe problem on a compact Riemannian manifold with nonnegative Ricci curvature and strictly convex boundary. By strict convexity we mean the second fundamental form Π of the boundary has a positive lower bound. By scaling we can always assume that the lower bound is 1. In its precise form, the conjecture in [W2] states the following:

Conjecture 1 ([W2])**.** *Let (M^n, g) be a smooth compact Riemannian manifold with $Ric \geq 0$ and $\Pi \geq 1$ on its nonempty boundary. Let $u \in C^\infty(M)$ be a positive solution to the following equation*

$$\begin{aligned} \Delta u &= 0 \quad \text{on} \quad M, \\ \frac{\partial u}{\partial \nu} + \lambda u &= u^q \quad \text{on} \quad \partial M, \end{aligned} \tag{1.1}$$

where the parameters λ and q are always assumed to satisfy $\lambda > 0$ and $1 < q \leq \frac{n}{n-2}$. If $\lambda \leq \frac{1}{q-1}$, then u must be constant unless $q = \frac{n}{n-2}$, M is isometric to $\overline{\mathbb{B}^n} \subset \mathbb{R}^n$ and u corresponds to

$$u_a(x) = \left[\frac{2}{n-2} \frac{1 - |a|^2}{1 + |a|^2 |x|^2 - 2x \cdot a} \right]^{(n-2)/2}$$

for some $a \in \mathbb{B}^n$.

This conjecture, if true, would have very interesting geometric consequences. We refer the readers to [W2] for further discussion. In this paper we will verify the conjecture in some special cases.

Theorem 2. *Let (M^n, g) be a smooth compact Riemannian manifold with nonnegative sectional curvature and the second fundamental form of the boundary $\Pi \geq 1$. Then the only positive solution to (1.1) is constant if $\lambda \leq \frac{1}{q-1}$, provided $2 \leq n \leq 8$ and $1 < q \leq \frac{4n}{5n-9}$.*

Although this result requires the stronger assumption on the sectional curvature and severe restriction on the dimension and the exponent, it does yield the conjectured sharp range for λ. This is a delicate issue as illustrated by the followng result on the model space $\overline{\mathbb{B}^n}$.

Proposition 1. *If $1 < q < \frac{n}{n-2}$ and $\lambda(q-1) > 1$ then the equation*

$$\begin{aligned} \Delta u &= 0 \quad \text{on} \quad \overline{\mathbb{B}^n}, \\ \frac{\partial u}{\partial \nu} + \lambda u &= u^q \quad \text{on} \quad \partial\overline{\mathbb{B}^n}, \end{aligned}$$

admits a positive, nonconstant solution.

It should be mentioned that on the model space $\overline{\mathbb{B}^n}$ with $n \geq 3$ the conjecture is verified in [GuW] in all dimensions when $\lambda \leq \frac{n-2}{2}$ by the method of moving planes. The approach to Theorem 2 is based on an integral method with a key idea borrowed from the recent work [XX] by Xia and Xiong, where a sharp lower bound for the first Steklov eigenvalue was proved.

For $n = 2$ Theorem 2 confirms the conjecture when $q \leq 8$. By an approach based on maximum principle in the spirit of [E1, P, W1], we can verify the conjecture in dimension 2 for $q \geq 2$. Combining both results we fully confirm Conjecture 1 in dimension 2.

Theorem 3. *Let (Σ, g) be a smooth compact surface with nonnegative Gaussian curvature and geodesic curvature $\kappa \geq 1$ on the boundary. Then the only positive solution to the following equation*

$$\begin{aligned} \Delta u = 0 \quad & on \quad \Sigma, \\ \tfrac{\partial u}{\partial \nu} + \lambda u = u^q \quad & on \quad \partial\Sigma, \end{aligned} \tag{1.2}$$

where $q > 1$ and $0 < \lambda \leq \frac{1}{q-1}$, is constant.

The paper is organized as follows. In Section 2 we derive some integral identities that will be used later. The proof of Theorem 2 is given in Section 3. In Section 4 we present the argument based on maximum principle in dimension two and prove Theorem 3. In the last section we make some further remarks about Conjecture 1 and deduce some corollaries from our Liouville type results.

2. Some integral identities

Let (M^n, g) be a smooth compact Riemannian manifold with boundary Σ and $v \in C^\infty(M)$ be a **positive** function. We write $f = v|_\Sigma$, $\chi = \frac{\partial v}{\partial \nu}$. Let w be another smooth functions on M satisfying the following boundary conditions

$$w|_\Sigma = 0, \frac{\partial w}{\partial \nu} = -1. \tag{2.1}$$

Proposition 2. *For any $b \in \mathbb{R}$*

$$\begin{aligned} &\int_M \left(1 - \frac{1}{n}\right)(\Delta v)^2 v^b w + \frac{b}{2} w v^{b-2} |\nabla v|^2 \left[3v\Delta v + (b-1)|\nabla v|^2\right] \\ &= \int_M v^b D^2 w(\nabla v, \nabla v) - |\nabla v|^2 v^b \Delta w - \frac{b}{2}|\nabla v|^2 v^{b-1}\langle \nabla v, \nabla w\rangle \\ &+ \left(\left|D^2 v - \frac{\Delta v}{n} g\right|^2 + Ric(\nabla v, \nabla v)\right) v^b w - \int_\Sigma f^b |\nabla f|^2. \end{aligned}$$

Proof. The following weighted Reilly formula was proved in [QX] for any smooth functions v and ϕ

$$\begin{aligned} &\int_M \left[\left(1 - \frac{1}{n}\right)(\Delta v)^2 - \left|D^2 v - \frac{\Delta v}{n} g\right|^2\right]\phi \\ &= \int_M D^2\phi(\nabla v, \nabla v) - |\nabla v|^2 \Delta\phi + Ric(\nabla v, \nabla v)\phi \\ &+ \int_\Sigma \phi\left[2\chi\Delta_\Sigma f + H\chi^2 + \Pi(\nabla f, \nabla f)\right] + \frac{\partial \phi}{\partial \nu}|\nabla f|^2, \end{aligned} \tag{2.2}$$

