A SIMPLER PROOF OF FRANK AND LIEB'S SHARP INEQUALITY ON THE HEISENBERG GROUP

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Abstract. We give a simpler proof of the sharp Frank-Lieb inequality on the Heisenberg group $\mathbb{H}^m$. The proof bypasses the sophisticated argument for existence of a minimizer and is based on the study of the 2nd variation of subcritical functionals using their fundamental techniques.

1. Introduction

In a ground-breaking work [FL1], Frank and Lieb determined the sharp constant and extremal functions for the Folland-Stein inequality on the Heisenberg group $\mathbb{H}^m$. Recall

$$\mathbb{H}^m = \{u = (z,t) : z \in \mathbb{C}^m, t \in \mathbb{R}\}$$

with the group law

$$u \cdot u' = (z,t) \cdot (z',t) = \left(z + z', t + t' + 2\operatorname{Im} z\overline{z'}\right).$$

The Haar measure on $\mathbb{H}^m$ is the standard Lebesgue measure $du = dzdt$. For $\delta > 0$ we write $\delta u = \left(\delta z, \delta^2 t\right)$ for the dilation. We denote the homogeneous norm on $\mathbb{H}^m$ by

$$|u| = |(z,t)| = \left(|z|^4 + t^2\right)^{1/4}.$$

Throughout the paper $Q = 2m + 2$. The Frank-Lieb inequality then states

Theorem 1. *Let $0 < \lambda < Q$ and $p = 2Q/(2Q - \lambda)$. Then for any $f, g \in L^p(\mathbb{H}^m)$*

$$\left|\int_{\mathbb{H}^m \times \mathbb{H}^m} \frac{\overline{f(u)} g(v)}{|u^{-1}v|^{\lambda}} dudv\right| \le \left(\frac{2\pi^{m+1}}{m!}\right)^{\lambda/Q} \frac{m!\Gamma((Q-\lambda)/2)}{\Gamma^2((2Q-\lambda)/4)} \|f\|_p \|g\|_p,$$

with equality if and only if

$$f(u) = cH\left(\delta\left(a^{-1}u\right)\right), g(u) = c'H\left(\delta\left(a^{-1}u\right)\right)$$

for some $c, c' \in \mathbb{C}$ and $a \in \mathbb{H}^m$ (unless $f \equiv 0$ or $g \equiv 0$). Here H is the function defined by

$$H(z,t) = \left(\left(1 + |z|^2\right)^2 + t^2\right)^{-(2Q-\lambda)/4}.$$

Via the Cayley transform, Theorem 1 is equivalent to the following formulation on the sphere $\mathbb{S}^{2m+1} = \left\{z \in \mathbb{C}^{m+1} : |z| = 1\right\}$.

Theorem 2. *Let $0 < \lambda < Q$ and $p = 2Q/(2Q - \lambda)$. Then for any $f, g \in L^p\left(\mathbb{S}^{2m+1}\right)$*

$$\left|\int_{\mathbb{S}^{2m+1} \times \mathbb{S}^{2m+1}} \frac{\overline{f(\xi)} g(\eta)}{|1 - \xi \cdot \overline{\eta}|^{\lambda/2}} d\sigma(\xi) d\sigma(\eta)\right| \le \left(\frac{2\pi^{m+1}}{m!}\right)^{\lambda/Q} \frac{m!\Gamma((Q-\lambda)/2)}{\Gamma^2((2Q-\lambda)/4)} \|f\|_p \|g\|_p,$$

with equality if and only if

$$f(\xi)=\frac{c}{|1-\xi\cdot\overline{\eta}|^{(2Q-\lambda)/2}},\, g(\xi)=\frac{c'}{|1-\xi\cdot\overline{\eta}|^{(2Q-\lambda)/2}}$$

for some $c,c'\in\mathbb{C}$ *and* $\eta\in\mathbb{C}^{m+1}$ *with* $|\eta|<1$ *(unless* $f\equiv 0$ *or* $g\equiv 0$*).*

Their proof consists of two major steps. The first step is to prove that the infimum

$$\inf\left\{\left|\int_{\mathbb{H}^m\times\mathbb{H}^m}\frac{\overline{f(u)}g(v)}{|u^{-1}v|^{\lambda}}dudv\right|:\|f\|_p=\|g\|_p=1\right\}$$

is achieved by some (f,g) and moreover $f=g$. This requires some sophisticated harmonic analysis on $\mathbb{H}^m$. The 2nd step is to work on $\mathbb{S}^{2m+1}$ and determine the extremal function $f=g$. By using the invariance of the problem under the CR automorphism group and a Hersch-type argument, they first arrange that f satisfies a moment zero condition. Then by exploiting masterfully the 2nd variation of the functional with test functions provided by the moment zero condition, they prove that such f must be constant.

In this paper, we present a shorter proof of the Frank-Lieb inequality which bypasses the subtle proof of existence and the Hersch-type argument. We use a scheme of subcritical approximation. The starting point is that the operator

$$I_\lambda f(\xi)=\int_{\mathbb{S}^{2m+1}}\frac{f(\eta)}{|1-\xi\cdot\overline{\eta}|^{\lambda/2}}d\sigma(\eta).$$

is compact from $L^p\left(\mathbb{S}^{2m+1}\right)$ to $L^{p'}\left(\mathbb{S}^{2m+1}\right)$, if $p>2Q/(2Q-\lambda)$ and $p'=p/(p-1)$. Therefore the minimization problem

$$\Lambda_p=\inf\left\{\|I_\lambda f\|_{p'}:\|f\|_p=1\right\}$$

has a minimizer u_p which can be taken to be nonnegative. Moreover, due to a symmetry-breaking, u_p automatically satisfies a moment zero condition. Therefore we can analyze the 2nd variation of the functional $\|I_\lambda f\|_{p'}/\|f\|_p$ at u_p by fully using Frank and Lieb's techniques. Though we could not prove that u_p is constant as we had expected, we are able to show that u_p converges to a constant in $L^{2Q/(2Q-\lambda)}\left(\mathbb{S}^{2m+1}\right)$ as $p\searrow 2Q/(2Q-\lambda)$. This is enough to yield Frank-Lieb's sharp inequality.

