# MANIFOLDS WITH $k$-POSITIVE RICCI CURVATURE

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## 1. Introduction

Let $(M, g)$ be an $n$-dimensional Riemannian manifold. We say $M$ has *$k$-positive Ricci curvature* if at each point $p \in M$ the sum of the $k$ smallest eigenvalues of the Ricci curvature at $p$ is positive. We say that the $k$-positive Ricci curvature is bounded below by $\alpha$ if the sum of the $k$ smallest eigenvalues is greater than $\alpha$. Note that $n$-positive Ricci curvature is equivalent to positive scalar curvature and one-positive Ricci curvature is equivalent to positive Ricci curvature. We first describe some basic connect sum and surgery results for $k$-positive Ricci curvature that are direct generalizations of the well known results for positive scalar curvature ($n$-positive Ricci curvature). Using these results we construct examples that motivate questions and conjectures in the cases of 2-positive and $(n-1)$-positive Ricci curvature. In particular:

**Conjecture 1:** If $M$ is a closed $n$-manifold that admits a metric with 2-positive Ricci curvature then the fundamental group, $\pi_1(M)$, is virtually free.

We formulate an approach to solving this conjecture based, at least philosophically, on the method used in the proof of the Bonnet-Myers theorem: A closed $n$-manifold that admits a metric with positive Ricci curvature (1-positive Ricci curvature) has finite fundamental group. The Bonnet-Myers theorem proves a diameter bound for a manifold with positive Ricci curvature bounded below and then uses covering spaces to conclude the result on the fundamental group. In our approach to Conjecture 1 we describe a suitable notion of "two-dimensional diameter bound", namely, the notion of *fill radius*. This idea was introduced in [G], [G-L2], [S-Y2]. We show that if a closed manifold satisfies a curvature condition that implies a fill radius bound then the fundamental group is virtually free. The proof of this statement uses covering space theory, like in the Bonnet-Myers argument, as well as some notions from geometric group theory. See [R-W] for full details and Section 4 of this paper for an overview. However, the question: "Does a manifold with 2-positive Ricci curvature bounded below by $\alpha$ satisfy an upper bound on its fill radius?" remains open.

It is clear that $k$-positive Ricci curvature implies $(k+1)$-positive Ricci curvature, so results on $n$-manifolds with $n$-positive Ricci curvature hold for $n$-manifolds with $k$-positive Ricci curvature, any $k, 0 < k < n$. As a result of the surgery theorem we pose some interesting questions on the difference between $n$-positive Ricci curvature and $(n-1)$-positive Ricci curvature.

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## 2. Manifolds with $k$-positive Ricci curvature

In this section we state some structure theorems for manifolds with $k$-positive Ricci curvature and provide some examples. Throughout we assume that $(M,g)$ is an $n$-dimensional Riemannian manifold with $n \geq 3$. Recall that $n$-positive Ricci curvature is positive scalar curvature and one-positive Ricci curvature is positive Ricci curvature. The following is a direct generalization of a well-known result of Gromov-Lawson [G-L1] and Schoen-Yau [S-Y1] on connect sum and surgeries of manifolds with positive scalar curvature (also see [R-S]).

**Theorem 2.1.** *Let $M$ be a compact $n$-manifold with a metric of $k$-positive Ricci curvature, $2 \leq k \leq n$. Then any manifold obtained from $M$ by performing surgeries in codimension $q$ with $q \geq \max\{n+2-k, 3\}$ also has a metric of $k$-positive Ricci curvature. If $M_1$ and $M_2$ are compact $n$-manifolds with metrics of $k$-positive Ricci curvature, $2 \leq k \leq n$, then their connect sum $M_1 \# M_2$ also carries a metric with $k$-positive Ricci curvature.*

