ON A SHARP INEQUALITY RELATING YAMABE INVARIANTS ON A POINCARE-EINSTEIN MANIFOLD

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ABSTRACT. For a Poincare-Einstein manifold under certain restrictions, X. Chen, M. Lai and F. Wang [CLW] proved a sharp inequality relating Yamabe invariants. We show that the inequality is true without any restriction.

A complete Riemannian manifold (X^n, g_+) is called conformally compact if X is the interior of a compact manifold $\overline{X}$ with nonempty boundary $\Sigma = \partial\overline{X}$ and for any defining function r on $\overline{X}$ the metric $\overline{g} := r^2 g_+$ extends to a smooth metric on $\overline{X}$ (we gloss over the boundary regularity of $\overline{g}$ as it does not play an essential role in our discussion). In particular, $\overline{g}|_\Sigma$ defines a Riemannian metric on Σ. Though a different choice of r gives rise to a different $\overline{g}$, the conformal classes $[\overline{g}]$ and $[\overline{g}|_\Sigma]$ are invariantly defined. $(\Sigma, [\overline{g}|_\Sigma])$ is called the conformal infinity of (X^n, g_+). If furthermore $Ric\,(g_+) = -\,(n-1)\,g_+$, then (X^n, g_+) is called Poincaré-Einstein (or conformally compact Einstein). Poincaré-Einstein manifolds have been studied intensively since the work of Graham-Lee [GL]. More recently the subject has gained new momentum from string theory or more specifically the AdS/CFT correspondence in which such manifolds serve as the framework for a connection between supergravity and conformal field theory.

A guiding principle for studying Poincaré-Einstein manifolds is to understand the interaction between the geometry on X and the conformal geometry on $\partial\overline{X}$. Recently Gursky-Han proved an eminent result of this type which relates the Yamabe invariant of $\left(\overline{X}, [\overline{g}]\right)$ and the Yamabe invariant of $(\Sigma, [\overline{g}|_\Sigma])$. Before stating their result, let us first recall some basics on the Yamabe problem.

On a compact Riemannian manifold (M^n, g) with nonempty boundary $\Sigma = \partial M$, there are in fact two types of Yamabe problems. We first consider the functional

$$E_g\,(u) = \int_M \left(\frac{4\,(n-1)}{n-2}\,|\nabla u|^2 + Ru^2\right) dv_g + 2\int_\Sigma Hu^2 d\sigma_g,$$

where R is the scalar curvature and H is the mean curvature of the boundary. (In our convention H is the trace of the 2nd fundamental form.) This functional has the important property of being conformally invariant: if $\widetilde{g} = \phi^{4/(n-2)} g$ is another metric, then $E_{\widetilde{g}}\,(u) = E_g\,(u\phi)$. The type I Yamabe invariant is defined as

$$Y\,(M, [g]) = \inf_{u \in H^1(M)\setminus\{0\}} \frac{E_g\,(u)}{\left(\int_M |u|^{2n/(n-2)}\,dv_g\right)^{(n-2)/n}}.$$

As the notation indicates, $Y\,(M, [g])$ depends only on the conformal class $[g]$. If the infimum is achieved, then a minimizer u properly scaled is smooth and positive and the metric $u^{4/(n-2)} g$ then has constant scalar curvature on M and zero mean curvature on Σ. The type I Yamabe problem whether the infimum is always achieved has not been completely solved. It has been solved in many cases (see [E1] and [BC]).

The type II Yamabe invariant is defined as

$$Q\left(M,\Sigma,[g]\right)=\inf_{u\in H^{1}(M)\setminus\{0\}}\frac{E_{g}\left(u\right)}{\left(\int_{\Sigma}|u|^{2(n-1)/(n-2)}\,d\sigma_{g}\right)^{(n-2)/(n-1)}}.$$

It should be noted that $Q\left(M,\Sigma,[g]\right)$ can be $-\infty$. If $Q\left(M,\Sigma,[g]\right)>-\infty$ and the infimum is achieved, then a minimizer u properly scaled is smooth and positive and the metric $u^{4/(n-2)}g$ then has zero scalar curvature on M and constant mean curvature on Σ. The type II Yamabe problem whether the infimum is achieved when $Q\left(M,\Sigma,[g]\right)>-\infty$ has been solved in various cases (see [E2],[M1] and [M2]). But there are still cases that remain open. Apart from the minimization problem, both $Y\left(M,[g]\right)$ and $Q\left(M,\Sigma,[g]\right)$ are important invariants for which it is useful to have good estimates.