The paper is organized as follows. In Section 2, we present a proof for Jerison-Lee's sharp Sobolev inequality which is equivalent to Frank-Lieb's inequality with $\lambda=Q-2$. The analysis is simpler in this special case. In Section 3, we collect some fundamental results on the operator I_λ. We present the proof for Frank-Lieb's inequality in Section 4. Finally, we make some further remarks and raise several open problems in the last Section.

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2. Proof of Jerison-Lee inequality

The Frank-Lieb inequality for $\lambda=Q-2$ is equivalent to the following sharp Sobolev inequality established by Jerison and Lee in 1990's

(2.1)
$$\int_{\mathbb{S}^{2m+1}}\left(|\nabla_b f|^2+\frac{m^2}{4}f^2\right)d\sigma\geq\frac{\pi m^2}{4}\left(\frac{2}{m!}\right)^{\frac{1}{m+1}}\left(\int_{\mathbb{S}^{2m+1}}|f|^{2(m+1)/m}\,d\sigma\right)^{\frac{m}{m+1}}.$$

Here we use the canonical pseudohermitian structure $\theta_c=\left(2\sqrt{-1}\partial\,|z|^2\right)|_{\mathbb{S}^{2m+1}}$ on $\mathbb{S}^{2m+1}$ with Webster scalar curvature $R=m\,(m+1)\,/2$ and the adapted Riemannian metric $4g_0$, where g_0 is the standard metric on $\mathbb{S}^{2m+1}$. (But to be consistent with the general case, we still use the standard measure $d\sigma$ which differs from the usual pseudohermitian volume form $\theta_c\wedge(d\theta_c)^m$ by a scaling constant.) To prove this fundamental result, Jerison and Lee [JL1] first proved that the sharp constant

$$\Lambda=\inf\left\{E\,(f):\|f\|_{2(m+1)/m}=1\right\}, \tag{2.2}$$

where $E\,(f)=\int_{\mathbb{S}^{2m+1}}\left(|\nabla_b f|^2+\frac{m^2}{4}f^2\right)d\sigma$, is achieved by a smooth and positive minimizer u satisfying the following PDE

$$-\Delta_b u+\frac{m^2}{4}u=\Lambda u^{(m+2)/m}\text{ on }\mathbb{S}^{2m+1}. \tag{2.3}$$

This was achieved by a blow-up analysis which we outline. For any $q<(m+2)\,/m$, the embedding from $S^{1,2}\left(\mathbb{S}^{2m+1}\right)$ to $L^{q+1}\left(\mathbb{S}^{2m+1}\right)$ is compact and therefore the minimization problem

$$\Lambda_q=\inf\left\{E\,(f):\|f\|_{q+1}=1\right\}$$

has a smooth and positive solution u_q which then satisfies the PDE

$$-\Delta_b u+\frac{m^2}{4}u=\Lambda_q u^q\text{ on }\mathbb{S}^{2m+1}. \tag{2.4}$$

and $\|u_q\|_{q+1}=1$. If $\{u_q\}$ does not blow up as $q\nearrow(m+2)\,/m$, the limit is a solution for (2.2). If it does blow up, then one can properly scale and extract a limit on $\mathbb{H}^m$. As $\mathbb{H}^m$ is equivalent to $\mathbb{S}^{2m+1}$, this limit also yields a solution to (2.2). In [JL2] they proved that any positive solution of the PDE (2.3) must be of the form

$$u\,(\xi)=c\,|\cosh t+(\sinh t)\,\xi\cdot\overline{\eta}|^{-2}$$

for some $t\geq0$ and $\eta\in\mathbb{S}^{2m+1}$. The inequality (2.1) then follows.

In Section 3 of [FL1], assuming existence of a minimizer for (2.2) Frank and Lieb presented a simpler proof for the classification of all extremal functions. We will apply their idea to show directly that $\{u_q\}$ converges to a constant as $q\nearrow(m+2)\,/m$. Therefore constant functions are minimizers for (2.2) and hence the sharp inequality(2.1).

For $\eta\in\mathbb{S}^{2m+1}$ and $t\geq0$ we define $\Phi_{t,\eta}:\mathbb{S}^{2m+1}\to\mathbb{S}^{2m+1}$ by

$$\Phi_{t,\eta}\,(\xi)=\frac{1}{\cosh t+\sinh t\xi\cdot\overline{\eta}}\left(\xi-(\xi\cdot\overline{\eta})\,\eta\right)+\frac{\sinh t+\cosh t\xi\cdot\overline{\eta}}{\cosh t+\sinh t\xi\cdot\overline{\eta}}\eta.$$

This is a one-parameter family of CR automorphisms with $\Phi_{0,\eta}=Id$. Moreover, $\Phi_{t,\eta}^*\theta_c=\phi_{t,\eta}\theta_c$, where

$$\phi_{t,\eta}\,(\xi)=\frac{1}{|\cosh t+(\xi\cdot\overline{\eta})\sinh t|^2}.$$

(For more details we refer to the appendices of [FL1].) Given a function f, we get a family $f_{t,\eta}$ defined by

$$f_{t,\eta}(\xi) = f \circ \Phi_{t,\eta}(\xi)\, \phi_{t,\eta}^{m/2}(\xi).$$

It is well known that the CR Yamabe functional is invariant under such transformations, more specifically

$$E(f_{t,\eta}) = E(f), \|f_{t,\eta}\|_{2(m+1)/m} = \|f\|_{2(m+1)/m}.$$

With this invariance, we can establish a CR analogue of the Kazdan-Warner identity [KW].