*Proof.* The proof follows easily from the procedure provided in the proof in [G-L1] of the similar statement for positive scalar curvature (the case of $k = n$). We note that this procedure fails when $k = 1$ (the case of positive Ricci curvature). For the sake of completeness we will give the proof here of the connect sum result following, sometimes verbatim, the method of Gromov-Lawson. Suppose that $M$ is an $n$-manifold, $n \geq 3$, and that $M$ has a metric with Ricci curvature that is $k$-positive, for $2 \leq k \leq n$ (i.e., at each point the sum of any $k$ eigenvalues of the Ricci curvature is positive). Fix a point $p \in M$ and consider a normal coordinate ball $D$ centered at $p$. Following [G-L1] we will change the metric in $D$ preserving that the Ricci curvature is $k$-positive such that the metric agrees with the original metric near $\partial D$ and such that near $p$ the metric is the standard metric on $S^{(n-1)}(\varepsilon) \times \mathbb{R}$, for any $\varepsilon$ sufficiently small. It follows from this that 1-handles can be added and connected sums taken preserving that the Ricci curvature is $k$-positive, for any $2 \leq k \leq n$.

The method of Gromov-Lawson proceeds as follows: We consider the Riemannian product $D \times \mathbb{R}$ with coordinates $(x,t)$, where $x$ are normal coordinates on $D$. We define a hypersurface $N \subset D \times \mathbb{R}$ by the relation

$$N = \{(x,t) : (x,t) \in \gamma\}$$

where $\gamma$ is a smooth curve in the $(r,t)$-plane that is monotonically decreasing, begins along the positive $r$-axis and ends as a straight line parallel to the $t$-axis. The metric induced on $N$ from $D \times \mathbb{R}$ extends the metric on $D$ near its boundary and ends with a product metric of the form $S^{n-1}(\varepsilon) \times \mathbb{R}$. If $\varepsilon$ is sufficiently small then by Lemma 1 of [G-L1] we can change the metric in this tubular piece to the standard metric on the Riemannian product of the standard $\varepsilon$-sphere with $\mathbb{R}$.

The key problem is to choose the curve $\gamma$ so that the metric induced on $N$ has strictly $k$-positive Ricci curvature at all points. To do this we begin as in [G-L1] by letting $\ell$ be a geodesic ray in $D$ beginning at the origin. Then the surface $\ell \times \mathbb{R}$ is totally geodesic in $D \times \mathbb{R}$ and the normal to $N$ along points of $N \cap (\ell \times \mathbb{R})$ lies in $\ell \times \mathbb{R}$. It follows that $\gamma_\ell = N \cap (\ell \times \mathbb{R})$ is a principal curve on $N$ and that the associated principal curvature at a point $(r,t) \in \gamma$ is the curvature $\kappa$ of $\gamma$ at that point. The remaining principal curvatures at such a point are of the form

$(-1/r + O(r)) \times \sin\theta$ where $\theta$ is the angle between the normal to the hypersurface and the $t$-axis.

Fix a point $q \in \gamma_\ell \subset N$ corresponding to a point $(r,t) \in \gamma$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_q(N)$ such that $e_1$ is the tangent vector to $\gamma_\ell$ and $\{e_2, \ldots, e_n\}$ (which are tangent vectors to $D$) are principal vectors for the second fundamental form of $N$. The Gauss curvature equation relates the curvature tensor $R_{ijlm}$ of $N$ with the curvature tensor $\overline{R}_{ijlm}$ of $D \times \mathbb{R}$. In particular, with respect to this basis, at $q$:

$$\begin{aligned} \overline{R}_{ijij} &= R_{ijij} - \lambda_i\lambda_j \quad \text{for } i \neq j \\ \overline{R}_{ijlj} &= R_{ijlj} \quad \text{for } \ i \neq l. \end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are the principal curvatures corresponding to the directions $e_1, \ldots, e_n$ respectively. As above, $\lambda_1 = \kappa$, the curvature of $\gamma_\ell$ in $\ell \times \mathbb{R}$ and $\lambda_j = (-1/r + O(r))\sin\theta$ for $j = 2, \ldots n$. Since $D \times \mathbb{R}$ has the product metric we have:

