A SHARP INEQUALITY RELATING YAMABE INVARIANTS ON ASYMPTOTICALLY POINCARE-EINSTEIN MANIFOLDS WITH A RICCI CURVATURE LOWER BOUND

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ABSTRACT. Let (X^n, g_+) be a conformally compact manifold with $Ric \geq -(n-1)$. If g_+ is asymptotically Poincare-Einstein, we establish a sharp inequality relating the type II Yamabe invariant of $\overline{X}$ and the Yamabe invariant of its conformal infinity.

1. INTRODUCTION

The Yamabe problem for closed Riemannian manifolds was completely solved by Aubin and Schoen (cf. [A, SY] for complete exposition). For compact Riemannian manifolds with boundary, there are two types of Yamabe problems and neither has bee completely solved. Let (M^n, g) be a compact Riemannian manifold (M^n, g) with nonempty boundary $\Sigma = \partial M$. The functional

$$E_g(u) = \int_M \left(\frac{4(n-1)}{n-2} |\nabla u|^2 + Ru^2 \right) dv_g + 2 \int_\Sigma Hu^2 d\sigma_g,$$

where R is the scalar curvature and H is the mean curvature of the boundary, has the important property of being conformally invariant: if $\widetilde{g} = \phi^{4/(n-2)}g$ is another metric, then $E_{\widetilde{g}}(u) = E_g(u\phi)$. The functional can be written as

$$E_g(u) = \int_M uL_g u \, dv_g + 2 \int_\Sigma \left(\frac{2(n-1)}{n-2} \frac{\partial u}{\partial \nu} + Hu \right) u \, d\sigma_g,$$

where $L_g u = -\frac{4(n-1)}{n-2}\Delta_g u + Ru$ is the conformal Laplacian. The type I Yamabe invariant is defined as

$$Y(M, [g]) = \inf_{u \in H^1(M)\backslash\{0\}} \frac{E_g(u)}{\left(\int_M |u|^{2n/(n-2)} \, dv_g \right)^{(n-2)/n}}.$$

The type I Yamabe problem is whether the infimum is always achieved. It is proved that $Y(M, [g]) \leq Y(\mathbb{S}^n_+)$ and moreover the infimum is achieved if the inequality is strict. The strategy to solve the type I Yamabe problem is to show that the strict inequality $Y(M, [g]) < Y(\mathbb{S}^n_+)$ is always true unless $(M, [g])$ is conformally diffeomorphic to $\mathbb{S}^n_+$. It has been confirmed in many cases (see [E1] and [BC]), but some exceptional cases remain open.

The type II Yamabe invariant is defined as

$$Q(M, \Sigma, [g]) = \inf_{u \in H^1(M)\backslash\{0\}} \frac{E_g(u)}{\left(\int_\Sigma |u|^{2(n-1)/(n-2)} \, d\sigma_g \right)^{(n-2)/(n-1)}}.$$

It should be noted that $Q\left(M, \Sigma, [g]\right)$ can be $-\infty$. If $Q\left(M, \Sigma, [g]\right) > -\infty$ and the infimum is achieved, then a minimizer u properly scaled is smooth and positive and the metric $u^{4/(n-2)}g$ then has zero scalar curvature on M and constant mean curvature on Σ. The type II Yamabe problem is whether the infimum is achieved when $Q\left(M, \Sigma, [g]\right) > -\infty$. Parallel to the type I Yamabe problem, it is proved that $Q\left(M, \Sigma, [g]\right) \leq Q\left(\overline{\mathbb{B}^n}, \mathbb{S}^{n-1}\right)$ and moreover the infimum is achieved if the inequality is strict. The strategy to solve the type II Yamabe problem is to show that the strict inequality $Q\left(M, \Sigma, [g]\right) < Q\left(\overline{\mathbb{B}^n}, \mathbb{S}^{n-1}\right)$ is always true unless $(M, [g])$ is conformally diffeomorphic to $\overline{\mathbb{B}^n}$. It has been confirmed in various cases (see [E2],[M1] and [M2]). But there are still cases that remain open.

Apart from the minimization problem, both $Y\left(M, [g]\right)$ and $Q\left(M, \Sigma, [g]\right)$ are important invariants and it is useful to have lower estimates for them. Let (X^n, g_+) be a Poincaré–Einstein manifold and $\Sigma = \partial \overline{X}$. We pick a fixed defining function r on $\overline{X}$ which gives rise to a metric $\overline{g} = r^2 g_+$ on $\overline{X}$. As $[\overline{g}]$ and $[\overline{g}|_\Sigma]$ are invariantly defined, the Yamabe invariants $Y\left(\overline{X}, [\overline{g}]\right), Q\left(\overline{X}, \Sigma, [\overline{g}]\right)$ and $Y\left(\Sigma, [\overline{g}|_\Sigma]\right)$ are natural invariants of (X^n, g_+). X. Chen, M. Lai and F. Wang proved the following elegant inequality relating these two Yamabe invariants.