Proposition 1. *If a positive function $f \in C^\infty(\mathbb{S}^{2m+1})$ satisfies the following PDE*

$$-\Delta_b f + \frac{m^2}{4} f = K f^{(m+2)/m}, \tag{2.5}$$

where K is a smooth function on $\mathbb{S}^{2m+1}$, then

$$\int_{\mathbb{S}^{2m+1}} \left[\langle \nabla K, \nabla \psi_\eta \rangle_0 + TKT\psi_\eta\right] f^{2(m+1)/m} d\sigma = 0, \tag{2.6}$$

where

$$\psi_\eta(\xi) = \operatorname{Re}(\xi \cdot \overline{\eta}), T = J\xi$$

and $\langle \cdot, \cdot \rangle_0$ stands for the standard metric on $\mathbb{S}^{2m+1}$.

Proof. As f satisfies the equation (2.5), it is a critical point of the functional

$$\mathcal{F}(u) = E(u) / \left(\int_{\mathbb{S}^{2m+1}} K u^{2(m+1)/m} d\sigma\right)^{m/(m+1)}.$$

Thus

$$\frac{d}{dt}|_{t=0} \mathcal{F}(f_{t,\eta}) = 0.$$

But $E(f_{t,\eta}) = E(f)$ and, by a change of variables,

$$\begin{aligned}\int_{\mathbb{S}^{2m+1}} K f_{t,\eta}^{2(m+1)/m} d\sigma &= \int_{\mathbb{S}^{2m+1}} K (f \circ \Phi_{t,\eta})^{2(m+1)/m} \phi_{t,\eta}^{m+1} d\sigma \\ &= \int_{\mathbb{S}^{2m+1}} K \circ \Phi_{t,\eta}^{-1} f^{2(m+1)/m} d\sigma.\end{aligned}$$

Therefore we obtain

$$\int_{\mathbb{S}^{2m+1}} \langle \nabla K, X \rangle_0 f^{2(m+1)/m} d\sigma = 0,$$

with $X = \frac{d}{dt}|_{t=0} \Phi_{t,\eta}$. Direct calculation yields

$$X = \nabla \psi_\eta + (T\psi_\eta) T$$

and hence the formula (2.6) $\square$

Corollary 1. *If $u > 0$ satisfies (2.4) with $1 < q < (m+2)/m$, then*

$$\int_{\mathbb{S}^{2m+1}} u(\xi)^{q+1} \xi d\sigma(\xi) = 0.$$

Proof. The equation (2.4) can be written as

$$-\Delta_b u+\frac{m^2}{4}u=Ku^{(m+2)/m},$$

with $K=\Lambda_q u^{q-(m+2)/m}$. By Proposition 1 and integration by parts

$$\begin{aligned}
0&=\int_{\mathbb{S}^{2m+1}}\left[\left\langle \nabla K,\nabla\psi_\eta\right\rangle_0+TKT\psi_\eta\right]u^{2(m+1)/m}d\sigma\\
&=\left(q-\frac{m+2}{m}\right)\Lambda_q\int_{\mathbb{S}^{2m+1}}\left[\left\langle \nabla u,\nabla\psi_\eta\right\rangle_0+TuT\psi_\eta\right]u^q d\sigma\\
&=\frac{1}{q+1}\left(q-\frac{m+2}{m}\right)\Lambda_q\int_{\mathbb{S}^{2m+1}}\left[\left\langle \nabla u^{q+1},\nabla\psi_\eta\right\rangle_0+Tu^{q+1}T\psi_\eta\right]d\sigma\\
&=-\frac{1}{q+1}\left(q-\frac{m+2}{m}\right)\Lambda_q\int_{\mathbb{S}^{2m+1}}u^{q+1}\left(\Delta\psi_\eta+T^2\psi_\eta\right)d\sigma\\
&=\frac{2(m+1)}{q+1}\left(q-\frac{m+2}{m}\right)\Lambda_q\int_{\mathbb{S}^{2m+1}}u^{q+1}\psi_\eta d\sigma.
\end{aligned}$$

Therefore $\int_{\mathbb{S}^{2m+1}}u^{q+1}\psi_\eta d\sigma=0$ for all $\eta\in\mathbb{S}^{2m+1}$. This yields the desired conclusion. □

By calculating the 2nd variation of the functional at the minimizer u_q, we have for any f with $\int_{\mathbb{S}^{2m+1}}u_q^q f d\sigma=0$

$$\int_{\mathbb{S}^{2m+1}}\left(\left|\nabla_b f\right|^2+\frac{m^2}{4}f^2\right)d\sigma\geq q\int_{\mathbb{S}^{2m+1}}\left(\left|\nabla_b u_q\right|^2+\frac{m^2}{4}u_q^2\right)d\sigma\int_{\mathbb{S}^{2m+1}}u_q^{q-1}f^2 d\sigma.$$

By Corollary 1, we can take $f(\xi)=u_q(\xi)x_i$ or $u_q(\xi)y_i$, where $x_i=\operatorname{Re}\xi_i, y_i=\operatorname{Im}\xi_i$. Therefore, summing the corresponding inequalities for all such f's, we obtain, in view of $\left\|u_q\right\|_{q+1}=1$

$$\begin{aligned}
q\int_{\mathbb{S}^{2m+1}}\left(\left|\nabla_b u_q\right|^2+\frac{m^2}{4}u_q^2\right)d\sigma&\leq\int_{\mathbb{S}^{2m+1}}\left[\sum_i\left|\nabla_b\left(u_q x_i\right)\right|^2+\left|\nabla_b\left(u_q y_i\right)\right|^2+\frac{m^2}{4}u_q^2\right]d\sigma\\
&=\int_{\mathbb{S}^{2m+1}}\left[\left|\nabla_b u_q\right|^2+u_q^2\left(\frac{m^2}{4}-\sum_i\left(x_i\Delta_b x_i+y_i\Delta_b y_i\right)\right)\right]d\sigma\\
&=\int_{\mathbb{S}^{2m+1}}\left[\left|\nabla_b u_q\right|^2+\frac{m(m+2)}{4}u_q^2\right]d\sigma.
\end{aligned}$$