$$\begin{aligned} \overline{R}_{ijlj} &= R^D_{ijlj}, && \text{for } i,j,l = 2, \ldots, n \\ \overline{R}_{1jij} &= R^D_{\frac{\partial}{\partial r}jij}\cos\theta, && \text{for } i,j = 2, \ldots, n \\ \overline{R}_{i1j1} &= R^D_{i\frac{\partial}{\partial r}j\frac{\partial}{\partial r}}\cos^2\theta, && \text{for } i,j = 2, \ldots, n. \end{aligned}$$

where $R^D$ is the curvature tensor of the metric on $D$. It follows that the Ricci curvature of $N$ at $(x,t)$ with respect to the frame $\{e_1, \ldots, e_n\}$ is given by:

(2.1)

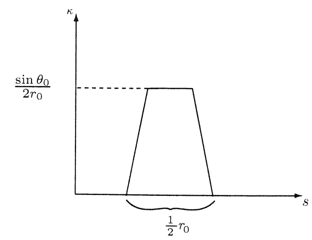
$$\begin{aligned} \mathrm{Ric}_{11} &= \mathrm{Ric}^D(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r})\cos^2\theta + \kappa\sum_{j=2}^n \lambda_j \\ \mathrm{Ric}_{1j} &= \mathrm{Ric}^D(\tfrac{\partial}{\partial r}, e_j)\cos\theta, && \text{for } j = 2, \ldots, n \\ \mathrm{Ric}_{ij} &= \mathrm{Ric}^D_{ij} - R^D_{i\frac{\partial}{\partial r}j\frac{\partial}{\partial r}}\sin^2\theta, && \text{for } i,j = 2, \ldots, n, \ \ i \neq j \\ \mathrm{Ric}_{ii} &= \mathrm{Ric}^D_{ii} - R^D_{i\frac{\partial}{\partial r}i\frac{\partial}{\partial r}}\sin^2\theta + \lambda_i(\sum_{j=2, j\neq i}^n \lambda_j + \kappa), && \text{for } i = 2, \ldots, n \end{aligned}$$

where $\mathrm{Ric}^D$ denotes the Ricci curvature of $D$.

The eigenvalues of a matrix depend continuously on the entries of the matrix. Therefore for $\theta$ sufficiently small, using that $\lambda_i = (-1/r + O(r))\sin\theta$, for $i \geq 2$, the Ricci curvature at $(x,t)$ in formula (2.1) is $k$-positive. In particular, there is an $\theta_0 > 0$ such that for $0 \leq \theta \leq \theta_0$ the Ricci curvature at $(x,t)$ is $k$-positive. As in [G-L1] we bend $\gamma$ to the angle $\theta_0$ and continue $\gamma$ as a straight line segment. Denote the straight line segment of $\gamma$ by $\gamma_0$. Along this curve $N$ has Ricci curvature that is $k$-positive. Since $\kappa \equiv 0$ along $\gamma_0$, we see that as $r$ becomes small the Ricci curvature of $N$ is of the form:

(2.2)
$$\begin{aligned} \mathrm{Ric}_{ij} &= O(1) && \text{for } i \neq j \\ \mathrm{Ric}_{ii} &= \frac{(n-2)\sin^2\theta_0}{r^2} + O(1) && \text{for } i = 2, \ldots, n \\ \mathrm{Ric}_{11} &= O(1) \end{aligned}$$

Choose $r_0 > 0$ small. From (2.2) it follows that the the Ricci curvature of the hypersurface $N$ has $(n-1)$ positive eigenvalues and that each of these eigenvalues is larger in absolute value than the one remaining eigenvalue (corresponding to the direction of $e_1$). Therefore the Ricci curvature remains $k$-positive and, in fact, becomes 2-positive. Consider the point $(r_0, t_0) \in \gamma_0$. Bend $\gamma_0$, beginning at this point, with the curvature function $\kappa(s)$ similar to the one used in [G-L1]:




Here $s$ is the arclength parameter. Since $\max \kappa = \frac{\sin\theta_0}{2r_0}$ we see that:

$$\lambda_i\left(\sum_{j=2, j\neq i}^{n} \lambda_j + \kappa\right) > |\kappa \sum_{j=2}^{n} \lambda_j| > 0, \qquad \text{for } i = 2, \ldots, n.$$

Using (2.1) (and $r_0$ sufficiently small) this implies that along the bend the Ricci curvature of $N$ remains 2-positive. During this bending process the curve will not cross the line $r = r_0/2$ since the length of the bend is $r_0/2$ and it begins at height $r_0$. The total amount of bending is:

$$\Delta\theta = \int \kappa ds \simeq \frac{\sin\theta_0}{4},$$

independent of $r_0$.

Continue the curve with a new straight line segment $\gamma_1$ at an angle $\theta_1 = \theta_0 + \Delta\theta$. Repeat the above procedure now using $\theta_1$ where previously we used $\theta_0$. The total bending of this procedure will then be $\frac{\sin\theta_1}{4} > \frac{\sin\theta_0}{4}$. By repeating this procedure a finite number of times (depending on $\sin\theta_0$) we can achieve a total bend of $\pi/2$. This completes the proof of the connect sum result.

For the general case of surgeries we again explain the modification of the Gromov-Lawson argument. Let $S^p \subset M$ be an embedded sphere with trivial normal bundle $B$ of dimension $q \geq 3$. Identify $B$ with $S^p \times \mathbb{R}^q$. Define $r : S^p \times \mathbb{R}^q \to \mathbb{R}_+$ by $r(y, x) = ||x||$, and set $S^p \times D^q(\rho) = \{(y, x) : r(y, x) \leq \rho\}$. Choose $\bar{r} > 0$ such that the normal exponential map $\exp : B \to M$ is an embedding on $S^p \times D^q(\bar{r}) \subset B$. Lift the metric on $M$ to $S^p \times D^q(\bar{r})$ by the exponential map. Then $r$ is the distance function to $S^p \times \{0\}$ in $S^p \times D^q(\bar{r})$, and curves of the form $\{y\} \times \ell$, where $\ell$ is a ray in $D^q(\bar{r})$ emanating from the origin, are geodesics in $S^p \times D^q(\bar{r})$.

We now consider hypersurfaces in the Riemannian product $(S^p \times D^q(\bar{r})) \times \mathbb{R}$ of the form:

$$N = \{(y, x, t) : (r(y, x), t) \in \gamma\}$$

where $\gamma$ is, as above, a curve in the $(r, t)$-plane. As in the connect sum case we must choose $\gamma$ so that the metric on $N$ has $k$-positive Ricci curvature at all points. We first remark that $\gamma_\ell = N \cap (\{y\} \times \ell \times \mathbb{R})$ is a principal curve on $N$ and the associated principal curvature at a point corresponding to $(r, t) \in \gamma$ is exactly the curvature $\kappa$ of $\gamma$ at that point.