Theorem 1. *(Chen-Lai-Wang* [CLW]*) Let (X^n, g_+) be a Poincaré–Einstein manifold s.t. If the type II Yamabe problem on $\left(\overline{X}, \overline{g}\right)$ has a minimizing solution, then*

$$Y\left(\Sigma, [\overline{g}|_\Sigma]\right) \leq \frac{n-2}{4\left(n-1\right)} Q\left(\overline{X}, \Sigma, [\overline{g}]\right)^2, \ \ if \ n \geq 4;$$

$$32\pi\chi\left(\Sigma\right) \leq Q\left(\overline{X}, \Sigma, [\overline{g}]\right)^2, \ \ if \ n = 3.$$

Moreover, the equality holds if and only if (X^n, g_+) is isometric to the hyperbolic space $(\mathbb{H}^n, g_{\mathbb{H}})$.

In our previous work [WW], we removed the restriction in Theorem 1 and proved that the inequality is valid for all Poincaré–Einstein manifolds. Since the inequality is vacuous when $Y\left(\partial X, [g]\right) \leq 0$, we prefer to state the result in the following way.

Theorem 2. *Let (X^n, g_+) be a Poincaré–Einstein manifold whose conformal infinity has nonnegative Yamabe invariant. Then*

$$Q\left(\overline{X}, \Sigma, [\overline{g}]\right) \geq 2\sqrt{\frac{(n-1)}{(n-2)} Y\left(\Sigma, [\overline{g}|_\Sigma]\right)} \ if \ n \geq 4;$$

$$Q\left(\overline{X}, \Sigma, [\overline{g}]\right) \geq 4\sqrt{2\pi\chi\left(\Sigma\right)} \ if \ n = 3.$$

Moreover, the equality holds iff (X^n, g_+) is isometric to the hyperbolic space $(\mathbb{H}^n, g_{\mathbb{H}})$.

In this paper we prove that the same inequality holds in a much broader context. It suffices for (X^n, g_+) to have Ricci curvature bounded from below $Ric\left(g_+\right) \geq -\left(n-1\right)g_+$ and satisfy an asymptotic condition near infinity. This seems to us to be the natural setting for the inequality and it fits well within the general framework of understanding the boundary effect under a Ricci curvature lower bound. We now explain the asymptotic condition precisely. Let (X^n, g_+) be a conformally compact manifold. As usual, we pick a fixed defining function r on $\overline{X}$ which gives rise to a metric $\overline{g} = r^2 g_+$ on $\overline{X}$. We say that (X^n, g_+) is asymptotically Poincare-Einstein if

$$Ric\left(g_+\right) + (n-1)g_+ = o\left(r^2\right).$$

We can now state our main result.

Theorem 3. *Let* (X^n, g_+) *be a conformally compact manifold whose conformal infinity has nonnegative Yamabe invariant. If* $Ric(g_+) \geq -(n-1)g_+$ *and* (X^n, g_+) *is asymptotically Poincare-Einstein, then*

$$Q\left(\overline{X}, \Sigma, [\overline{g}]\right) \geq 2\sqrt{\frac{(n-1)}{(n-2)}Y\left(\Sigma, [\overline{g}|_\Sigma]\right)} \text{ if } n \geq 4;$$

$$Q\left(\overline{X}, \Sigma, [\overline{g}]\right) \geq 4\sqrt{2\pi\chi\left(\Sigma\right)} \text{ if } n = 3.$$

Moreover, the equality holds iff (X^n, g_+) *is isometric to the hyperbolic space* $(\mathbb{H}^n, g_{\mathbb{H}})$.

Remark 1. *When* $Y\left(\Sigma, [\overline{g}|_\Sigma]\right) = Y\left(\mathbb{S}^{n-1}\right) = (n-1)(n-2)\omega_{n-1}^{2/(n-1)}$, *here* ω_{n-1} *is the volume of* $\mathbb{S}^{n-1}$, *the right hand side then equals* $2(n-1)\omega_{n-1}^{1/(n-1)} = Q\left(\overline{\mathbb{B}^n}, \mathbb{S}^{n-1}\right)$. *Thus in this case we must have equality and hence rigidity. This rigidity results was proved by* [DJ] *and* [LQS]. *Therefore our ineqaulity can be viewd as a quantative version of their rigidity result: when the conformal infinity is closed to* $\mathbb{S}^{n-1}$ *in terms of the Yamabe invariant,* $\left(\overline{X}, [\overline{g}]\right)$ *is close to the ball* $\overline{\mathbb{B}^n}$ *in terms of the type II Yamabe invariant.*