We now come back to Poincaré–Einstein manifolds. Let (X^{n},g_{+}) be a Poincaré–Einstein manifold and $\Sigma=\partial\overline{X}$. We pick a fixed defining function r on $\overline{X}$ which gives rise to a metric $\overline{g}=r^{2}g_{+}$ on $\overline{X}$. As $[\overline{g}]$ and $[\overline{g}|_{\Sigma}]$ are invariantly defined, the Yamabe invariants $Y\left(\overline{X},[\overline{g}]\right),Q\left(\overline{X},\Sigma,[\overline{g}]\right)$ and $Y\left(\Sigma,[\overline{g}|_{\Sigma}]\right)$ are natural invariants of (X^{n},g_{+}). We can now state the result of Gursky-Han.

Theorem 1. *(Gursky-Han* [GH]*) Suppose $\overline{X}$ satisfies one of the following two conditions*

(1) $3\leq n\leq5$*;*
(2) $n\geq6$ *and X is spin.*

Let $\widetilde{g}$ be a type I Yamabe minimizer in $[\overline{g}]$. Then

$$\frac{n}{n-2}Y\left(\Sigma,[\overline{g}|_{\Sigma}]\right)\leq\frac{n-2}{4\left(n-1\right)}Y\left(\overline{X},[\overline{g}]\right)I^{2},\ \ \textit{if } n\geq4;$$
$$12\pi\chi\left(\Sigma\right)\leq\frac{n-2}{4\left(n-1\right)}Y\left(\overline{X},[\overline{g}]\right)I^{2},\ \ \textit{if } n=3,$$

where I is the isoperimetric ratio for $\widetilde{g}$ (cf. [GH] *for the precise definition). Moreover, if the equality holds, then $\widetilde{g}$ is Einstein and $\widetilde{g}|_{\Sigma}$ has constant scalar curvature.*

Modifying Gursky-Han's method, X. Chen, M. Lai and F. Wang [CLW] proved the following result for type II Yamabe invariant.

Theorem 2. *(Chen-Lai-Wang* [CLW]*) Let (X^{n},g_{+}) be a Poincaré–Einstein manifold s.t. $\left(\overline{X},\overline{g}\right)$ satisfies one of the conditions*

(a) *the dimension $3\leq n\leq7$;*
(b) *the dimension $n\geq8$ and X is spin;*
(c) *the dimension $n\geq8$ and X is locally conformally flat.*

Then

$$Y\left(\Sigma,[\overline{g}|_{\Sigma}]\right)\leq\frac{n-2}{4\left(n-1\right)}Q\left(\overline{X},\Sigma,[\overline{g}]\right)^{2},\ \ \textit{if } n\geq4;$$
$$32\pi\chi\left(\Sigma\right)\leq Q\left(\overline{X},\Sigma,[\overline{g}]\right)^{2},\ \ \textit{if } n=3.$$

Moreover, the equality holds if and only if (X^{n},g_{+}) is isometric to the hyperbolic space $(\mathbb{H}^{n},g_{\mathbb{H}})$.

The inequality in Theorem 2 is very elegant as both sides are natural invariants. But the proof makes use of a type II Yamabe minimizer which exists under the extra conditions. Due to the fact that the type II Yamabe problem is not completely solved yet, the result is not proved for all Poincaré-Einstein manifolds.

It should be noted that Theorem 2 without any restriction is claimed in [R]. In the exceptional cases not covered by Theorem 2, he applies the same method in [CLW] to a metric in $[\overline{g}]$ with zero scalar curvature on $\overline{X}$ and constant mean curvature on Σ, whose existence is proved in [MN]. But it is clear that this metric is in general just a critical point of the Yamabe functional not a minimizer. As such the argument does not yield any information on the Yamabe invariant. Therefore the proof in [R] for the exceptional cases is invalid.

In this note, we remove the restrictions in Theorem 2 by an indirect route. Note the inequality is vacuous when $Y\left(\Sigma,[\overline{g}|_{\Sigma}]\right)\leq 0$. Therefore we state the result in the following way.

Theorem 3. *Let (X^n,g_+) be a Poincaré–Einstein manifold whose conformal infinity has nonnegative Yamabe invariant. Then*

$$Q\left(\overline{X},\Sigma,[\overline{g}]\right)\geq 2\sqrt{\frac{(n-1)}{(n-2)}Y\left(\Sigma,[\overline{g}|_{\Sigma}]\right)}\ \textit{if } n\geq 4;$$

$$Q\left(\overline{X},\Sigma,[\overline{g}]\right)\geq 4\sqrt{2\pi\chi\left(\Sigma\right)}\ \textit{if } n=3.$$

Moreover, the equality holds iff (X^n,g_+) is isometric to the hyperbolic space $(\mathbb{H}^n,g_{\mathbb{H}})$.