Now fix a point $q \in \gamma_\ell \subset N$ corresponding to a point $(r,t) \in \gamma$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_q(N)$ such that $e_1$ is the tangent vector to $\gamma_\ell$, and $e_2, \ldots, e_n$ are principal vectors for the second fundamental form of $N$. If the metric on $S^p \times D^q(\bar{r})$ at $q$ is the product metric then the vectors $e_2, \ldots, e_q$ can be chosen to be tangent to $\{y\} \times D^q(\bar{r})$ and the vectors $e_{q+1}, \ldots, e_n$ to be tangent to $S^p \times \{x\}$. The principal curvatures $\lambda_2, \ldots, \lambda_q$ are then of the form $(-\frac{1}{r} + O(r))\sin\theta$, where $\theta$ is the angle between the normal to the hypersurface and the $t$-axis and the principal curvatures $\lambda_{q+1}, \ldots, \lambda_n$ are of the form $O(1)\sin\theta$ (i.e., are independent of $r$). In the general case such a simple description is not possible. However if $r$ is small the $q-1$ largest principal curvatures are of the form $(-\frac{c}{r} + O(1))\sin\theta$, where $c$ is a positive constant that can be bounded away from zero and the remaining principal curvatures are of this form $O(1)\sin\theta$. (Since the product of the principal curvatures grows like $\frac{1}{r^{q-1}}$.) We will denote the $q-1$ largest principal curvatures by $\lambda_2, \ldots, \lambda_q$ corresponding to the directions $e_2, \ldots, e_q$ and the remainder by $\lambda_{q+1}, \ldots, \lambda_n$ corresponding to the directions $e_{q+1}, \ldots, e_n$.

The Gauss curvature equation relates the curvature tensor $R_{ijlm}$ of $N$ with the curvature tensor $\overline{R}_{ijlm}$ of $(S^p \times D^q(\bar{r})) \times \mathbb{R}$. In particular, with respect to this basis, at $q$:

$$\begin{aligned}
\overline{R}_{ijij} &= R_{ijij} - \lambda_i\lambda_j \quad \text{for } i \neq j \\
\overline{R}_{ijlj} &= R_{ijlj} \quad \text{for } \ i \neq l.
\end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are the principal curvatures corresponding to the directions $e_1, \ldots, e_n$ respectively. As above, $\lambda_1 = \kappa$, the curvature of $\gamma_\ell$ in $\ell \times \mathbb{R}$ and $\lambda_j = (-\frac{c}{r} + O(1))\sin\theta$ for $j = 2, \ldots q$. The remaining principal curvatures $\lambda_{q+1}, \ldots, \lambda_n$ are of the form $O(1)\sin\theta$. For this reason, unlike in the connect sum case, they play no useful role in the following computation. Since $(S^p \times D^q(\bar{r})) \times \mathbb{R}$ has the product metric in the second factor we have:

$$\begin{aligned}
\overline{R}_{ijlj} &= R^{S^p\times D^q}_{ijlj}, && \text{for } i,j,l = 2,\ldots,n \\
\overline{R}_{1jij} &= R^{S^p\times D^q}_{\frac{\partial}{\partial r}jij}\cos\theta, && \text{for } i,j = 2,\ldots,n \\
\overline{R}_{i1j1} &= R^{S^p\times D^q}_{i\frac{\partial}{\partial r}j\frac{\partial}{\partial r}}\cos^2\theta, && \text{for } i,j = 2,\ldots,n.
\end{aligned}$$

where $R^{S^p\times D^q}$ is the curvature tensor of the metric on $S^p \times D^q$. It follows that the Ricci curvature of $N$ at $(y,x,t)$ with respect to the frame $\{e_1, \ldots, e_n\}$ is given by:

(2.3)

$$\begin{aligned}
\mathrm{Ric}_{11} &= \mathrm{Ric}^{S^p\times D^q}(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r})\cos^2\theta + \kappa\sum_{j=2}^n \lambda_j \\
\mathrm{Ric}_{1j} &= \mathrm{Ric}^{S^p\times D^q}(\tfrac{\partial}{\partial r}, e_j)\cos\theta, && \text{for } j = 2,\ldots,n \\
\mathrm{Ric}_{ij} &= \mathrm{Ric}^{S^p\times D^q}_{ij} - R^{S^p\times D^q}_{i\frac{\partial}{\partial r}j\frac{\partial}{\partial r}}\sin^2\theta, && \text{for } i,j = 2,\ldots,n, \ \ i \neq j \\
\mathrm{Ric}_{ii} &= \mathrm{Ric}^{S^p\times D^q}_{ii} - R^{S^p\times D^q}_{i\frac{\partial}{\partial r}i\frac{\partial}{\partial r}}\sin^2\theta + \lambda_i(\sum_{j=2, j\neq i}^n \lambda_j + \kappa), && \text{for } i = 2,\ldots,n
\end{aligned}$$

where $\mathrm{Ric}^{S^p\times D^q}$ denotes the Ricci curvature of $S^p \times D^q$.