The method in [CLW] is based on ideas introduced in Gursky-Han [GH] in which they studied the type I Yamabe invariant on $\overline{X}$. Let $g \in [\overline{g}]$ be a type II Yamabe minimizer and write $g_+ = \rho^{-2}g$. The following identity plays an important role in the proof of Theorem 1 as well as Theorem 2

$$T_+ = T + (n-2)\rho^{-1}\left(D^2\rho - \frac{\Delta\rho}{n}g\right),$$

where T_+ and T are the traceless Ricci tensor of g_+ and g, respectively. As g_+ is Einstein, $T_+ = 0$ and hence

$$\frac{1}{n-2}\rho T = -\left(D^2\rho - \frac{\Delta\rho}{n}g\right).$$

By an integration by part over $X_\varepsilon = \{r \geq \varepsilon\}$, using the fact that g has constant scalar curvature, we obtain

$$\frac{1}{n-2}\int_{X_\varepsilon} \rho\,|T|^2\,dv_g = -\int_{\partial X_\varepsilon} T\left(\nabla\rho, \nu\right)d\sigma_g.$$

The rest of the proof is to analyze the limit of the boundary term as $\varepsilon \to 0$.

When g_+ is not Einstein, the above approach breaks down at the beginning. Instead, we study a modified Yamabe problem which produces a positive function u satisfying the equation

$$-\Delta_{g_+}u = \frac{n(n-2)}{4}u.$$

Write $u = v^{-(n-2)/2}$ and set $\Phi = v^{-1}\left(|\nabla v|^2 - v^2\right)$. The following calculation is crucial for our proof

$$\text{div}\left(v^{-(n-2)}\nabla\Phi\right) = 2v^{-(n-2)}Q,$$

where

$$Q = \left|D^2v - \frac{\Delta v}{n}g_+\right|^2 + Ric\left(\nabla v, \nabla v\right) + (n-1)|\nabla v|^2.$$

We then integrate the above identity over X_ε. The analysis of the boundary term follows the same strategy in [CLW].

The paper is organized as follows. In Section 2 we discuss some background material. In Section 3, we study a modified Yamabe problem and estimate the corresponding invariants. As a corollary we prove Theorem 3. We discuss the related problem on compact manifolds in the last Section.

2. Preliminaries

Throughout this paper (X^n, g_+) is asymptotically hyperbolic of order $C^{m,\alpha}$: if r is smooth defining function on $\overline{X}$, the metric $\overline{g} = r^2 g$ extends to a $C^{m,\alpha}$ metric on $\overline{X}$ and $|d\rho|_{\overline{g}}^2 = 1$ along $\Sigma := \partial\overline{X}$. For all the analysis it suffices to have $m \geq 4$. We also assume

$$Ric(g_+) \geq -(n-1)g_+$$

and that g_+ is asymptotically Poincare-Einstein in the following sense

$$Ric(g_+) + (n-1)g_+ = o\left(r^2\right).$$

Let $h \in [\overline{g}|_\Sigma]$ be a metric on Σ. It is proved in [Lee] that there is a defining function r s.t. in a collar neighborhood of Σ

$$(2.1) \qquad\qquad g_+ = r^{-2}\left(dr^2 + h_r\right),$$

where h_r is an r-dependent family of metrics on $\partial\overline{X}$ with $h_r|_{r=0} = h$. Moreover we have the following expansion (see, e.g. [GW])

$$h_r = h + h_2 r^2 + o\left(r^2\right),$$

where

$$h_2 = \begin{cases} -\frac{1}{n-3}\left(Ric(h) - \frac{R_h}{2(n-2)}h\right), & \text{if } n \geq 4; \\ -\frac{1}{4}h, & \text{if } n = 3. \end{cases}$$

It follows that $\overline{g} = r^2 g_+$ has totally geodesic boundary. As we assume $Y\left(\Sigma, [\overline{g}|_\Sigma]\right) \geq 0$, we choose h to have scalar curvature $R_h \geq 0$.

Lee [Lee] constructed a positive smooth function ϕ on X s.t. $\Delta\phi = n\phi$ and near $\partial\overline{X}$

$$\phi = r^{-1} + \frac{R_h}{4(n-1)(n-2)}r + o\left(r^2\right).$$

Under the condition $R_h \geq 0$, he further proved that $|d\phi|_{g_+}^2 \leq \phi^2$. Consider the metric $\widetilde{g} := \phi^{-2}g_+$ on $\overline{X}$. Its scalar curvature is given by

$$(2.2) \qquad \widetilde{R} = \phi^2\left(R + 2(n-1)\phi^{-1}\Delta\phi - n(n-1)\phi^{-2}|d\phi|^2\right)$$

$$\geq \phi^2\left(R + n(n-1)\right).$$

Moreover, by a direct calculation the boundary is totally geodesic. We consider the following modified energy functional

$$\widetilde{E}(f) = E_{\overline{g}}(f) - \int_X \left(R_+ + n(n-1)\right)\phi^2 f^2 dv_{\overline{g}}.$$