Take $\phi = v^b w$. We calculate

$$\begin{aligned}
\nabla\phi &= v^b \nabla w + bwv^{b-1}\nabla v \\
D^2\phi &= v^b D^2 w + bv^{b-1}\left(dv \otimes dw + dw \otimes dv\right) + bwv^{b-1} D^2 v \\
&\quad + b\left(b-1\right) wv^{b-2} dv \otimes dv, \\
\Delta\phi &= v^b \Delta w + 2bv^{b-1}\left\langle \nabla v, \nabla w\right\rangle + bwv^{b-1}\Delta v + b\left(b-1\right) wv^{b-2}\left|\nabla v\right|^2, \\
D^2\phi\left(\nabla v, \nabla v\right) &= v^b D^2 w\left(\nabla v, \nabla v\right) + 2bv^{b-1}\left|\nabla w\right|^2\left\langle \nabla v, \nabla w\right\rangle + bwv^{b-1} D^2 v\left(\nabla v, \nabla v\right) \\
&\quad + b\left(b-1\right) wv^{b-2}\left|\nabla v\right|^4.
\end{aligned}$$

Plugging these equations into (2.2) and using (2.1) yields

$$\begin{aligned}
&\int_M \left[\left(1-\frac{1}{n}\right)\left(\Delta v\right)^2 - \left|D^2 v - \frac{\Delta v}{n} g\right|^2\right] v^b w \\
&= \int_M v^b D^2 w\left(\nabla v, \nabla v\right) + bwv^{b-1} D^2 v\left(\nabla v, \nabla v\right) - \left|\nabla v\right|^2\left(v^b \Delta w + bwv^{b-1}\Delta v\right) \\
&+ Ric\left(\nabla v, \nabla v\right) v^b w - \int_\Sigma f^b \left|\nabla f\right|^2.
\end{aligned}$$

We calculate

$$\begin{aligned}
wv^{b-1} D^2 v\left(\nabla v, \nabla v\right) &= \frac{1}{2} wv^{b-1}\left\langle \nabla v, \nabla\left|\nabla v\right|^2\right\rangle \\
&= \frac{1}{2}\left[\operatorname{div}\left(wv^{b-1}\left|\nabla v\right|^2 \nabla v\right) - \left|\nabla v\right|^2 \operatorname{div}\left(wv^{b-1}\nabla v\right)\right] \\
&= \frac{1}{2}\left[\operatorname{div}\left(wv^{b-1}\left|\nabla v\right|^2 \nabla v\right) - w\left|\nabla v\right|^2 v^{b-1}\Delta v\right. \\
&\quad \left. - \left(b-1\right) wv^{b-2}\left|\nabla v\right|^4 - \left|\nabla v\right|^2 v^{b-1}\left\langle \nabla v, \nabla w\right\rangle\right].
\end{aligned}$$

Integrating yields

$$\int_M wv^{b-1} D^2 v\left(\nabla v, \nabla v\right) = -\frac{1}{2}\int_M w\left|\nabla v\right|^2 v^{b-1}\Delta v + \left(b-1\right) wv^{b-2}\left|\nabla v\right|^4 + \left|\nabla v\right|^2 v^{b-1}\left\langle \nabla v, \nabla w\right\rangle.$$

Plugging this into the previous integral identity yields

$$\begin{aligned}
&\int_M \left[\left(1-\frac{1}{n}\right)\left(\Delta v\right)^2 - \left|D^2 v - \frac{\Delta v}{n} g\right|^2\right] v^b w \\
&= \int_M v^b D^2 w\left(\nabla v, \nabla v\right) - \left|\nabla v\right|^2 v^b \Delta w - \frac{b}{2} wv^{b-2}\left|\nabla v\right|^2\left[3v\Delta v + \left(b-1\right)\left|\nabla v\right|^2\right] \\
&- \frac{b}{2}\left|\nabla v\right|^2 v^{b-1}\left\langle \nabla v, \nabla w\right\rangle + Ric\left(\nabla v, \nabla v\right) v^b w - \int_\Sigma f^b\left|\nabla f\right|^2.
\end{aligned}$$

Reorganizing yields the desired identity. □

Proposition 3. *Under the same assumptions as in Proposition 2, we have*

$$\int_M v^b D^2 w\left(\nabla v, \nabla v\right) + \left(v\Delta v + \frac{b}{2}\left|\nabla v\right|^2\right) v^{b-1}\left\langle \nabla v, \nabla w\right\rangle - \frac{1}{2} v^b \left|\nabla v\right|^2 \Delta w$$
$$= \frac{1}{2}\int_\Sigma f^b \left(\left|\nabla f\right|^2 - \chi^2\right).$$

Proof. For any vector field X the following identity holds

$$\left\langle \nabla_{\nabla v} X, \nabla v\right\rangle + Xv\Delta v - \frac{1}{2}\left|\nabla v\right|^2 \operatorname{div} X = \operatorname{div}\left(Xv\nabla v - \frac{1}{2}\left|\nabla v\right|^2 X\right).$$

In the following we take $X = \nabla w$. Note that $\nabla w = -\nu$ on Σ. Multiplying both sides of the above identity by v^b and integrating yields

$$\int_M v^b D^2 w\left(\nabla v, \nabla v\right) + v^b \Delta v\left\langle \nabla v, \nabla w\right\rangle - \frac{1}{2} v^b \left|\nabla v\right|^2 \Delta w$$
$$= \int_M v^b \operatorname{div}\left(\left\langle \nabla v, \nabla w\right\rangle \nabla v - \frac{1}{2}\left|\nabla v\right|^2 \nabla w\right)$$
$$= \int_M -bv^{b-1}\left(\left\langle \nabla v, \nabla w\right\rangle \left|\nabla v\right|^2 - \frac{1}{2}\left|\nabla v\right|^2 \left\langle \nabla v, \nabla w\right\rangle\right) + \int_\Sigma f^b \left(\left\langle \nabla v, \nabla w\right\rangle \chi - \frac{1}{2}\left|\nabla v\right|^2 \frac{\partial w}{\partial \nu}\right)$$
$$= -\frac{b}{2}\int_M v^{b-1}\left\langle \nabla v, \nabla w\right\rangle \left|\nabla v\right|^2 + \int_\Sigma f^b\left(-\chi^2 + \frac{1}{2}\left|\nabla v\right|^2\right).$$