Therefore

$$(q-1)\int_{\mathbb{S}^{2m+1}}\left|\nabla_b u_q\right|^2 d\sigma\leq\frac{m^2}{4}\left(\frac{m+2}{m}-q\right)\int_{\mathbb{S}^{2m+1}}u_q^2 d\sigma.$$

As $\left\|u_q\right\|_{q+1}=1$, the above inequality implies that $\int_{\mathbb{S}^{2m+1}}\left|\nabla_b u_q\right|^2 d\sigma\to 0$ as $q\nearrow\frac{m+2}{m}$. It follows that we can choose a sequence $q_i\nearrow\frac{m+2}{m}$ s.t. the sequence $\{u_i=u_{q_i}\}$ converges to a nonzero constant c, i.e.

$$\int_{\mathbb{S}^{2m+1}}\left|\nabla_b\left(u_i-c\right)\right|^2 d\sigma\to 0,\left\|u_i-c\right\|_{2(m+1)/m}\to 0.$$

Therefore the constant function c is a minimizer of (2.2) and the inequality (2.1) follows.

3. Preliminaries for the general case

In this Section, we present some fundamental technical results needed for the proof of the Frank-Lieb inequality in the general case. We first recall the Funk-Hecke theorem on $\mathbb{S}^{2m+1}$ (cf. [FL1, BLM] and references therein for more details). The space $L^2\left(\mathbb{S}^{2m+1}\right)$ can be decomposed into its $U(m+1)$-irreducible components

$$L^2\left(\mathbb{S}^{2m+1}\right)=\oplus_{j,k\geq 0}\mathcal{H}_{j,k}, \tag{3.1}$$

Here $\mathcal{H}_{j,k}$ is the space of restrictions to $\mathbb{S}^{2m+1}$ of harmonic polynomials $p(z,\overline{z})$ on $\mathbb{C}^{m+1}$ that are homogeneous of degree j in z and degree k in $\overline{z}$. For an integrable function K on the unit disc in $\mathbb{C}$ we can define an integral operator with kernel $K(\xi\cdot\overline{\eta})$ on $\mathbb{S}^{2m+1}$ by

$$(If)(\xi)=\int_{\mathbb{S}^{2m+1}}K(\xi\cdot\overline{\eta})\,f(\eta)\,d\sigma(\eta).$$

The Funk-Hecke theorem states that such operators are diagonal with respect to the decomposition (3.1). We need the following explicit results.

Proposition 2. *(Corollary 5.3 in* [FL1]*) Let $-1<\alpha<(m+1)/2$.*

1. *The eigenvalue of the operator with kernel $|1-\xi\cdot\overline{\eta}|^{-2\alpha}$ on the subspace $\mathcal{H}_{j,k}$ is*
$$E_{j,k}=\frac{2\pi^{m+1}\Gamma(m+1-2\alpha)}{\Gamma^2(\alpha)}\frac{\Gamma(j+\alpha)}{\Gamma(j+m+1-\alpha)}\frac{\Gamma(k+\alpha)}{\Gamma(k+m+1-\alpha)}$$
2. *The eigenvalue of the operator with kernel $|\xi\cdot\overline{\eta}|^2|1-\xi\cdot\overline{\eta}|^{-2\alpha}$ on the subspace $\mathcal{H}_{j,k}$ is*
$$E_{j,k}\left(1-\frac{(\alpha-1)(m+1-2\alpha)(2jk+n(j+k-1+\alpha))}{(j-1+\alpha)(j+m+1-\alpha)(k-1+\alpha)(k+m+1-\alpha)}\right)$$

When $\alpha=0$ or 1, the formulas are to be understood by taking limits with fixed j and k.

Remark 1. *It is clear that $E_{j,k}>0$ for all j and k if $\alpha=\lambda/4$ with $\lambda\in(0,Q)$. Therefore we draw the following important corollary: the operator I_λ is positive in the sense that*

$$\langle I_\lambda f,f\rangle=\int_{\mathbb{S}^{2m+1}\times\mathbb{S}^{2m+1}}\frac{\overline{f(\xi)}f(\eta)}{|1-\xi\cdot\overline{\eta}|^{\lambda/2}}d\sigma(\xi)\,d\sigma(\eta)\geq 0.$$

From Proposition 2 Frank and Lieb deduced the following inequality which plays a crucial role.

Theorem 3. *Let $0<\lambda<Q=2(m+1)$, then there exist $C>0$ s.t. for any f on $\mathbb{S}^{2m+1}$ one has*

$$\begin{aligned}&\int_{\mathbb{S}^{2m+1}}\frac{\overline{f(\xi)}f(\eta)\operatorname{Re}\xi\cdot\overline{\eta}}{|1-\xi\cdot\overline{\eta}|^{\lambda/2}}d\sigma(\eta)\,d\sigma(\xi)\\&\geq\frac{\lambda}{4(m+1)-\lambda}\langle I_\lambda f,f\rangle+\frac{C(2(m+1)-\lambda)}{4(m+1)-\lambda}\langle I_\lambda(f-a_f),f-a_f\rangle,\end{aligned}$$

where $a_f=\frac{1}{|\mathbb{S}^{2m+1}|}\int_{\mathbb{S}^{2m+1}}f(\eta)\,d\sigma(\eta)$ is the average of f.