We remark that by [WY] and [CG] Σ is connected when it has nonnegative Yamabe invariant. We first address the equality case. First suppose $n\geq 4$. If $Y\left(\Sigma,[\overline{g}|_{\Sigma}]\right)<Y\left(\mathbb{S}^{n-1},g_{\mathbb{S}^{n-1}}\right)$, then the equality implies

$$Q\left(\overline{X},\Sigma,[\overline{g}]\right)<Q\left(\overline{\mathbb{B}^n},\partial\overline{\mathbb{B}^n},[g_0]\right),$$

the Yamabe invariant of the model case. It is well known that $Q\left(\overline{X},\Sigma,[\overline{g}]\right)$ is then achieved and therefore the rigidity is covered by the original argument in [CLW]. If $Y\left(\Sigma,[\overline{g}|_{\Sigma}]\right)=Y\left(\mathbb{S}^{n-1},g_{\mathbb{S}^{n-1}}\right)$, then by the solution of the Yamabe problem on closed manifolds, $\left(\Sigma,[\overline{g}|_{\Sigma}]\right)$ is conformally diffeomorphic to $\mathbb{S}^{n-1}$. Then (X^n,g_+) is isometric to the hyperbolic space $(\mathbb{H}^n,g_{\mathbb{H}})$ by [DJ] and [LQS]. The same argument works when $n=3$.

To prove the inequality, we first recall some basic facts on Poincaré-Einstein manifolds. Let $h\in[\overline{g}|_{\Sigma}]$ be a metric on Σ. It is proved in [Lee] that there is a defining function r s.t. in a collar neighborhood of Σ

$$g_+=r^{-2}\left(dr^2+h_r\right),$$

where h_r is an r-dependent family of metrics on $\partial\overline{X}$ with $h_r|_{r=0}=h$. Moreover we have the following expansion (see, e.g. [GW])

$$h_r=h+h_2r^2+o\left(r^2\right),$$

where

$$h_2=\begin{cases}-\frac{1}{n-3}\left(Ric\left(h\right)-\frac{R_h}{2(n-2)}h\right), & \text{if } n\geq 4;\\ -\frac{1}{4}h, & \text{if } n=3.\end{cases}$$

It follows that $\overline{g}=r^2g_+$ has totally geodesic boundary. As we assume $Y\left(\Sigma,[\overline{g}|_{\Sigma}]\right)\geq 0$, we can choose h to have $R_h\geq 0$.

Lee [Lee] constructed a positive smooth function ϕ on X s.t. $\Delta\phi = n\phi$ and near $\partial\overline{X}$

$$\phi = r^{-1} + \frac{R_h}{4\,(n-1)\,(n-2)} r + O\left(r^2\right).$$

Under the condition $R_h \geq 0$, he further proved that $|d\phi|^2_{g+} \leq \phi^2$. It was observed by Qing [Q] that $\widetilde{g} := \phi^{-2} g_+$ is then a metric on $\overline{X}$ with nonnegative scalar curvature and totally geodesic boundary and hence $E_{\widetilde{g}}\,(u) \geq 0$. By the conformal invariance, we also have $E_{\overline{g}}\,(u) \geq 0$. For $1 < q \leq n/\,(n-2)$, consider

$$\lambda_q := \inf \frac{E_{\overline{g}}\,(u)}{\left(\int_\Sigma |u|^{q+1}\, d\sigma_{\overline{g}}\right)^{2/(q+1)}}.$$

Notice that $\lambda_q = Q\,(X, \partial X, [\overline{g}])$ when $q = n/\,(n-2)$. As $E_{\overline{g}}\,(u) \geq 0$, it is easy to see that $\lim_{q\to n/(n-2)} \lambda_q = Q\,(X, \partial X, [\overline{g}])$. Therefore Theorem 3 follows from the following

Theorem 4. *Let (X^n, g_+) be a Poincaré–Einstein manifold whose conformal infinity has positive Yamabe invariant. For $1 < q < n/\,(n-2)$ the invariant λ_q satisfies*

$$\lambda_q \geq 2\sqrt{\frac{(n-1)}{(n-2)} Y\left(\Sigma, [\overline{g}|_\Sigma]\right)} V\,(\Sigma, \overline{g})^{-\frac{(n-q(n-2))}{(n-3)(q+1)}} \quad \textit{if } n \geq 4;$$

$$\lambda_q \geq 4\sqrt{2\pi\chi\,(\Sigma)} V\,(\Sigma, \overline{g})^{-\frac{3-q}{2(q+1)}} \quad \textit{if } n = 3.$$

The rest of the note is devoted to the proof of this theorem. We recall the following lemma from [CLW] which follows by a direct calculation.