Since $\lambda_i = (-\frac{c}{r} + O(1))\sin\theta$, for $i = 2, \ldots, q$ and $\lambda_i = O(1)\sin\theta$, for $i = q+1, \ldots, n$, if $\theta$ sufficiently small, the Ricci curvature at $(y,x,t)$ in formula (2.3) is $k$-positive. In particular, there is an $\theta_0 > 0$ such that for $0 \leq \theta \leq \theta_0$ the Ricci curvature at $(y, x, t)$ is $k$-positive. As above we bend $\gamma$ to the angle $\theta_0$ and continue $\gamma$ as a straight line segment. Denote the straight line segment of $\gamma$ by $\gamma_0$. Along this curve $N$ has Ricci curvature that is $k$-positive. Since $\kappa \equiv 0$ along $\gamma_0$, we see that as $r$ becomes small the Ricci curvature of $N$ is of the form:

$$\begin{aligned} \mathrm{Ric}_{ij} &= O(1) && \text{for } i \neq j \\ \mathrm{Ric}_{ii} &= \frac{(q-2)c^2 \sin^2 \theta_0}{r^2} + O(\tfrac{1}{r}) && \text{for } i = 2, \ldots, q \\ \mathrm{Ric}_{ii} &= O(\tfrac{1}{r}) && \text{for } i = q+1, \ldots, n \\ \mathrm{Ric}_{11} &= O(1) \end{aligned} \tag{2.4}$$

(Note that since $\lambda_{q+1}, \ldots, \lambda_n$ are bounded for $r$ small, we cannot conclude any more than $\mathrm{Ric}_{ii} = O(\frac{1}{r})$ for $i = q+1, \ldots, n$.) Therefore, provided $q \geq 3$, for $r$ sufficiently small the Ricci curvature has $q-1$ positive eigenvalues that strongly dominate all other eigenvalues. In particular, provided $k > n - q + 1$ the Ricci curvature remains $k$-positive. We can then bend $\gamma$ to a line parallel to the $t$-axis and preserve $k$-positive Ricci curvature as above.

This construction of $N$ determines a "tube" of $k$-positive Ricci curvature with two boundary components. The initial boundary component has a collar neighborhood isometric to a tubular neighborhood of $S^p$ in $M$. The final boundary component has a collar neighborhood isometric to $\partial(S^p \times D^q(\varepsilon)) \times \mathbb{R} = S^p \times S^{q-1}(\varepsilon) \times \mathbb{R}$ for the product metric with the $\mathbb{R}$ factor. This allows us to glue the initial end of $N$ to $M \setminus (S^p \times D^q(r))$ to construct a manifold $M'$ with $k$-positive Ricci curvature and with one boundary component $S^p \times S^{q-1}(\varepsilon)$.

As in [G-L1] we next observe that the metric on $\partial(S^p \times D^q(\varepsilon)) = S^p \times S^{q-1}(\varepsilon)$ can be homotoped through metrics with $k$-positive Ricci curvature to the standard product metric of euclidean spheres on $S^p \times S^{q-1}(\varepsilon)$. The argument that accomplishes this is the same as that in Gromov-Lawson: For $\varepsilon$ sufficiently small the metric on $\partial(S^p \times D^q(\varepsilon))$ can be homotoped through metrics with $k$-positive Ricci curvature to a metric on $S^p \times S^{q-1}(\varepsilon)$ that is a Riemannian submersion with totally geodesic fibers that have the euclidean metric of curvature $\frac{1}{\varepsilon^2}$. When $\varepsilon$ sufficiently small the terms $\mathrm{Ric}(v, v)$ for vertical vectors $v$ strongly dominate all other terms of the Ricci curvature and therefore we can deform this metric, preserving $k$-positive Ricci curvature (for $k > n - q + 1$), through Riemannian submersions to one with the standard metric on $S^p$.