Remark 2. *Taking $C=0$, this is precisely Theorem 5.1 in* [FL1]*. By inspecting their proof, it is easy to get the above strengthened version.*

Proposition 3. *If* $f_{t,\eta} = f \circ \Phi_{t,\eta} \phi_{t,\eta}^{(m+1)/p}$, *then*

$$I_\lambda \left(f_{t,\eta} \right) = I_\lambda \left(f \phi_{t,-\eta}^{(m+1)/p' - \lambda/4} \right) \circ \Phi_{t,\eta} \phi_{t,\eta}^{\lambda/4}.$$

Proof. By direction calculation, the following identity holds

$$\left| 1 - \Phi_{t,\eta} \left(\xi \right) \cdot \overline{\Phi_{t,\eta} \left(\zeta \right)} \right|^2 = \left| 1 - \xi \cdot \overline{\zeta} \right|^2 \phi_{t,\eta} \left(\xi \right) \phi_{t,\eta} \left(\zeta \right). \tag{3.2}$$

We compute by a change of variables

$$\begin{aligned} I_\lambda \left(f_{t,\eta} \right) \left(\xi \right) &= \int_{\mathbb{S}^{2m+1}} \frac{f \circ \Phi_{t,\eta} \left(\zeta \right) \phi_{t,\eta}^{(m+1)/p} \left(\zeta \right)}{\left| 1 - \xi \cdot \overline{\zeta} \right|^{\lambda/2}} d\sigma \left(\zeta \right) \\ &= \int_{\mathbb{S}^{2m+1}} \frac{f \left(\zeta \right) \left[\phi_{t,\eta} \circ \Phi_{t,\eta}^{-1} \left(\zeta \right) \right]^{-(m+1)/p'}}{\left| 1 - \xi \cdot \overline{\Phi_{t,\eta}^{-1} \left(\zeta \right)} \right|^{\lambda/2}} d\sigma \left(\zeta \right). \end{aligned}$$

It is easy to see that $\phi_{t,\eta} \circ \Phi_{t,\eta}^{-1} \left(\zeta \right) = 1/\phi_{t,-\eta} \left(\zeta \right)$ while using (3.2) we have

$$\begin{aligned} \left| 1 - \xi \cdot \overline{\Phi_{t,\eta}^{-1} \left(\zeta \right)} \right|^2 &= \left| 1 - \Phi_{t,\zeta} \left(\xi \right) \cdot \overline{\zeta} \right|^2 \phi_{t,-\eta} \circ \Phi_{t,\eta} \left(\xi \right) \left(\xi \right) \phi_{t,-\eta} \left(\zeta \right) \\ &= \left| 1 - \Phi_{t,\zeta} \left(\xi \right) \cdot \overline{\zeta} \right|^2 \phi_{t,-\eta} \left(\zeta \right) / \phi_{t,\eta} \left(\xi \right) \end{aligned}$$

Therefore

$$\begin{aligned} I_\lambda \left(f_{t,\eta} \right) \left(\xi \right) &= \left[\phi_{t,\eta} \left(\xi \right) \right]^{\lambda/4} \int_{\mathbb{S}^{2m+1}} \frac{f \left(\zeta \right) \phi_{t,-\eta} \left(\zeta \right)^{(m+1)/p' - \lambda/4}}{\left| 1 - \Phi_{t,\eta} \left(\xi \right) \cdot \overline{\zeta} \right|^{\lambda/2} \left(\xi \right)} d\sigma \left(\zeta \right) \\ &= I_\lambda \left(f \phi_{t,-\eta}^{(m+1)/p' - \lambda/4} \right) \circ \Phi_{t,\eta} \left(\xi \right) \phi_{t,\eta}^{\lambda/4} \left(\xi \right). \end{aligned}$$

□

4. Proof of the sharp inequality

We fix $\lambda \in (0, Q)$. Recall that the operator I_λ is defined by

$$I_\lambda f \left(\xi \right) = \int_{\mathbb{S}^{2m+1}} \frac{f \left(\eta \right)}{\left| 1 - \xi \cdot \overline{\eta} \right|^{\lambda/2}} d\sigma \left(\eta \right).$$

Given $1 < p < \frac{Q}{Q-\lambda}$, set $p^* = \frac{Qp}{Q - p(Q-\lambda)} > 1$. By the work of Folland-Stein [FS], I_λ is a bounded operator from $L^p \left(\mathbb{S}^{2m+1} \right)$ to $L^{p^*} \left(\mathbb{S}^{2m+1} \right)$. In other words, there exists a positive constant C s.t. for all $f \in L^p \left(\mathbb{S}^{2m+1} \right)$.

$$\left\| I_\lambda f \right\|_{p^*} \leq C \left\| f \right\|_p.$$

The contribution of Frank and Lieb [FL1] (Theorem 2) is the determination of the sharp constant and extremal functions when $p = \frac{2Q}{2Q-\lambda}$ and hence $p^* = \frac{2Q}{2Q+\lambda}$. Indeed, it is easy to verify that the sharp inequality in Theorem 2 is equivalent to

Theorem 4. *For* $f \in L^{\frac{2Q}{2Q-\lambda}} \left(\mathbb{S}^{2m+1} \right)$

$$\left\| I_\lambda f \right\|_{\frac{2Q}{2Q+\lambda}} \leq \left(\frac{2\pi^{m+1}}{m!} \right)^{\lambda/Q} \frac{m! \Gamma \left(\left(Q - \lambda \right)/2 \right)}{\Gamma^2 \left(\left(2Q - \lambda \right)/4 \right)} \left\| f \right\|_{\frac{2Q}{2Q-\lambda}}.$$

To present our proof of the above sharp inequality, we start with the following

Proposition 4. *For* $1 < p < \frac{Q}{Q-\lambda}, 1 < q < p^* = \frac{Qp}{Q-p(Q-\lambda)}$, *the operator* $I_\lambda : L^p\left(\mathbb{S}^{2m+1}\right) \to L^q\left(\mathbb{S}^{2m+1}\right)$ *is compact.*