Lemma 1. *Suppose $g \in [\overline{g}]$ is scalar flat. Near $\Sigma = \partial\overline{X}$, we can write*

$$g = dr^2 + g_{ij}\,(r, x)\, dx_i dx_j,$$

where $\{x_1, \cdots, x_{n-1}\}$ are local coordinates on Σ. If we write $g_+ = \rho^{-2} g$, then

$$\rho = r - \frac{H}{2\,(n-1)} r^2 + \frac{1}{6}\left(\frac{R^\Sigma}{n-2} - \frac{H^2}{n-1}\right) r^3 + o\left(r^3\right).$$

In particular,

$$\rho^{-1}\left(\frac{\partial}{\partial\nu}|\nabla\rho|^2 + \rho^{-1}\left(1 - |\nabla\rho|^2\right)\frac{\partial\rho}{\partial\nu}\right)|_\Sigma = \frac{R^\Sigma}{n-2} - \frac{H^2}{n-1}.$$

Since the trace operator $H^1\left(\overline{X}\right) \to L^{q+1}\,(\Sigma)$ is compact for $q < n/\,(n-2)$, by standard elliptic theory, the above infimum λ_q is achieved by a smooth, positive function u s.t.

$$\int_\Sigma u^{q+1} d\overline{\sigma} = 1 \tag{0.1}$$

and

$$\begin{cases} -\frac{4(n-1)}{n-2}\overline{\Delta} u + \overline{R} u = 0 & \text{on } \overline{X}, \\ \frac{4(n-1)}{n-2}\frac{\partial u}{\partial\overline{\nu}} + 2\overline{H} u = \lambda_q u^q & \text{on } \Sigma. \end{cases}$$

The equations imply that the metric $g = u^{4/(n-2)}\overline{g}$ has zero scalar curvature and the mean curvature on the boundary is given by

$$H = \frac{\lambda_q}{2} u^{q - \frac{n}{n-2}}. \tag{0.2}$$

We now apply the method of [GH] and [CLW] to g. Write $g_+ = \rho^{-2} g$. As g_+ is Einstein, we have

$$E_g = -(n-2)\,\rho^{-1}\left[D^2\rho - \frac{\Delta\rho}{n} g\right].$$

Let r be the distance function to Σ w.r.t. g and denote $X_\varepsilon = \{r \geq \varepsilon\}$. From the above identity, we have

$$\begin{aligned}
\int_{X_\varepsilon} \rho\,|E_g|^2\,dv_g &= -(n-2)\int_{X_\varepsilon}\left\langle D^2\rho - \frac{\Delta\rho}{n} g, E_g\right\rangle dv_g \\
&= -(n-2)\int_{X_\varepsilon}\left\langle D^2\rho, E_g\right\rangle dv_g \\
&= -(n-2)\int_{\partial X_\varepsilon} E\left(\nabla\rho,\nu\right) dv_g \\
&= (n-2)^2\int_{\partial X_\varepsilon}\rho^{-1}\left(D^2\rho\left(\nabla\rho,\nu\right) - \frac{\Delta\rho}{n}\frac{\partial\rho}{\partial\nu}\right) dv_g \\
&= \frac{(n-2)^2}{2}\int_{\partial X_\varepsilon}\rho^{-1}\left(\frac{\partial}{\partial\nu}\left|\nabla\rho\right|^2 - \frac{2\Delta\rho}{n}\frac{\partial\rho}{\partial\nu}\right) dv_g.
\end{aligned}$$

As g has zero scalar curvature and $g_+ = \rho^{-2} g$ has scalar curvature $-n\,(n-1)$, we have

$$-\frac{2}{n}\rho\Delta\rho = 1 - \left|\nabla\rho\right|^2.$$

Thus

$$\int_{X_\varepsilon}\rho\,|E_g|^2\,dv_g = \frac{(n-2)^2}{2}\int_{\partial X_\varepsilon}\rho^{-1}\left(\frac{\partial}{\partial\nu}\left|\nabla\rho\right|^2 + \rho^{-1}\left(1 - \left|\nabla\rho\right|^2\right)\frac{\partial\rho}{\partial\nu}\right) dv_g.$$