Therefore

$$\int_M v^b D^2 w\left(\nabla v, \nabla v\right) + \left(v\Delta v + \frac{b}{2}\left|\nabla v\right|^2\right) v^{b-1}\left\langle \nabla v, \nabla w\right\rangle - \frac{1}{2} v^b \left|\nabla v\right|^2 \Delta w$$
$$= \frac{1}{2}\int_\Sigma f^b \left(\left|\nabla f\right|^2 - \chi^2\right).$$

□

3. The proof of Theorem 2

Throughout this section (M^n, g) is a smooth compact Riemannian manifold with nonempty boundary Σ. We study **positive** solutions of the following equation

$$\begin{array}{rcl} \Delta u = 0 & \text{on} & M, \\ \frac{\partial u}{\partial \nu} + \lambda u = u^q & \text{on} & \Sigma, \end{array}$$

We write $u = v^{-a}$ with $a \neq 0$ a constant to be determined later. Then v satisfies the following equation

$$\begin{array}{rcl} \Delta v = (a+1)\, v^{-1}\left|\nabla v\right|^2 & \text{on} & M, \\ \chi = \frac{1}{a}\left(\lambda f - f^{1+a-aq}\right) & \text{on} & \Sigma, \end{array} \tag{3.1}$$

where $f = v|_{\partial\Sigma}$, $\chi = \frac{\partial v}{\partial \nu}$. Multiplying both sides by v^s and integrating over M yields

$$(a+s+1)\int_M \left|\nabla v\right|^2 v^{s-1} = \int_\Sigma f^s \chi. \tag{3.2}$$

By Proposition 2

$$\left[\left(1-\frac{1}{n}\right)(a+1)^2+\frac{b(3a+b+2)}{2}\right]\int_M v^{b-2}|\nabla v|^4 w$$
$$=\int_M v^b D^2 w(\nabla v,\nabla v)-|\nabla v|^2 v^b\Delta w-\frac{b}{2}|\nabla v|^2 v^{b-1}\langle\nabla v,\nabla w\rangle-\int_\Sigma f^b|\nabla f|^2+Q,$$

where

$$Q=\int_M\left(\left|D^2 v-\frac{\Delta v}{n}g\right|^2+Ric(\nabla v,\nabla v)\right)v^b w.$$

By Proposition 3

$$\int_M v^b D^2 w(\nabla v,\nabla v)+\left(a+1+\frac{b}{2}\right)v^{b-2}|\nabla v|^2\langle\nabla v,\nabla w\rangle-\frac{1}{2}v^b|\nabla v|^2\Delta w$$
$$=\frac{1}{2}\int_\Sigma f^b\left(|\nabla f|^2-\chi^2\right).$$

We use the above identity to eliminate the term involving $\langle\nabla v,\nabla w\rangle$ in the previous identity and obtain

$$\left[\left(1-\frac{1}{n}\right)(a+1)^2+\frac{b(3a+b+2)}{2}\right]\int_M v^{b-2}|\nabla v|^4 w$$
$$=\int_M\frac{a+1+b}{a+1+\frac{b}{2}}v^b D^2 w(\nabla v,\nabla v)-\frac{a+1+\frac{3}{4}b}{a+1+\frac{b}{2}}|\nabla v|^2 v^b\Delta w$$
$$+\int_\Sigma\frac{\frac{1}{4}b}{a+1+\frac{b}{2}}f^b\chi^2-\frac{a+1+\frac{3}{4}b}{a+1+\frac{b}{2}}f^b|\nabla f|^2+Q.$$

We choose $b=-\frac{4}{3}(a+1)$. Then

$$\frac{[5n-9-(n+9)a](a+1)}{9n}\int_M v^{b-2}|\nabla v|^4 w \tag{3.3}$$
$$=-\int_M v^b D^2 w(\nabla v,\nabla v)-\int_\Sigma f^b\chi^2+Q.$$

Let $\rho=d(\cdot,\Sigma)$ be the distance function to the boundary. It is Lipschitz on M and smooth away from the cut locus $\mathrm{Cut}(\Sigma)$ which is a closed set of measure zero in the interior of M. We consider $\psi:=\rho-\frac{\rho^2}{2}$. Notice that ψ is smooth near Σ and satisfies

$$\psi|_\Sigma=0,\ \frac{\partial\psi}{\partial\nu}=-1.$$

From now on we assume that M has nonnegative sectional curvature and $\Pi\geq 1$ on Σ. By the Hessian comparison theorem (cf. [K]) $\rho\leq 1$ hence $\psi\geq 0$ and

$$-D^2\psi\geq g$$

in the support sense. The new idea that ψ can be used as a good weight function is introduced in [XX] to study the first Steklov eigenvalue. To overcome the difficulty that ψ is not smooth, they constructed smooth approximations.

Proposition 4 ([XX]). *Fix a neighborhood $\mathcal{C}$ of* $\mathrm{Cut}(\Sigma)$ *in the interior of M. Then for any $\varepsilon > 0$, there exists a smooth nonnegative function ψ_ε on M s.t. $\psi_\varepsilon = \psi$ on $M \backslash \mathcal{C}$ and*

$$-D^2\psi_\varepsilon \geq (1-\varepsilon)\, g$$

The construction is based on the work [GW1, GW2, GW3].

In (3.3) taking the weight $w = \psi_\varepsilon$ yields

$$\begin{aligned}&\frac{[5n-9-(n+9)\,a]\,(a+1)}{9n}\int_M v^{b-2}\,|\nabla v|^4\,\psi_\varepsilon\\ &\geq (1-\varepsilon)\int_{\mathcal{C}} v^b\,|\nabla v|^2 - \int_{M\backslash\mathcal{C}} v^b D^2\psi\,(\nabla v,\nabla v) - \int_\Sigma f^b\chi^2 + Q_\varepsilon,\end{aligned}$$

where

$$Q_\varepsilon = \int_M \left(\left|D^2 v - \frac{\Delta v}{n} g\right|^2 + Ric\,(\nabla v,\nabla v)\right) v^b\psi_\varepsilon.$$