This result is more or less standard. We outline the proof. Write $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*}$ with $\theta \in (0,1)$. By the Holder inequality, we have

$$\left\|I_\lambda f\right\|_q \le \left\|I_\lambda f\right\|_p^\theta \left\|I_\lambda f\right\|_{p^*}^{1-\theta}$$

Therefore it suffices to prove that I_λ from $L^p\left(\mathbb{S}^{2m+1}\right)$ to $L^p\left(\mathbb{S}^{2m+1}\right)$ is compact. For any kernel $K(\xi,\eta)$, the following inequality for the integral operator I_K is well known (in a much more general setting)

$$\left\|I_K f\right\|_p \le C_K \left\|f\right\|_p,$$

where

$$C_K = \max\left\{\sup_\xi \int_{\mathbb{S}^{2m+1}} \left|K(\xi,\eta)\right| d\sigma(\eta), \sup_\eta \int_{\mathbb{S}^{2m+1}} \left|K(\xi,\eta)\right| d\sigma(\xi)\right\}.$$

If K is continuous, it can be approximated uniformly by polynomials in ξ, η by the Stone-Weierstrass theorem. Therefore $I_K : L^p\left(\mathbb{S}^{2m+1}\right) \to L^p\left(\mathbb{S}^{2m+1}\right)$ is compact as it can be approximated by operators of finite rank. In our case $K(\xi,\eta) = \left|1 - \xi \cdot \overline{\eta}\right|^{-\lambda/2}$. It is easy to see that it can be approximated by continuous kernels

$$K^\varepsilon(\xi,\eta) = \begin{cases} \varepsilon^{-\lambda/2}, & \text{if } \left|1-\xi\cdot\overline{\eta}\right| \le \varepsilon, \\ \left|1-\xi\cdot\overline{\eta}\right|^{-\lambda/2} & \text{if } \left|1-\xi\cdot\overline{\eta}\right| \ge \varepsilon. \end{cases}$$

Therefore $I_\lambda : L^p\left(\mathbb{S}^{2m+1}\right) \to L^p\left(\mathbb{S}^{2m+1}\right)$ is compact.

We now take $p > \frac{2Q}{2Q-\lambda}$. By a simple calculation, its dual $p' := \frac{p}{p-1} < p^*$. Therefore the following minimization problem

$$\Lambda_p = \sup\left\{\left\|I_\lambda f\right\|_{p'} / \left\|f\right\|_p : f \in L^q\left(\mathbb{S}^{2m+1}\right), f \ne 0\right\} \tag{4.1}$$

has a solution u_p by Proposition 4. To simplify the presentation, we will drop the subscript p temporarily. We can obviously assume that $u \ge 0$. It satisfies the following Euler-Lagrange equation (when properly scaled)

$$\begin{cases} v^{p-1} = I_\lambda(u) \\ u^{p-1} = I_\lambda(v) \end{cases}.$$

Then

$$\begin{aligned} \left\langle I_\lambda(u-v), u-v\right\rangle &= \left\langle v^{p-1} - u^{p-1}, u-v\right\rangle \\ &= -\int_{\mathbb{S}^{2m+1}} \left(u^{p-1} - v^{p-1}\right)(u-v)\, d\sigma \\ &\le 0. \end{aligned}$$

By the positivity of I_λ (Remark 1), we must have $u = v$, i.e.

$$I_\lambda(u) = u^{p-1}.$$

Then

$$\Lambda_p = \frac{\left\|I_\lambda u\right\|_{p'}}{\left\|u\right\|_p} = \left\|u\right\|_p^{p-2}$$

or $\left\|u\right\|_p = \Lambda_p^{1/(p-2)}$. It is clear that $\lim_{p \to \frac{2Q}{2Q-\lambda}} \Lambda_p^{1/(p-2)} = \Lambda^{-2(Q-\lambda)/(2Q-\lambda)}$.

Lemma 1. *The function u satisfies*

$$\int_{\mathbb{S}^{2m+1}} u(\xi)^p \xi d\sigma(\xi) = 0.$$

Proof. We consider for any $\eta \in \mathbb{S}^{2m+1}$ the family $u_{t,\eta} = u \circ \Phi_{t,\eta} \phi_{t,\eta}^{(m+1)/p}$. Clearly $\|u_{t,\eta}\|_p = \|u\|_p$. By Proposition 3, we have

$$I_\lambda(u_{t,\eta}) = I_\lambda\left(u\phi_{t,-\eta}^{(m+1)/p'-\lambda/4}\right) \circ \Phi_{t,\eta} \phi_{t,\eta}^{\lambda/4}.$$

Therefore

$$\begin{aligned}
\|I_\lambda(u_t)\|_{p'}^{p'} &= \int_{\mathbb{S}^{2m+1}} \phi_{t,\eta}^{\lambda p'/4}(\xi) \left(I_\lambda\left(u\phi_{t,-\eta}^{(m+1)/p'-\lambda/4}\right) \circ \Phi_{t,\eta}(\xi)\right)^{p'} d\sigma(\xi) \\
&= \int_{\mathbb{S}^{2m+1}} \phi_{t,\eta}^{\lambda p'/4-m+1} \circ \Phi_{t,\eta}^{-1}(\xi) \left(I_\lambda\left(u\phi_{t,-\eta}^{(m+1)/p'-\lambda/4}\right)(\xi)\right)^{p'} d\sigma(\xi) \\
&= \int_{\mathbb{S}^{2m+1}} \phi_{t,-\eta}^{m+1-\lambda p'/4}(\xi) \left(I_\lambda\left(u\phi_{t,-\eta}^{(m+1)/p'-\lambda/4}\right)(\xi)\right)^{p'} d\sigma(\xi).
\end{aligned}$$