Note that $(R + n(n-1))\phi^2 \in C^{m-3,\alpha}\left(\overline{X}\right)$ under our assumptions. More explicitly, by (2.2)

$$\widetilde{E}(f) = \int_X \left[\frac{4(n-1)}{n-2}|df|_{\widetilde{g}}^2 + \left(\widetilde{R} - (R + n(n-1))\phi^2\right)f^2\right] dv_{\overline{g}} \geq 0.$$

Since $R_+ + n(n-1) \geq 0$, we have

$$(2.3) \qquad\qquad E_{\overline{g}}(f) \geq \widetilde{E}(f).$$

3. Estimate on modified Yamabe Quotients

For $1 < q \leq n/(n-2)$, consider

$$\widetilde{\lambda}_q := \inf \frac{\widetilde{E}(f)}{\left(\int_\Sigma |f|^{q+1} \, d\sigma_{\overline{g}}\right)^{2/(q+1)}}.$$

Theorem 4. *Let (X^n, g_+) be a Poincaré–Einstein manifold whose conformal infinity has positive Yamabe invariant. For $1 < q \leq n/(n-2)$ the invariant λ_q satisfies*

$$\widetilde{\lambda}_q \geq 2\sqrt{\frac{(n-1)}{(n-2)}} Y\left(\Sigma, [\overline{g}|_\Sigma]\right) V\left(\Sigma, \overline{g}\right)^{-\frac{(n-q(n-2))}{(n-3)(q+1)}} \quad if\ n \geq 4;$$

$$\widetilde{\lambda}_q \geq 4\sqrt{2\pi\chi(\Sigma)} V\left(\Sigma, \overline{g}\right)^{-\frac{3-q}{2(q+1)}} \quad if\ n = 3.$$

Since $\widetilde{E}(f) \geq 0$, it is easy to see that $\lim_{q \nearrow n/(n-2)} \widetilde{\lambda}_q = \widetilde{\lambda}_{n/(n-2)}$. Therefore, it suffices to prove the above theorem for $q < n/(n-2)$.

Since the trace operator $H^1(\overline{X}) \to L^{q+1}(\Sigma)$ is compact for $q < n/(n-2)$, by standard elliptic theory, the above infimum λ_q is achieved by a smooth, positive function f s.t.

$$(3.1) \qquad\qquad \int_\Sigma f^{q+1} d\overline{\sigma} = 1$$

and

$$(3.2) \qquad \begin{cases} -\frac{4(n-1)}{n-2}\overline{\Delta}f + \overline{R}f = (R + n(n-1))\phi^2 f & \text{on } \overline{X}, \\ \frac{4(n-1)}{n-2}\frac{\partial f}{\partial \nu} = \lambda_q f^q & \text{on } \Sigma. \end{cases}$$

By the conformal invariance of the conformal Laplacian, we have

$$L_g\left(f\phi^{-(n-2)/2}\right) = \phi^{-(n+2)/2} L_{\overline{g}}(f)$$

$$= (R + n(n-1)) f\phi^{-(n-2)/2}.$$

In other words, $u := f\phi^{-(n-2)/2}$ satisfies the following equation

$$(3.3) \qquad\qquad -\Delta_{g_+} u = \frac{n(n-2)}{4}u.$$

Write $u = v^{-(n-2)/2}$. Then

$$\Delta_{g_+} v = \frac{n}{2}v^{-1}\left(|dv|^2_{g_+} + v^2\right).$$

Equivalently $\Delta_{g_+} v - nv = \frac{n}{2}\Phi$ with $\Phi = v^{-1}\left(|dv|^2_{g_+} - v^2\right)$.

Lemma 1. *We have*

$$(3.4) \qquad\qquad \mathrm{div}\left(v^{-(n-2)}\nabla\Phi\right) = 2v^{-(n-2)}Q,$$

where

$$Q = \left|D^2 v - \frac{\Delta v}{n}g_+\right|^2 + Ric(\nabla v, \nabla v) + (n-1)|\nabla v|^2 \geq 0.$$

All the computation is done with respect to g_+, but we drop the subscript to simplify the presentation.

Proof. As $v\Phi = |\nabla v|^2 - v^2$, we have, by using the Bochner formula

$$\frac{1}{2}\left(v\Delta\phi + 2\langle\nabla v, \nabla\phi\rangle + \phi\Delta v\right) = \left|D^2 v\right|^2 + \langle\nabla v, \nabla\Delta v\rangle + Ric\left(\nabla v, \nabla v\right) - v\Delta v - |\nabla v|^2$$

$$= \frac{(\Delta v)^2}{n} + \langle\nabla v, \nabla\Delta v\rangle + v\Delta v - n|\nabla v|^2 + Q$$

$$= \frac{\Delta v}{n}\left(\Delta v - nv\right) + \langle\nabla v, \nabla\left(\Delta v - nv\right)\rangle + Q$$

$$= \frac{1}{2}\Phi\Delta v + \frac{n}{2}\langle\nabla v, \nabla\Phi\rangle + Q$$