Letting $\varepsilon \to 0$ and shrinking the neighborhood yields

$$\begin{aligned}&\frac{[5n-9-(n+9)\,a]\,(a+1)}{9n}\int_M v^{b-2}\,|\nabla v|^4\,\psi\\ &\geq \int_{\mathcal{C}} v^b\,|\nabla v|^2 - \int_{M\backslash\mathcal{C}} v^b D^2\psi\,(\nabla v,\nabla v) - \int_\Sigma f^b\chi^2 + Q\end{aligned}$$

where

$$Q = \int_M \left(\left|D^2 v - \frac{\Delta v}{n} g\right|^2 + Ric\,(\nabla v,\nabla v)\right) v^b\psi.$$

On $M\backslash\mathcal{C}$ the function ψ is smooth and satisfies $-D^2\psi \geq g$. Therefore

$$\begin{aligned}&\frac{[5n-9-(n+9)\,a]\,(a+1)}{9n}\int_M v^{b-2}\,|\nabla v|^4\,\psi\\ &\geq \int_M v^b\,|\nabla v|^2 - \int_\Sigma f^b\chi^2 + Q.\end{aligned}$$

Using the boundary condition for v we obtain

$$\begin{aligned}&\frac{[5n-9-(n+9)\,a]\,(a+1)}{9n}\int_M v^{b-2}\,|\nabla v|^4\,\psi\\ &\geq \int_M v^b\,|\nabla v|^2 - \frac{1}{a}\int_\Sigma \left(\lambda f^{b+1} - f^{b+1+a-aq}\right)\chi + Q\\ &= \int_M v^b\,|\nabla v|^2 - \frac{(a+b+2)\,\lambda}{a} w^b\,|\nabla w|^2 + \frac{2a+b+2-aq}{a} w^{b+a-aq}\,|\nabla w|^2 + Q\\ &= \int_M \left(1 - \frac{\lambda\,(2-a)}{3a}\right) v^b\,|\nabla v|^2 + \left(\frac{2}{3} - q + \frac{2}{3a}\right) v^{b+a-aq}\,|\nabla v|^2 + Q,\end{aligned}$$

which can be written as

$$A\int_M v^{b-2}\,|\nabla v|^4\,\psi + B\int_M v^b\,|\nabla v|^2 + C\int_M v^{b+a-aq}\,|\nabla v|^2 \geq Q, \tag{3.4}$$

where, with $x = a^{-1}$

$$A = \frac{[5n-9-(n+9)\,a]\,(a+1)}{9n} = \frac{[(5n-9)\,x-(n+9)]\,(x+1)}{9nx^2},$$
$$B = \frac{\lambda\,(2-a)}{3a} - 1 = \frac{\lambda}{3}\,(2x-1) - 1$$
$$C = q - \frac{2}{3} - \frac{2}{3a} = q - \frac{2}{3} - \frac{2}{3}x$$

We want to choose a s.t. $A, B, C \le 0$, i.e.

$$\left(x - \frac{n+9}{5n-9}\right)(x+1) \le 0,$$
$$\frac{\lambda}{3}\,(2x-1) - 1 \le 0,$$
$$q - \frac{2}{3} - \frac{2}{3}x \le 0.$$

By simple calculations these inequalities become

$$-1 \le x \le \frac{n+9}{5n-9},$$
$$\frac{3}{2}q - 1 \le x \le \frac{3}{2}\frac{1}{\lambda} + \frac{1}{2}.$$

The choice is possible when $\frac{3}{2}q - 1 \le \frac{3}{2}\frac{1}{\lambda} + \frac{1}{2}$ and $\frac{3}{2}q - 1 \le \frac{n+9}{5n-9}$ i.e. when $(q-1)\,\lambda \le 1$ and $q \le \frac{4n}{5n-9}$. As $q > 1$ we must have $2 \le n \le 8$. Then when $q \le \frac{4n}{5n-9}$ and $(q-1)\,\lambda \le 1$ by choosing $\frac{1}{a} = \frac{3}{2}q - 1$ we have

$$C = 0, B = (q-1)\,\lambda - 1 \le 0, A = \frac{5n-9}{6n}q\left(\frac{3}{2}q - 1\right)^2\left(q - \frac{4n}{5n-9}\right) \le 0.$$

Thus the left hand side of (3.4) is nonpositive while the right hand side is nonnegative. It follows that both sides of (3.4) are zero and we must have

$$D^2 v = \frac{a+1}{n}v^{-1}\left|\nabla v\right|^2 g, \quad Ric\,(\nabla v, \nabla v) = 0. \tag{3.5}$$

If $q < \frac{4n}{5n-9}$ or $\lambda\,(q-1) < 1$ we have $A < 0$ or $B < 0$, respectively and hence v must be constant. It remains to prove that v must also be constant when

$$q = \frac{4n}{5n-9}, \quad \lambda\,(q-1) = 1. \tag{3.6}$$

Under this assumption, we have

$$a = \frac{1}{\frac{3}{2}q - 1} = \frac{5n-9}{n+9}.$$

As $Ric \ge 0$ the second equation in (3.5) implies $Ric\,(\nabla v, \cdot) = 0$. We denote

$$h = \frac{a+1}{n}v^{-1}\left|\nabla v\right|^2 = \frac{6}{n+9}v^{-1}\left|\nabla v\right|^2.$$

Then $D^2 v = hg$. Working with a local orthonormal frame we differentiate

$$\begin{aligned} h_j &= v_{ij,i} = v_{ii,j} - R_{jiil} v_l \\ &= (\Delta v)_j + R_{jl} v_l \\ &= n h_j. \end{aligned}$$

Thus $h_j = 0$, i.e. h is constant. To continue, we observe that since

$$|\nabla v|^2 = \frac{n+9}{6} hv,$$

differentiating both sides we get

$$\frac{n+9}{6} h v_j = 2 v_i v_{ij} = 2 h v_j.$$

Therefore

$$(n-3)\, h \nabla v = 0.$$

Taking inner product on both sides with ∇v and using the fact $v > 0$, we see $(n-3)\, h^2 = 0$. When $n \neq 3$, we have $h = 0$ and hence $\nabla v = 0$ and v must be a constant function.