Differentiating $\log\left(\|I_\lambda(u_t)\|_{p'} / \|u_t\|_p\right)$ at $t = 0$ yields,

$$\begin{aligned}
0 &= \int_{\mathbb{S}^{2m+1}} \left[\left(m+1-\frac{\lambda p'}{4}\right) \dot{\phi} I_\lambda(u)^{p'} + p'\left(\frac{m+1}{p'} - \frac{\lambda}{4}\right) I_\lambda(u)^{p'-1} I_\lambda(u\dot{\phi})\right] d\sigma \\
&= \left(m+1-\frac{\lambda p'}{4}\right) \int_{\mathbb{S}^{2m+1}} \left[I_\lambda(u)^{p'} \dot{\phi} + I_\lambda(u)^{p'-1} I_\lambda(u\dot{\phi})\right] d\sigma,
\end{aligned}$$

where

$$\dot{\phi}(\xi) = \frac{d}{dt}|_{t=0} \phi_{t,-\eta}(\xi) = 2\operatorname{Re}(\xi \cdot \overline{\eta}).$$

By the Euler-Lagrange equation, $I_\lambda(u) = u^{p-1}$. Therefore we have

$$\begin{aligned}
0 &= \int_{\mathbb{S}^{2m+1}} \left[u^p \dot{\phi} + u I_\lambda(u\dot{\phi})\right] d\sigma \\
&= \int_{\mathbb{S}^{2m+1}} \left[u^p \dot{\phi} + I_\lambda(u)(u\dot{\phi})\right] d\sigma \\
&= 2\int_{\mathbb{S}^{2m+1}} u^p \dot{\phi} d\sigma,
\end{aligned}$$

i.e. $\int_{\mathbb{S}^{2m+1}} u^p(\xi) \operatorname{Re}(\xi \cdot \overline{\eta}) d\sigma(\xi) = 0$ for all $\eta \in \mathbb{S}^{2m+1}$. The conclusion follows. □

By the 2nd variation, we have for all real f with $\int_{\mathbb{S}^n} u^{p-1} f = 0$

$$(p'-1) \int_{\mathbb{S}^{2m+1}} (I_\lambda u)^{p'-2} (I_\alpha f)^2 \le (p-1)^2 \int_{\mathbb{S}^{2m+1}} u^{p-2} f^2.$$

Using the Euler-Lagrange equation $I_\lambda(u) = u^{p-1}$, this simplifies as

$$\int_{\mathbb{S}^{2m+1}} u^{2-p} (I_\lambda f)^2 \le (p-1)^2 \int_{\mathbb{S}^{2m+1}} u^{p-2} f^2. \tag{4.2}$$

By the Holder inequality

$$\begin{aligned}\int_{\mathbb{S}^{2m+1}} f I_\lambda f &= \int_{\mathbb{S}^{2m+1}} u^{(p-2)/2} f u^{(2-p)/2} I_\lambda f \\ &\le \left(\int_{\mathbb{S}^{2m+1}} u^{p-2} f^2\right)^{1/2} \left(\int_{\mathbb{S}^{2m+1}} u^{2-p} \left(I_\lambda f\right)^2\right)^{1/2} \\ &\le (p-1) \int_{\mathbb{S}^{2m+1}} u^{p-2} f^2.\end{aligned}$$

In summary, we have for all real f with $\int_{\mathbb{S}^n} u^{p-1} f = 0$

$$\int_{\mathbb{S}^{2m+1}\times\mathbb{S}^{2m+1}} \frac{f(\xi) f(\eta)}{|1-\xi\cdot\overline{\eta}|^{\lambda/2}} d\sigma(\xi)\, d\sigma(\eta) \le (p-1) \int_{\mathbb{S}^{2m+1}} u^{p-2} f^2.$$

By Lemma 1, we can take $f(z) = u(z) \operatorname{Re} z_i$ or $u(z) \operatorname{Im} z_i, i = 1, \cdots, m+1$ and adding these inequalities yields

$$\int_{\mathbb{S}^{2m+1}\times\mathbb{S}^{2m+1}} \frac{\operatorname{Re}\xi\cdot\overline{\eta}}{|\xi-\eta|^{\lambda/2}} u(\xi) u(\eta) d\sigma(\xi)\, d\sigma(\eta) \le (p-1) \int_{\mathbb{S}^{2m+1}} u^p.$$

Combined with Theorem 3, this implies

$$\begin{aligned}(p-1)\int_{\mathbb{S}^{2m+1}} u^p &\ge \frac{\lambda}{4(m+1)-\lambda} \langle I_\lambda u, u\rangle + \frac{C(2(m+1)-\lambda)}{4(m+1)-\lambda} \langle I_\lambda(u-a), (u-a)\rangle \\ &= \frac{\lambda}{4(m+1)-\lambda} \int_{\mathbb{S}^{2m+1}} u^p + \frac{C(2(m+1)-\lambda)}{4(m+1)-\lambda} \langle I_\lambda(u-a), (u-a)\rangle,\end{aligned}$$

where a is the average of u. Therefore we have

$$\left(p - \frac{4(m+1)}{4(m+1)-\lambda}\right) \int_{\mathbb{S}^{2m+1}} u^p \ge \frac{C(2(m+1)-\lambda)}{4(m+1)-\lambda} \langle I_\lambda(u-a), (u-a)\rangle.$$

Note that $\frac{4(m+1)}{4(m+1)-\lambda} = \frac{2Q}{2Q-\lambda}$.