Thus,

$$\Delta\Phi = (n-2)v^{-1}\langle\nabla v, \nabla\Phi\rangle + 2Q$$

or

$$\text{div}\left(v^{-(n-2)}\nabla\Phi\right) = 2v^{-(n-2)}Q.$$

$\square$

We now consider the metric $g = u^{4/(n-2)}g_+$. Since $u = f\phi^{-(n-2)/2}$, we also have

$$g = f^{4/(n-2)}\phi^{-2}g_+ = f^{4/(n-2)}\widetilde{g}.$$

As $\partial\overline{X}$ is totally geodesic w.r.t. $\widetilde{g}$ and g is conformal to $\widetilde{g}$, we know that $\partial\overline{X}$ is umbilic w.r.t. g and its mean curvature, in view of the boundary condition of (3.2), is given by

$$(3.5) \qquad\qquad H = \frac{\lambda_q}{2}f^{q-\frac{n}{n-2}}.$$

Set $\rho = u^{2/(n-2)} = v^{-1}$. By a direct calculation, the equation (3.3) becomes, using g as the background metric

$$(3.6) \qquad\qquad 2\rho\Delta\rho = n\left(|\nabla\rho|^2 - 1\right).$$

Let t be the geodesic distance to Σ w.r.t. g. We need the following lemma which is essentially contained in [CLW].

Lemma 2. *Near $\Sigma = \partial\overline{X}$, we can write*

$$g = dt^2 + g_{ij}\left(t, x\right)dx_i dx_j,$$

where $\{x_1, \cdots, x_{n-1}\}$ are local coordinates on Σ. Then

$$\rho = t - \frac{H}{2(n-1)}t^2 + \frac{1}{6}\left(\frac{R^\Sigma}{n-2} - \frac{H^2}{n-1}\right)t^3 + o\left(t^3\right).$$

In particular,

$$\frac{\partial}{\partial\nu}\left[\rho^{-1}\left(|\nabla\rho|^2 - 1\right)\right]|_\Sigma = \frac{R^\Sigma}{n-2} - \frac{H^2}{n-1}.$$

Proof. For completeness, we present the proof showing that the Einstein condition is not required. In local coordinates

$$|\nabla \rho|^2 = \left(\frac{\partial \rho}{\partial t}\right)^2 + g^{ij} \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j},$$

$$\Delta \rho = \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial \log \sqrt{G}}{\partial t} \frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{G}} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{G} \frac{\partial \rho}{\partial x_j}\right).$$

Restricting (3.6) on Σ on which both ρ and r vanish with order 1 yields $\frac{\partial \rho}{\partial t}|_\Sigma = 1$.

Differentiating (3.6) in t yields

$$(3.7) \qquad \frac{2}{n}\left(\frac{\partial \rho}{\partial t} \Delta \rho + \rho \frac{\partial}{\partial t} \Delta \rho\right) = 2\frac{\partial \rho}{\partial t}\frac{\partial^2 \rho}{\partial t^2} + 2g^{ij}\frac{\partial^2 \rho}{\partial x_i \partial t}\frac{\partial \rho}{\partial x_j} - g^{ik}g^{jl}\frac{\partial g_{kl}}{\partial t}\frac{\partial \rho}{\partial x_i}\frac{\partial \rho}{\partial x_j}.$$

Evaluating both sides on Σ yields

$$\frac{2}{n}\left(\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial \log \sqrt{G}}{\partial t}\right)|_\Sigma = 2\frac{\partial^2 \rho}{\partial t^2}|_\Sigma.$$

Thus

$$\frac{\partial^2 \rho}{\partial t^2}|_\Sigma = \frac{1}{n-1}\frac{\partial \log \sqrt{G}}{\partial t}|_\Sigma = -\frac{H}{n-1}.$$

Differentiating the formula for $\Delta \rho$ we get

$$\frac{\partial}{\partial t}\Delta \rho|_\Sigma = \left(\frac{\partial^3 \rho}{\partial t^3} + \frac{\partial^2 \log \sqrt{G}}{\partial t^2} + \frac{\partial \log \sqrt{G}}{\partial t}\frac{\partial^2 \rho}{\partial t^2}\right)|_\Sigma$$

$$= \left(\frac{\partial^3 \rho}{\partial t^3} + \frac{\partial^2 \log \sqrt{G}}{\partial t^2} + \frac{H^2}{n-1}\right)|_\Sigma$$

Differentiating (3.7) in r and evaluating on Σ, we obtain

$$\frac{2}{n}\left(\frac{\partial^2 \rho}{\partial t^2}\Delta \rho + 2\frac{\partial}{\partial t}\Delta \rho\right)|_\Sigma = 2\left(\frac{\partial^2 \rho}{\partial t^2}\right)^2|_\Sigma + 2\frac{\partial^3 \rho}{\partial t^3}|_\Sigma = \frac{2H^2}{(n-1)^2} + 2\frac{\partial^3 \rho}{\partial t^3}|_\Sigma.$$