It remains to handle the case $n = 3$, $q = 2$ and $\lambda = 1$. We need to further inspect the proof and observe that we used the inequality $-D^2 \psi (\nabla v, \nabla v) \geq |\nabla v|^2$ on $M \backslash \mathcal{C}$. Therefore this must be an equality. Then this implies that

$$-D^2 \psi (\nabla v, \cdot) = \langle \nabla v, \cdot \rangle .$$

As $-\nabla \psi = \nu$ on the boundary the above identity implies $\Pi (\nabla f, \cdot) = \langle \nabla f, \cdot \rangle$ on Σ. As $D^2 v = hg$ we have for $X \in T\Sigma$

$$\begin{aligned} 0 &= D^2 v (X, \nu) \\ &= X\chi - \Pi (\nabla f, X) \\ &= X\chi - Xf. \end{aligned}$$

Thus $\chi - f$ is constant. But as $\chi = 2\left(f - f^{1/2}\right)$ by the boundary condition we conclude f is constant. Therefore v is constant.

4. Maximum principle argument in dimension 2

It is unfortunate that the integral argument in previous section only works for $1 < q \leq 8$ in dimension 2. On the other hand, in [E1, P], an approach based on maximum principle is developed to derive a sharp lower bound of the first Steklov eigenvalue on a compact surface with boundary. This idea is also used in [W1] to prove the limiting case $q = \infty$. Surprisingly this type of argument works for any power $q \geq 2$.

Throughout this section (Σ, g) is a smooth compact surface with Gaussian curvature $K \geq 0$ and geodesic curvature $\kappa \geq 1$ on the boundary. Our goal is to prove the following uniqueness result.

Theorem 4. *Let $u > 0$ be a smooth function on Σ satisfying the following equation*

$$\begin{aligned} \Delta u &= 0 \quad &&\text{on} \quad \Sigma, \\ \frac{\partial u}{\partial \nu} + \lambda u &= u^q \quad &&\text{on} \quad \partial\Sigma, \end{aligned}$$

where λ is a positive constant and $q \geq 2$. Then u must be a constant function if $\lambda \leq \frac{1}{q-1}$.

Theorem 3 follows by combining the above theorem and Theorem 2.

To prove Theorem 4 we write $u = v^{-a}$, with $a \neq 0$ to be determined. Then v satisfies

$$\begin{aligned} \Delta v &= (a+1)\, v^{-1} \left|\nabla v\right|^2 \quad \text{on} \quad \Sigma, \\ \chi &= \tfrac{1}{a}\left(\lambda f - f^{1+a-aq}\right) \quad \text{on} \quad \partial\Sigma, \end{aligned}$$

where $f = v|_{\partial\Sigma}, \chi = \frac{\partial v}{\partial \nu}$. Let $\phi = v^b \left|\nabla v\right|^2$ with b to be determined.

Proposition 5. *We have*

$$\Delta\phi - 2\,(a+b+1)\, v^{-1} \langle \nabla v, \nabla\phi \rangle \geq \left[a\,(a-b) - (b+1)^2\right] v^{-b-2}\phi^2. \tag{4.1}$$

Proof. We have $\left|\nabla v\right|^2 = v^{-b}\phi$. We compute

$$\begin{aligned} \Delta \left|\nabla v\right|^2 &= v^{-b}\Delta\phi - 2bv^{-b-1} \langle \nabla v, \nabla\phi \rangle + \phi\Delta v^{-b} \\ &= v^{-b}\Delta\phi - 2bv^{-b-1} \langle \nabla v, \nabla\phi \rangle + \phi\left[-bv^{-b-1}\Delta v + b\,(b+1)\, v^{-b-2}\left|\nabla v\right|^2\right] \\ &= v^{-b}\Delta\phi - 2bv^{-b-1} \langle \nabla v, \nabla\phi \rangle + b\,(b-a)\, v^{-2b-2}\phi^2. \end{aligned}$$

Using the Bochner formula we obtain

$$\begin{aligned} & v^{-b}\Delta\phi - 2bv^{-b-1} \langle \nabla v, \nabla\phi \rangle + b\,(b-a)\, v^{-2b-2}\phi^2 \\ &\geq 2\left|D^2 v\right|^2 + 2\langle \nabla v, \nabla\Delta v\rangle \\ &\geq (\Delta v)^2 + 2\langle \nabla v, \nabla\Delta v\rangle \\ &= (a+1)^2 v^{-2b-2}\phi^2 + 2\,(a+1)\left[v^{-b-1}\langle \nabla v, \nabla\phi\rangle - (b+1)\, v^{-2b-2}\phi^2\right] \\ &= (a+1)\,(a-2b-1)\, v^{-2b-2}\phi^2 + 2\,(a+1)\, v^{-b-1}\langle \nabla v, \nabla\phi\rangle. \end{aligned}$$

Therefore

$$\Delta\phi - 2\,(a+b+1)\, v^{-1} \langle \nabla v, \nabla\phi \rangle \geq \left[a\,(a-b) - (b+1)^2\right] v^{-b-2}\phi^2.$$

□

We impose the following condition on a and b

$$a\,(a-b) - (b+1)^2 > 0. \tag{4.2}$$

As a result, $\Delta\phi - 2\,(a+b+1)\, v^{-1}\langle \nabla v, \nabla\phi\rangle \geq 0$. By the maximum principle, ϕ achieves its maximum somewhere on the boundary. We use the arclength s to parametrize the boundary. Suppose that ϕ achieves its maximum at s_0 on the boundary. Then we have

$$\phi'(s_0) = 0, \phi''(s_0) \leq 0, \frac{\partial\phi}{\partial\nu}(s_0) \geq 0.$$

Moreover by the Hopf lemma, the 3rd inequality is strict unless ϕ is constant.