By the positivity of I_λ, the RHS is nonnegative. As a consequence (now reattaching the subscript p), we have $\langle I_\lambda(u_p - a_p), (u_p - a_p)\rangle \to 0$ as $p \searrow \frac{2Q}{2Q-\lambda}$. By the Euler-Lagrange equation $I_\lambda(u) = u^{p-1}$again and the observation that I_λ maps a constant function to a constant function, this means $\int_{\mathbb{S}^{2m+1}} u_p^{p-1}(u_p - a_p) \to 0$ as $p \searrow \frac{2Q}{2Q-\lambda}$. Thus, as $p \searrow \frac{2Q}{2Q-\lambda}$

$$\begin{aligned}\int_{\mathbb{S}^{2m+1}} u_p^p &= a_p \int_{\mathbb{S}^{2m+1}} u_p^{p-1} + o(1) \\ &\le a_p \left|\mathbb{S}^{2m+1}\right|^{1/p} \left(\int_{\mathbb{S}^{2m+1}} u_p^p\right)^{(p-1)/p} + o(1).\end{aligned}$$

This implies

$$\|u_p\|_p \le a_p \left|\mathbb{S}^{2m+1}\right|^{1/p} + o(1).$$

On the other hand, we have $\|u_p\|_p \ge a_p \left|\mathbb{S}^{2m+1}\right|^{1/p}$ by the Holder inequality. Therefore $\lim_{p\searrow\frac{2Q}{2Q-\lambda}} \|u_p\|_p - a_p \left|\mathbb{S}^{2m+1}\right|^{1/p} = 0$. We assume that $u_p \xrightarrow{w^*} v$ in

$L^{\frac{2Q}{2Q-\lambda}}(\mathbb{S}^{2m+1})$. Clearly, $\lim_{p\searrow\frac{2Q}{2Q-\lambda}} a_p = a$, the average of v. We have

$$\begin{aligned}\|v\|_{\frac{2Q}{2Q-\lambda}} &\leq \lim_{p\searrow\frac{2Q}{2Q-\lambda}} \|u_p\|_{\frac{2Q}{2Q-\lambda}}\\ &\leq \lim_{p\searrow\frac{2Q}{2Q-\lambda}} \|u_p\|_p \left|\mathbb{S}^{2m+1}\right|^{\left(p-\frac{2Q}{2Q-\lambda}\right)/p}\\ &= \lim_{p\searrow\frac{2Q}{2Q-\lambda}} a_p \left|\mathbb{S}^{2m+1}\right|^{1/p}\\ &= a\left|\mathbb{S}^{2m+1}\right|^{\frac{2Q-\lambda}{2Q}}.\end{aligned}$$

In view of the Holder inequality, we must have $v = a$, i.e. v is constant and all the inequalities in the formula above are equalities, i.e.

$$a\left|\mathbb{S}^{2m+1}\right|^{\frac{2Q-\lambda}{2Q}} = \|v\|_{\frac{2Q}{2Q-\lambda}} = \lim_{p\searrow\frac{2Q}{2Q-\lambda}} \|u_p\|_{\frac{2Q}{2Q-\lambda}}.$$

The weak $*$-convergence plus the convergence of the norms implies strong convergence $u_p \to a$ in $L^{\frac{2Q}{2Q-\lambda}}(\mathbb{S}^{2m+1})$. Therefore the constant function is a minimizer for

$$\inf \|I_\lambda f\|_{\frac{2Q}{2Q+\lambda}} / \|f\|_{\frac{2Q}{2Q-\lambda}}.$$

Theorem 4 then follows from a simple calculation.

5. Further remarks

In a later paper [FL2], Frank and Lieb showed that the new method developed in [FL1] can be adapted to give a new, rearrangement-free proof of the following sharp Hardy-Littlewood-Sobolev inequality on $\mathbb{R}^n$ which was proved originally by Lieb [L] using rearrangement arguments.

Theorem 5. *Let $0 < \lambda < n$ and $p = 2n/(2n-\lambda)$. Then for any $f, g \in L^p(\mathbb{S}^n)$*

$$\left|\int_{\mathbb{S}^n\times\mathbb{S}^n} \frac{\overline{f(\xi)}g(\eta)}{|\xi-\eta|^\lambda} d\sigma(\xi)\, d\sigma(\eta)\right| \leq \pi^{\lambda/2}\frac{\Gamma((n-\lambda)/2)}{\Gamma(n-\lambda/2)}\left(\frac{\Gamma(n)}{\Gamma(n/2)}\right)^{1-\lambda/n} \|f\|_p \|g\|_p,$$

with equality if and only if

$$f(\xi) = \frac{c}{|1-\xi\cdot a|^{(2n-\lambda)/2}}, g(\xi) = \frac{c'}{|1-\xi\cdot a|^{(2n-\lambda)/2}}$$

for some $c, c' \in \mathbb{C}$ and $a \in \mathbb{R}^{n+1}$ with $|a| < 1$ (unless $f \equiv 0$ or $g \equiv 0$).

Our method can also be adapted to give a simpler proof of Lieb's theorem. In this case, we work with the operator

$$I_\lambda f(\xi) = \int_{\mathbb{S}^n} \frac{f(\eta)}{|\xi-\eta|^\lambda} d\sigma(\eta)$$

and consider, for $p > 2n/(2n-\lambda)$, the minimization problem

$$\Lambda_p = \sup\left\{\|I_\lambda f\|_{p'} / \|f\|_p : f \in L^p(\mathbb{S}^n), f \neq 0\right\}. \tag{5.1}$$

The rest of proof requires minor modifications and we omit the details.

We end with some open problems. In the CR cases, it would be interesting to classify all positive solutions to the Euler-Lagrange equation

$$I_{\lambda}(u) = u^{p-1}$$

for all $p \geq 2Q/(2Q-\lambda)$ on $\mathbb{S}^{2m+1}$, not merely extremal functions of the corresponding inequality. On $\mathbb{S}^n$, this kind of classification results can be established by the powerful method of moving planes or moving spheres (cf. [CLO, Li]). On $\mathbb{S}^{2m+1}$, the classification was known in the critical case $p = 2Q/(2Q-\lambda)$ only when $\lambda = Q-2$ by the work of Jerison-Lee [JL2]. The critical case when $\lambda \neq Q-2$ and all the subcritical cases seem to be largely open on $\mathbb{S}^{2m+1}$ as far as we know (cf. [W1, W2] for discussions about the significance of such classification problems).

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