Using the previous formulas, we arrive at

$$\frac{\partial^3 \rho}{\partial t^3}|_\Sigma = \frac{2}{n-2}\left(\frac{H^2}{n-1} + \frac{\partial^2 \log \sqrt{G}}{\partial t^2}|_\Sigma\right).$$

By a direct calculation, we also have

$$\frac{\partial^2 \log \sqrt{G}}{\partial t^2}|_\Sigma = -Ric\,(\nu,\nu) - \frac{H^2}{n-1}.$$

Therefore

$$\frac{\partial^3 \rho}{\partial t^3}|_\Sigma = -\frac{2}{n-2}Ric\,(\nu,\nu)$$

$$= \frac{R^\Sigma}{n-2} - \frac{H^2}{n-1},$$

where we used the Gauss equation in the last step.

The second identity follows from a direct calculation. $\qquad\square$

We can now prove Theorem 4. Integrating the identity (3.4) on $X_\varepsilon = \{t \geq \varepsilon\}$ yields

$$2 \int_{X_\varepsilon} v^{-(n-2)} Q dv_{g_+} = \int_{\partial X_\varepsilon} v^{-(n-2)} \frac{\partial \Phi}{\partial \nu} d\sigma_{g_+}.$$

Since $g_+ = \rho^{-2} g$, we obtain by a direct calculation

$$\int_{\partial X_\varepsilon} v^{-(n-2)} \frac{\partial \Phi}{\partial \nu+} d\sigma_{g_+} = \int_{\partial X_\varepsilon} \frac{\partial}{\partial \nu} \left[\rho^{-1} \left(|\nabla \rho|^2 - 1 \right) \right] d\sigma_g.$$

Therefore

$$2 \int_{X_\varepsilon} v^{-(n-2)} Q dv_{g_+} = \int_{\partial X_\varepsilon} \frac{\partial}{\partial \nu} \left[\rho^{-1} \left(|\nabla \rho|^2 - 1 \right) \right] d\sigma_g.$$

Letting $\varepsilon \to 0$, we obtain, in view of Lemma 2

$$(3.8) \qquad 2 \int_X v^{-(n-2)} Q dv_{g_+} = \int_\Sigma \left(\frac{R^\Sigma}{n-2} - \frac{H^2}{n-1} \right) d\sigma_g$$

The rest of the argument is the same as in [WW]. We present it for completeness. By (3.5) and the Holder inequality again

$$\int_\Sigma H^2 d\sigma = \left(\frac{\lambda_q}{2} \right)^2 \int_\Sigma f^{2\left(q - \frac{n}{n-2}\right)} f^{2(n-1)/(n-2)} d\overline{\sigma}$$

$$= \left(\frac{\lambda_q}{2} \right)^2 \int_\Sigma f^{2\left(q - \frac{1}{n-2}\right)} d\overline{\sigma}$$

$$\leq \left(\frac{\lambda_q}{2} \right)^2 \left(\int_\Sigma f^{q+1} d\overline{\sigma} \right)^{2\left(q - \frac{1}{n-2}\right)/(q+1)} V\left(\Sigma, \overline{g}\right)^{\left(\frac{n}{n-2} - q\right)/(q+1)}$$

$$= \left(\frac{\lambda_q}{2} \right)^2 V\left(\Sigma, \overline{g}\right)^{\left(\frac{n}{n-2} - q\right)/(q+1)}.$$

Plugging the above inequality into (3.8), we obtain

$$(3.9) \quad 2 \int_{X_\varepsilon} v^{-(n-2)} Q dv_{g_+} \leq \frac{\lambda_q^2}{4(n-1)} V\left(\Sigma, \overline{g}\right)^{\left(\frac{n}{n-2} - q\right)/(q+1)} - \frac{1}{n-2} \int_\Sigma R^\Sigma d\sigma.$$

When $n = 3$, this implies

$$\lambda_q^2 V\left(\Sigma, \overline{g}\right)^{(3-q)/(q+1)} \geq 32\pi \chi\left(\Sigma\right).$$

In the following, we assume $n > 3$. By (3.1) and the Holder inequality

$$1 = \int_\Sigma f^{q+1} d\overline{\sigma}$$

$$\leq \left(\int_\Sigma f^{2(n-1)/(n-2)} d\overline{\sigma} \right)^{\frac{(q+1)(n-2)}{2(n-1)}} V\left(\Sigma, \overline{g}\right)^{\frac{n-q(n-2)}{2(n-1)}}$$

$$= V\left(\Sigma, g\right)^{\frac{(q+1)(n-2)}{2(n-1)}} V\left(\Sigma, \overline{g}\right)^{\frac{n-q(n-2)}{2(n-1)}}$$

Thus

$$V\left(\Sigma, \overline{g}\right)^{-\frac{n-q(n-2)}{(n-2)(q+1)}} \leq V\left(\Sigma, g\right).$$