Denote the induced metric on $S^p \times S^{q-1}(\varepsilon)$ by $g_0$ and the product of the standard metrics on $S^p \times S^{q-1}(\varepsilon)$ by $g_1$. The final step of the Gromov-Lawson argument shows that the homotopy constructed in the previous paragraph can be used to find a metric of $k$-positive Ricci curvature on $S^p \times S^{q-1}(\varepsilon) \times [0, a]$, for some $a > 0$, whose restriction to a collar neighborhood of $S^p \times S^{q-1}(\varepsilon) \times \{0\}$ is $g_0 + dt^2$ and whose restriction to a collar neighborhood of $S^p \times S^{q-1}(\varepsilon) \times \{a\}$ is $g_1 + dt^2$. We will give a proof below in Proposition 2.2. Using this result we can assume that the end of the manifold $M'$ is isometric to $S^p \times S^{q-1}(\varepsilon) \times \mathbb{R}$ equipped with the product of the standard metrics on the spheres. From this the surgery result follows immediately. $\square$

To complete the proof of Theorem 2.1 we prove the following proposition motivated by Proposition 3.3 of [R-S], Lemma 3 of [G-L] and [Ga].

**Proposition 2.2.** *Let $X$ be a compact $n$-manifold. Suppose that there is a smooth family of Riemannian metrics on $X$, $\{g_t\}$, $0 \le t \le 1$, each with $k$-positive Ricci curvature for some $k$, $2 \le k \le n$. Then there is a Riemannian metric $g$ on $X \times [0,a]$, for some $a > 0$, with $k$-positive Ricci curvature such that the restriction of $g$ on a collar neighborhood of $X \times \{0\}$ is $g_0 + dt^2$ and the restriction of $g$ on a collar neighborhood of $X \times \{a\}$ is $g_1 + dt^2$.*

*Proof.* Consider the metric $g_{f(t)} + dt^2$ on $X \times [0,a]$, where $f(t)$ is a $C^2([0,a])$ function that is monotonically increasing from 0 to 1. We will explicitly determine $f(t)$ below. We compute the curvature of $g_{f(t)} + dt^2$ at the point $(x_0,t_0) \in X \times [0,a]$. Let $e_0$ be the unit normal along the hypersurface $X \times \{t_0\}$ pointing in the direction $\frac{\partial}{\partial t}$. Let $\{e_1,\dots,e_n\}$ be an orthonormal frame along $X \times \{t_0\}$ near $(x_0,t_0)$. Then $\{e_0,\dots,e_n\}$ is an orthonormal frame along $X \times [0,1]$ near $(x_0,t_0)$. Let $\{\omega_0,\dots,\omega_n\}$ be the dual coframe. For notational convenience we use the index ranges $\alpha,\beta,\gamma = 0,1,\dots,n$ and $i,j,k = 1,2,\dots,n$. The connection one-form for the coframe $\{\omega_0,\dots,\omega_n\}$ is $\{\omega_{\alpha\beta}\}$. The one-forms $\omega_{ij}$ depend only on the metric $g_{f(t)}$. However the one-forms $\omega_{0i}$ depend on $f'(t)$ so at $(x_0,t_0)$ we have:

$$\begin{aligned} \omega_{0i} &= O(|f'|), \\ d\omega_{0i} &= O(|f''|) + O(|f'|) + O(|f'|^2). \end{aligned} \tag{2.5}$$

The curvature two-form $\Omega_{\alpha\beta}$ on $X \times [0,a]$ is determined by the structure equation:

$$\Omega_{\alpha\beta} = d\omega_{\alpha\beta} - \sum_{\gamma} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}. \tag{2.6}$$