Proposition 6. *We have*

$$\frac{\partial\phi}{\partial\nu} \leq 2f^b\left[\left(\left(\frac{b}{2} + a + 1\right)\frac{\chi}{f} - 1\right)\left((f')^2 + \chi^2\right) + f'\chi' - \chi f''\right].$$

Proof. We compute

$$\begin{aligned}\frac{\partial \phi}{\partial \nu} &= 2f^b D^2 v\left(\nabla v, \nu\right) + b f^{b-1}\chi\left(\left(f'\right)^2 + \chi^2\right)\\ &= 2f^b\left[\chi D^2 v\left(\nu,\nu\right) + f' D^2 v\left(\frac{\partial}{\partial s}, \nu\right) + \frac{b\chi}{2f}\left(\left(f'\right)^2+\chi^2\right)\right].\end{aligned}$$

On one hand

$$\begin{aligned}D^2 v\left(\frac{\partial}{\partial s}, \nu\right) &= \left\langle \nabla_{\frac{\partial}{\partial s}}\nabla v, \nu\right\rangle\\ &= \chi' - \left\langle \nabla v, \nabla_{\frac{\partial}{\partial s}}\nu\right\rangle\\ &= \chi' - f'\left\langle \frac{\partial}{\partial s}, \nabla_{\frac{\partial}{\partial s}}\nu\right\rangle\\ &= \chi' - \kappa f'.\end{aligned}$$

On the other hand from the equation of v we have on $\partial\Sigma$

$$D^2 v\left(\nu,\nu\right) + \kappa\chi + f'' = \left(a+1\right) f^{-1}\left(\left(f'\right)^2+\chi^2\right).$$

Plugging the above two identities into the formula for $\frac{\partial \phi}{\partial \nu}$ yields

$$\frac{\partial \phi}{\partial \nu} = 2f^b\left[\left(\left(\frac{b}{2}+a+1\right)\frac{\chi}{f} - \kappa\right)\left(\left(f'\right)^2+\chi^2\right) + f'\chi' - \chi f''\right].$$

where in the last step we use the assumption $\kappa \geq 1$. □

As

$$\phi\left(s\right) := \phi|_{\partial\Sigma} = f\left(s\right)^b\left(f'\left(s\right)^2 + \chi\left(s\right)^2\right),$$

we obtain

$$\phi'\left(s\right) = 2f^b f'\left[f'' + \frac{1}{a}\chi\left(\lambda - \left(1+a-aq\right)f^{a-aq}\right) + \frac{b}{2f}\left(f'^2+\chi^2\right)\right].$$

If $f'\left(s_0\right) \neq 0$ then at s_0

$$f'' = -\frac{1}{a}\chi\left(\lambda - \left(1+a-aq\right)f^{a-aq}\right) - \frac{b}{2f}\left(f'^2+\chi^2\right).$$

Therefore

$$\begin{aligned}\frac{\partial \phi}{\partial \nu} &\leq 2f^b\left[\left(\left(\frac{b}{2}+a+1\right)\frac{\chi}{f} - 1\right)\left(\left(f'\right)^2+\chi^2\right) + f'\chi'\right.\\ &\quad\left.+\frac{1}{a}\chi^2\left(\lambda - \left(1+a-aq\right)f^{a-aq}\right) + \frac{b}{2}\frac{\chi}{f}\left(f'^2+\chi^2\right)\right]\\ &= 2f^b\left(\left(f'\right)^2+\chi^2\right)\left[\frac{a+b+1}{a}\left(\lambda - f^{a-aq}\right) - 1 + \frac{1}{a}\left(\lambda - \left(1+a-aq\right)f^{a-aq}\right)\right]\\ &= 2f^b\left(\left(f'\right)^2+\chi^2\right)\left[\frac{a+b+2}{a}\lambda - 1 - \frac{\left(2-q\right)a+b+2}{a}f^{a-aq}\right].\end{aligned}$$

We want

$$\frac{a+b+2}{a}\lambda-1\leq 0,$$
$$(2-q)\,a+b+2=0.$$

Therefore we choose $b=(q-2)\,a-2$. Then the 1st equation is simply $(q-1)\,\lambda\leq 1$. The condition (4.2) becomes

$$\left(q^2-3q+1\right)a^2-2\,(q-1)\,a+1<0.$$

A solution always exists as the discriminant equals $4q>0$. Under such choices for a and b we have $\frac{\partial\phi}{\partial\nu}(s_0)\leq 0$. Therefore ϕ is constant.

If $f'(s_0)=0$ then at s_0

$$\phi''(s_0)=2f^bf''\left[f''+\frac{1}{a}\chi\left(\lambda-(1+a-aq)\,f^{a-aq}\right)+\frac{b}{2}\frac{\chi^2}{f}\right]\leq 0.$$

Therefore we have at s_0

$$\left(f''\right)^2+f''\chi\left[(q-1)\,\lambda-\frac{qa}{2}\frac{\chi}{f}\right]\leq 0.$$

while the condition $\frac{\partial\phi}{\partial\nu}(s_0)\geq 0$ becomes

$$\left(\frac{qa}{2}\frac{\chi}{f}-1\right)\chi^2-\chi f''\geq 0.$$

Set $A=(q-1)\,\lambda-\frac{qa}{2}\frac{\chi}{f}$. We have $\frac{qa}{2}\frac{\chi}{f}-1\leq\frac{qa}{2}\frac{\chi}{f}-(q-1)\,\lambda=-A$. Therefore the above two inequalities imply

$$\chi\left(A\chi+f''\right)\leq 0,$$
$$f''\left(A\chi+f''\right)\leq 0.$$

We have

$$A=(q-1)\,\lambda-\frac{q}{2}\left(\lambda-f^{a-qa}\right)=\left(\frac{q}{2}-1\right)\lambda+\frac{q}{2}f^{a-qa}\geq 0$$

if $q\geq 2$. Combining the two inequalities we then get $\left(A\chi+f''\right)^2\leq 0$. Therefore $A\chi+f''=0$. Then again we have $\frac{\partial\phi}{\partial\nu}(s_0)\leq 0$ and ϕ must be constant.

In all cases we have proved that ϕ is constant. As the coefficient on the right hand side of (4.1) is positive, we must have $\phi\equiv 0$. Therefore u is constant. This finishes the proof of Theorem 4.