Plugging this inequality into (3.9) yields

$$2 \int_{X_\varepsilon} v^{-(n-2)} Q dv_{g_+}$$

$$\leq \frac{V(\Sigma, g)^{\frac{n-1}{n-3}}}{4(n-1)} \left[\widetilde{\lambda}_q^2 V(\Sigma, \overline{g})^{\frac{2(n-q(n-2))}{(n-3)(q+1)}} - \frac{4(n-1)}{(n-2) V(\Sigma, g)^{\frac{n-1}{n-3}}} \int_\Sigma R^\Sigma d\sigma \right]$$

$$\leq \frac{V(\Sigma, g)^{\frac{n-1}{n-3}}}{4(n-1)} \left[\widetilde{\lambda}_q^2 V(\Sigma, \overline{g})^{\frac{2(n-q(n-2))}{(n-3)(q+1)}} - \frac{4(n-1)}{(n-2)} Y(\Sigma, [\gamma]) \right].$$

Therefore

$$\widetilde{\lambda}_q^2 \geq \frac{4(n-1)}{(n-2)} Y(\Sigma) V(\Sigma, \overline{g})^{-\frac{2(n-q(n-2))}{(n-3)(q+1)}}.$$

This finishes the proof of Theorem 4

We are now ready to prove our main result.

Theorem 5. *Let (X^n, g_+) be a conformally compact manifold whose conformal infinity has nonnegative Yamabe invariant. If $Ric(g_+) \geq -(n-1) g_+$ and (X^n, g_+) is asymptotically Poincare-Einstein, then*

$$Q(\overline{X}, \Sigma, [\overline{g}]) \geq 2 \sqrt{\frac{(n-1)}{(n-2)} Y(\Sigma, [\overline{g}|_\Sigma])} \ \text{if} \ n \geq 4;$$

$$Q(\overline{X}, \Sigma, [\overline{g}]) \geq 4 \sqrt{2\pi \chi(\Sigma)} \ \text{if} \ n = 3.$$

Moreover, the equality holds iff (X^n, g_+) is isometric to the hyperbolic space $(\mathbb{H}^n, g_\mathbb{H})$.

Proof. By (2.3), we have $Q(\overline{X}, \Sigma, [\overline{g}]) \geq \widetilde{\lambda}_{n/(n-2)}$. Therefore the inequality follows immediately from Theorem 4.

Suppose the equality holds. We present the argument for $n \geq 4$ and the same argument works for $n = 3$ with trivial modification. If $Y(\Sigma, [\overline{g}|_\Sigma]) < Y(\mathbb{S}^{n-1})$, the equality then implies

$$Q(\overline{X}, \Sigma, [\overline{g}]) = \widetilde{\lambda}_{n/(n-2)} < Q(\overline{\mathbb{B}^n}, \mathbb{S}^{n-1}).$$

Just like in the original Yamabe problem, this strict inequality implies that $\widetilde{\lambda}_{n/(n-2)}$ is achieved. Therefore in the proof of Theorem 4, we can take $q = n/(n-2)$ and obtain

$$2 \int_{X_\varepsilon} v^{-(n-2)} Q dv_{g_+} \leq \frac{V(\Sigma, g)^{\frac{n-1}{n-3}}}{4(n-1)} \left[\widetilde{\lambda}_{n/(n-2)}^2 - \frac{4(n-1)}{(n-2)} Y(\Sigma, [\gamma]) \right] = 0.$$

Thus $Q = 0$. In particular, $v > 0$ satisfies the over-determined system

$$D^2 v = \frac{\Delta v}{n} g_+.$$

This implies that (X^n, g_+) is isometric to the hyperbolic space (cf. [CLW] for the argument).

If $Y(\Sigma, [\overline{g}|_\Sigma]) = Y(\mathbb{S}^{n-1})$, then $(\Sigma, [\overline{g}|_\Sigma])$ is conformally equivalent to $\mathbb{S}^{n-1}$, by the solution of the Yamabe problem for closed manifolds. Then (X^n, g_+) is isometric to the hyperbolic space $(\mathbb{H}^n, g_\mathbb{H})$ by [DJ] and [LQS]. $\square$

4. Some Discussions on Compact Manifolds with Boundary

It is a natural question if the inequality in Theorem 3 holds for a compact Riemannian manifold (M^n, g) with $Ric \geq -(n-1)$ and $\Pi \geq 1$. We are motivated by the observation that some results for conformally compact manifolds follow from results for compact Riemannian manifolds by a limiting process. As an illustration, consider the following theorem by Lee.

Theorem 6. *(Lee [Lee]) Let (X^n, g_+) be a conformally compact manifold whose conformal infinity has nonnegative Yamabe invariant. If $Ric(g_+) \geq -(n-1)g_+$ and (X^n, g_+) is asymptotically Poincare-Einstein, then the bottom of spectrum $\lambda_0(X^n, g_+) = (n-1)^2/4$.*

When the Yambabe invariant of the conformal infinity is positive, Lee's theorem follows from the following result for compact Riemannian manifolds.