Note that $\nu = -\frac{\partial}{\partial r}$ along ∂X_ε. Letting $\varepsilon \to 0$, by Lemma 1 we obtain

$$\frac{2}{(n-2)^2}\int_X \rho\,|E_g|^2\,dv_g = \int_\Sigma\left(\frac{H^2}{n-1} - \frac{R^\Sigma}{n-2}\right) d\sigma. \tag{0.3}$$

By (0.1) and the Holder inequality again

$$\begin{aligned}
\int_\Sigma H^2 d\sigma &= \left(\frac{\lambda_q}{2}\right)^2\int_\Sigma u^{2\left(q-\frac{n}{n-2}\right)} u^{2(n-1)/(n-2)} d\overline{\sigma} \\
&= \left(\frac{\lambda_q}{2}\right)^2\int_\Sigma u^{2\left(q-\frac{1}{n-2}\right)} d\overline{\sigma} \\
&\leq \left(\frac{\lambda_q}{2}\right)^2\left(\int_\Sigma u^{q+1} d\overline{\sigma}\right)^{2\left(q-\frac{1}{n-2}\right)/(q+1)} V\left(\Sigma,\overline{g}\right)^{\left(\frac{n}{n-2}-q\right)/(q+1)} \\
&= \left(\frac{\lambda_q}{2}\right)^2 V\left(\Sigma,\overline{g}\right)^{\left(\frac{n}{n-2}-q\right)/(q+1)}.
\end{aligned}$$

Plugging the above inequality into (0.3), we obtain

$$\frac{2}{(n-2)^2}\int_X \rho\,|E_g|^2\,dv_g \leq \frac{\lambda_q^2}{4\,(n-1)} V\left(\Sigma,\overline{g}\right)^{\left(\frac{n}{n-2}-q\right)/(q+1)} - \frac{1}{n-2}\int_\Sigma R^\Sigma d\sigma. \tag{0.4}$$

When $n = 3$, this implies

$$\lambda_q^2 V\left(\Sigma,\overline{g}\right)^{(3-q)/(q+1)} \geq 32\pi\chi\left(\Sigma\right).$$

In the following, we assume $n > 3$. By (0.1) and the Holder inequality

$$\begin{aligned} 1 &= \int_{\Sigma} u^{q+1} d\overline{\sigma} \\ &\leq \left(\int_{\Sigma} u^{2(n-1)/(n-2)} d\overline{\sigma}\right)^{\frac{(q+1)(n-2)}{2(n-1)}} V(\Sigma, \overline{g})^{\frac{n-q(n-2)}{2(n-1)}} \\ &= V(\Sigma, g)^{\frac{(q+1)(n-2)}{2(n-1)}} V(\Sigma, \overline{g})^{\frac{n-q(n-2)}{2(n-1)}} \end{aligned}$$

Thus

$$V(\Sigma, \overline{g})^{-\frac{n-q(n-2)}{(n-2)(q+1)}} \leq V(\Sigma, g).$$

Plugging this inequality into (0.4) yields

$$\begin{aligned} &\frac{2}{(n-2)^2} \int_M \rho \left|E_g\right|^2 dv_g \\ \leq\ & \frac{V(\Sigma, g)^{\frac{n-1}{n-3}}}{4(n-1)} \left[\lambda_q^2 V(\Sigma, \overline{g})^{\frac{2(n-q(n-2))}{(n-3)(q+1)}} - \frac{4(n-1)}{(n-2) V(\Sigma, g)^{\frac{n-1}{n-3}}} \int_{\Sigma} R^{\Sigma} d\sigma\right] \\ \leq\ & \frac{V(\Sigma, g)^{\frac{n-1}{n-3}}}{4(n-1)} \left[\lambda_q^2 V(\Sigma, \overline{g})^{\frac{2(n-q(n-2))}{(n-3)(q+1)}} - \frac{4(n-1)}{(n-2)} Y(\Sigma, [\overline{g}|_{\Sigma}])\right]. \end{aligned}$$

Therefore

$$\lambda_q^2 \geq \frac{4(n-1)}{(n-2)} Y(\Sigma, [\overline{g}|_{\Sigma}]) V(\Sigma, \overline{g})^{-\frac{2(n-q(n-2))}{(n-3)(q+1)}}.$$

This finishes the proof of Theorem 4

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