5. Further discussions

Let (M^n,g) be a smooth compact Riemannian manifold with boundary Σ. We consider for $1<q\leq\frac{n}{n-2}$ and $\lambda>0$ the functional

$$J_{q,\lambda}(u)=\frac{\int_M|\nabla u|^2+\lambda\int_\Sigma u^2}{\left(\int_\Sigma|u|^{q+1}\right)^{\frac{2}{q+1}}},\ u\in H^1(M)\setminus\{0\}.$$

The first variation in the direction of $\dot{u}$ is

$$2\left[\frac{\int_M\left\langle \nabla u,\nabla\dot{u}\right\rangle+\lambda\int_\Sigma u\dot{u}}{\left(\int_\Sigma |u|^{q+1}\right)^{\frac{2}{q+1}}}-\frac{\int_M|\nabla u|^2+\lambda\int_\Sigma u^2}{\left(\int_\Sigma |u|^{q+1}\right)^{1+\frac{2}{q+1}}}\int_\Sigma |u|^q\dot{u}\right]$$
$$=\frac{2}{\left(\int_\Sigma |u|^{q+1}\right)^{\frac{2}{q+1}}}\left[-\int_M\dot{u}\Delta u+\int_\Sigma\left(\frac{\partial u}{\partial\nu}+\lambda u\right)\dot{u}-\frac{\int_M|\nabla u|^2+\lambda\int_\Sigma u^2}{\int_\Sigma |u|^{q+1}}\int_\Sigma |u|^q\dot{u}\right]$$

Thus a positive u is a critical point iff

$$\begin{aligned}\Delta u&=0 &&\text{on}\quad M,\\ \frac{\partial u}{\partial\nu}+\lambda u&=cu^q &&\text{on}\quad \Sigma,\end{aligned}$$

with $c=\frac{\int_M|\nabla u|^2+\lambda\int_\Sigma u^2}{\int_\Sigma |u|^{q+1}}$. In particular $u_0\equiv 1$ is a critical point. The second variation at u_0 in the direction of $\dot{u}$ with $\int_\Sigma \dot{u}=0$ is

$$\frac{2}{|\Sigma|^{\frac{2}{q+1}}}\left[-\int_M\dot{u}\Delta\dot{u}+\int_\Sigma\left(\frac{\partial\dot{u}}{\partial\nu}+\lambda\dot{u}\right)\dot{u}-\lambda q\left(\dot{u}\right)^2\right]$$
$$=\frac{2}{|\Sigma|^{\frac{2}{q+1}}}\left[\int_M\left|\nabla\dot{u}\right|^2-\lambda\left(q-1\right)\int_\Sigma\left(\dot{u}\right)^2\right].$$

Therefore u_0 is stable iff $\lambda(q-1)\le\sigma_1$, the first Steklov eigenvalue. On $\overline{\mathbb{B}^n}$ the first Steklov eigenvalue is 1. Therefore u_0 is not stable on $\overline{\mathbb{B}^n}$ when $\lambda(q-1)>1$. As the trace operator $H^1(M)\to L^q(\Sigma)$ is compact when $q<\frac{n}{n-2}$, $\inf J_{q,\lambda}$ is always achieved. Therefore we get the following

Proposition 7. *If $q<\frac{n}{n-2}$ and $\lambda(q-1)>1$ then the equation*

$$\begin{aligned}\Delta u&=0 &&\text{on}\quad \overline{\mathbb{B}^n},\\ \frac{\partial u}{\partial\nu}+\lambda u&=u^q &&\text{on}\quad \partial\overline{\mathbb{B}^n},\end{aligned}$$

admits a positive, nonconstant solution.

In the general case, under the assumption that $Ric\ge 0$ and $\Pi\ge 1$ on Σ, Conjecture 1 claims that u_0, up to scaling, is the only positive critical point of $J_{q,\lambda}$ if $\lambda(q-1)\le 1$. In particular we must have $\sigma_1\ge 1$ if the conjecture is true for a single exponent q. Therefore Conjecture 1 implies the following conjecture of Escobar [E2] .

Conjecture 2 ([E2])**.** *Let (M^n,g) be a compact Riemannian manifold with boundary with $Ric\ge 0$ and $\Pi\ge 1$ on Σ. Then the 1st Steklov eigenvalue $\sigma_1\ge 1$.*

In [E1], the conjecture is confirmed when $n=2$, extending the method of [P], where the same estimate for a planar domain is derived. In other dimensions, under the stronger assumption that M has nonnegative sectional curvature, the conjecture was proved recently in [XX]. By the previous discussion, Theorem 2 implies estimate in [XX] when $2\le n\le 8$ and can be viewed as a nonlinear generalization. Theorem 2 also gives us the following sharp Sobolev inequalities (see also the discussions in [W2]).

Corollary 1. *Let (M^n, g) be a smooth compact Riemannian manifold with nonnegative sectional curvature and $\Pi \geq 1$ on the boundary Σ. Assume $2 \leq n \leq 8$ and $1 < q \leq \frac{4n}{5n-9}$. Then*

$$\left(\frac{1}{|\Sigma|}\int_\Sigma |u|^{q+1}\right)^{2/(q+1)} \leq \frac{q-1}{|\Sigma|}\int_M |\nabla u|^2 + \frac{1}{|\Sigma|}\int_\Sigma u^2. \tag{5.1}$$

In the limiting case we can deduce the following logarithmic inequality.

Corollary 2. *Let (M^n, g) be a compact Riemannian manifold with nonnegative sectional curvature and $\Pi \geq 1$ on the boundary Σ. Assume $2 \leq n \leq 8$. Then for any $u \in C^\infty(M)$ with $\frac{1}{|\Sigma|}\int_\Sigma u^2 = 1$, we have*

$$\frac{1}{|\Sigma|}\int_\Sigma |u|^2 \log u^2 \leq \frac{2}{|\Sigma|}\int_M |\nabla u|^2.$$

Proof. Under the assumption on u (5.1) can be written as

$$\frac{1}{q-1}\left[\left(\frac{1}{|\Sigma|}\int_\Sigma |u|^{q+1}\right)^{2/(q+1)} - 1\right] \leq \frac{1}{|\Sigma|}\int_M |\nabla u|^2.$$

Taking limit $q \downarrow 1$ and applying L'Hospital's rule yields the desired inequality. □

Remark 1. *Linearization of the above inequality around $u_0 \equiv 1$ yields the inequality $\sigma_1 \geq 1$, i.e. if $\int_\Sigma u = 0$, then*

$$\int_\Sigma u^2 \leq \int_M |\nabla u|^2.$$

In dimension two we have a complete result in Theorem 3. As a corollary we have

Corollary 3. *Let (Σ, g) be a smooth compact surface with nonnegative Gaussian curvature and geodesic curvature $\kappa \geq 1$. Then for any $u \in H^1(\Sigma)$ and $q \geq 1$, we have*

$$L^{(q-1)/(q+1)}\left(\int_{\partial\Sigma} |u|^{q+1}\right)^{2/(q+1)} \leq (q-1)\int_\Sigma |\nabla u|^2 + \int_{\partial\Sigma} u^2.$$

Here L is the length of $\partial\Sigma$. Moreover, equality holds iff u is a constant function.