Theorem 7. *Let (M^n, g) be a compact Riemannian manifold with $Ric \geq -(n-1)$. If along the boundary $\Sigma := \partial M$ we have the mean curvature $H \geq n-1$, then the first Dirichlet eigenvalue*

$$\lambda_0(M) \geq \frac{(n-1)^2}{4}.$$

This theorem has a simple proof. Let r be the distance function to Σ. By standard method in Riemannian geometry, we have

$$\Delta r \leq -(n-1)$$

in the support sense. A direct calculation yields

$$\Delta e^{(n-1)r/2} \leq -\frac{(n-1)^2}{4}e^{(n-1)r/2}.$$

This implies $\lambda_0(M) \geq \frac{(n-1)^2}{4}$ (for technical details see [Wa1]).

We can deduce Lee's theorem from Theorem 7 when the conformal infinity has positive Yamabe invariant in the following way. As explained in Section 2, we pick a metric h on the conformal infinity with positive scalar curvature and then we have a good defining function r s.t. near the conformal infinity g_+ has a nice expansion (2.1). Then a simple calculation shows that the mean curvature of the boundary of $X_\varepsilon := \{r \geq \varepsilon\}$ satisfies

$$H = n - 1 + \frac{R_h}{2(n-2)}\varepsilon^2 + o(\varepsilon^2).$$

As $R_h > 0$, we have $H > n-1$ if ε is small enough. By Theorem, $\lambda_0(X_\varepsilon) \geq \frac{(n-1)^2}{4}$. It follows that $\lambda_0(X) \geq \frac{(n-1)^2}{4}$. As the opposite inequality was known by [Ma], we have $\lambda_0(X) = \frac{(n-1)^2}{4}$. When the conformal infinity has zero Yamabe invariant, the situation is more subtle. But by an idea in Cai-Galloway[CG], a similar argument still works (cf. [Wa1]).

We now come back to Theorem 3. By the asymptotic expansion (2.1) the second fundamental form of ∂X_ε satisfies

$$\Pi_+ = (1 + O(\varepsilon))g_+,$$

i.e. all the principal curvatures are close to 1. This leads us to consider a compact Riemannian manifold (M^n, g) with $Ric \geq -(n-1)$ and $\Pi \geq 1$ on its boundary Σ

and ask the question whether the inequality

$$(4.1) \qquad Q\left(M, \Sigma, g\right) \geq 2\sqrt{\frac{(n-1)}{(n-2)} Y\left(\Sigma\right)} \text{ if } n \geq 4;$$

$$Q\left(M, \Sigma, g\right) \geq 4\sqrt{2\pi\chi\left(\Sigma\right)} \text{ if } n = 3$$

holds. The answer turns out to be no in general. To construct a counter example, we consider the hyperbolic space using the ball model $\mathbb{B}^n$ with the metric $g_{\mathbb{H}} = \frac{4}{\left(1-|x|^2\right)^2} dx^2$. For $0 < R < 1$, the Euclidean ball

$$\left\{ x \in \mathbb{B}^n : |x|^2 = \sum_{i=1}^{n} x_i^2 \leq R \right\}$$

is a geodesic ball in $\left(\mathbb{B}^n, g_{\mathbb{H}}\right)$ and the boundary has 2nd fundamental form $\Pi = \frac{1+R^2}{2R} I$. We now consider

$$M = \left\{ x \in \mathbb{B}^n : |x|^2 = \sum_{i=1}^{n-1} x_i^2 + k x_n^2 \leq R \right\},$$

where $k > 0$ is close to 1. Then $(M, g_{\mathbb{H}})$ is a compact hyperbolic manifold with boundary and on its boundary we have $\Pi \geq 1$ if k is sufficiently close to 1 by continuity. Since Σ with the induced metric is rotationally symmetric, it is conformally equivalent to the standard sphere $\mathbb{S}^{n-1}$. Thus, $Y\left(\Sigma\right) = Y\left(\mathbb{S}^{n-1}\right)$. But when $k \neq 1$, the boundary is not umbilic with respect to the Euclidean metric and hence not with respect to $g_{\mathbb{H}}$ either. By [E2] and [M2], $Q\left(M, \Sigma, g_{\mathbb{H}}\right) < Q\left(\overline{\mathbb{B}^n}, \mathbb{S}^{n-1}\right)$. It follows that the inequality (4.1) is false.

Therefore, for a compact Riemannian manifold (M^n, g) with $Ric \geq -(n-1)$ and $\Pi \geq 1$ on its boundary Σ, it is more subtle to estimate its type II Yamabe invariant in terms of the boundary geometry. It is an interesting question and we do not have an explicit conjecture. Let us mention that in a similar setting, namely for a compact (M^n, g) with $Ric \geq 0$ and $\Pi \geq 1$ on its boundary Σ, there is a well-formulated conjecture [Wa2] on the type II Yamabe invariant in terms of the boundary area.

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