Finally we recall the following Moser-Trudinger-Onofri type inequality on the disc $\overline{\mathbb{B}^2}$ derived in [OPS]: for any $u \in H^1(\mathbb{B}^2)$,

$$\log\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} e^u\right) \leq \frac{1}{4\pi}\int_{\mathbb{B}^2} |\nabla u|^2 + \frac{1}{2\pi}\int_{\mathbb{S}^1} u. \tag{5.2}$$

In [W1] the following generalization was proved

Theorem 5 ([W1])**.** *Let (Σ, g) be a smooth compact surface with nonnegative Gaussian curvature and geodesic curvature $\kappa \geq 1$. Then for any $u \in H^1(\Sigma)$,*

$$\log\left(\frac{1}{L}\int_{\partial\Sigma} e^f\right) \leq \frac{1}{2L}\int_\Sigma |\nabla f|^2 + \frac{1}{L}\int_{\partial\Sigma} f.$$

Here L is the length of $\partial\Sigma$. Moreover if equality holds at a nonconstant function, then Σ is isometric to $\overline{\mathbb{B}^2}$ and all extremal functions are of the form

$$u(x) = \log \frac{1 - |a|^2}{1 + |a|^2 |x|^2 - 2x \cdot a} + c,$$

for some $a \in \mathbb{B}^2$ and $c \in \mathbb{R}$.

The argument in [W1] is by a variational approach based on the inequality (5.2). We can deduce the above inequality directly from Corollary 3. Indeed, taking $u = 1 + \frac{f}{q+1}$ in Corollary 3 we obtain

$$\left(\frac{1}{L}\int_{\partial\Sigma}\left(1+\frac{f}{q+1}\right)^{q+1}\right)^{2/(q+1)} \leq \frac{(q-1)}{(q+1)^2}\frac{1}{L}\int_{\Sigma}|\nabla f|^2 + \frac{1}{L}\int_{\partial\Sigma}\left(1+\frac{f}{q+1}\right)^2.$$

This can be rewritten as

$$\begin{aligned}&(q+1)\left\{\exp\left[\frac{2}{q+1}\log\frac{1}{L}\int_M\left(1+\frac{f}{q+1}\right)^{q+1}\right]-1\right\}\\ &\leq \frac{(q-1)}{(q+1)}\frac{1}{L}\int_\Sigma|\nabla f|^2 + \frac{2}{L}\int_{\partial\Sigma} f + \frac{1}{q+1}\frac{1}{L}\int_{\partial\Sigma} f^2.\end{aligned}$$

Letting $q \to \infty$ we get

$$\log\frac{1}{L}\int_M e^f \leq \frac{1}{2L}\int_\Sigma |\nabla f|^2 + \frac{1}{L}\int_{\partial\Sigma} f.$$

Acknowledgements

We thank the referee for a careful reading of the paper. The 3rd named author is partially supported by Simons Foundation Collaboration Grant for Mathematicians #312820.

References

[BVV] M. Bidaut-Véron and L. Véron. Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations. Invent. Math. 106 (1991), no. 3, 489–539.

[E1] J. Escobar. The geometry of the first non-zero Stekloff eigenvalue. J. Funct. Anal. 150 (1997), no. 2, 544–556.

[E2] J. Escober. An isoperimetric inequality and the first Steklov eigenvalue, J. Funct. Anal. 165 (1999), no. 1, 101–116.

[GW1] R. E. Greene and H. Wu. On the subharmonicity and plurisubharmonicity of geodesically convex functions, Indiana Univ. Math. J. 22 (1972/73), 641–653.

[GW2] R. E. Greene and H. Wu. C^1 convex functions and manifolds of positive curvature, Acta Math. 137 (1976), no. 3-4, 209–245.

[GW3] R. E. Greene and H. Wu. C^1 approximations of convex, subharmonic, and plurisubharmonic functions, Ann. Sci. Ecole Norm. Sup. (4) 12 (1979), no. 1, 47–84.

[GuW] Q. Guo and X. Wang. Uniqueness results for positive harmonic functions on $\overline{\mathbb{B}^n}$ satisfying a nonlinear boundary condition, arXiv: 1912.05568

[I] S. Ilias. Inégalités de Sobolev et résultats d'isolement pour les applications harmoniques. J. Funct. Anal. 139 (1996), no. 1, 182–195.

[K] A. Kasue. A Laplacian comparison theorem and function theoretic properties of a complete Riemannian manifold, Japan. J. Math. (N.S.) 8 (1982), no. 2, 309–341.

[OPS] B. Osgood, R. Phillips and P. Sarnak. Extremals of determinants of Laplacians. J. Funct. Anal. 80 (1988), no. 1, 148–211.

[P] L. E. Payne. Some isoperimetric inequalities for harmonic functions, SIAM J. Math. Anal. 1 (1970), 354–359.

[QX] G. Qiu and C. Xia. A generalization of Reilly's formula and its applications to a new Heintze-Karcher type inequality, Int. Math. Res. Not. IMRN 2015, no. 17, 7608–7619.

[W1] X. Wang. Uniqueness results on surfaces with boundary. Calc. Var. Partial Differential Equations 56 (2017), no. 3, Art. 87, 11 pp.

[W2] X. Wang. On compact Riemannian manifolds with convex boundary and Ricci curvature bounded from below, arXiv:1908.03069, to appear in J. Geom. Anal.

[XX] C. Xia and C. Xiong. Escobar's Conjecture on a sharp lower bound for the first nonzero Steklov eigenvalue, arXiv: 1907.07340

Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, China

E-mail address: gqianqiao@nwpu.edu.cn

Courant Institute, 251 Mercer Street, New York, NY 10012

E-mail address: fengbo@cims.nyu.edu

Department of Mathematics, Michigan State University, East Lansing, MI 48864

E-mail address: xwang@math.msu.edu