Denote the curvature tensor of $g_{f(t)} + dt^2$ on $X \times [0,a]$ by $\bar{R}_{\alpha\beta\gamma\delta}$ and the curvature tensor of $g_{f(t_0)}$ on $X \times \{t_0\}$ by $R_{ijkl}$. Then by the structure equation (2.6) (or by the Gauss equation) at $(x_0,t_0)$:

$$\bar{R}_{ijkl} = R_{ijkl} + O(|f'|^2). \tag{2.7}$$

By the structure equation and (2.5)

$$\begin{aligned} \bar{R}_{0jkl} &= O(|f''|) + O(|f'|) + O(|f'|^2), \\ \bar{R}_{0j0l} &= O(|f''|) + O(|f'|) + O(|f'|^2). \end{aligned}$$

Denote the Ricci curvature of $g_{f(t)} + dt^2$ on $X \times [0,a]$ by $\overline{\mathrm{Ric}}$ and the Ricci curvature of $g_{f(t_0)}$ on $X \times \{t_0\}$ by Ric. Then

$$\begin{aligned} \overline{\mathrm{Ric}}_{ij} &= \mathrm{Ric}_{ij} + O(|f''|) + O(|f'|) + O(|f'|^2), \\ \overline{\mathrm{Ric}}_{i0} &= O(|f''|) + O(|f'|) + O(|f'|^2), \\ \overline{\mathrm{Ric}}_{00} &= O(|f''|) + O(|f'|) + O(|f'|^2). \end{aligned} \tag{2.8}$$

Given constants, $0 < \varepsilon, \lambda << 1$, choose $a > 0$ sufficiently large so that there is a $C^2$ function $f : [0,a] \to [0,1]$ satisfying:

1. $f$ is monotonically increasing,
2. $f(t) = 0$, for $0 \le t \le \varepsilon$,
3. $f(t) = 1$, for $a - \varepsilon \le t \le a$,
4. $|f'(t)|, |f''(t)| < \lambda$, for all $t$.

Using (2.8) since $(X, g_t)$ has $k$-positive Ricci curvature for all $t \in [0,1]$, the compactness of $X$ and $[0,1]$ allow that $\lambda$ can be chosen sufficiently small to ensure that the metric $g_{f(t)} + dt^2$ (for $f$ as given above) has $k$-positive Ricci curvature everywhere on $X \times [0,a]$. The result follows. $\square$

Let $X$ be a compact Riemannian manifold with positive Ricci curvature. Then the manifold $X \times S^1$ has a metric of 2-positive Ricci curvature (and of non-negative Ricci curvature). Therefore, by Theorem 2.1, if $X_i$, $i = 1, \ldots, l$, are compact Riemannian $(n-1)$-manifolds with positive Ricci curvature, the manifolds:

$$\#_i^l (X_i \times S^1), \quad l \geq 2, \tag{2.9}$$

admit metrics with 2-positive Ricci curvature. From this we see that compact manifolds with 2-positive Ricci curvature can have large fundamental groups. In fact the fundamental groups of the examples (2.9) are virtually free. This implies that the manifolds (2.9), when $l \geq 2$, do not admit metrics of non-negative Ricci curvature since the universal covers of (2.9) have infinitely many ends. In contrast the universal cover of a compact manifold with non-negative Ricci curvature splits isometrically as a product $N \times \mathbb{R}^p$, where $N$ is a compact manifold, and therefore has one or two ends. Many other topologically distinct examples of compact manifolds that admit metrics of 2-positive Ricci curvature can be constructed by taking the connect sum of manifolds of positive Ricci curvature with the manifolds of (2.9).

Consider the round metric on the sphere of radius $r$, $S_r^{n-2}$, and the hyperbolic metric on the Riemann surface $\Sigma_g$ of genus $g \geq 2$. When $r$ is sufficiently small the manifolds $S_r^{n-2} \times \Sigma_g$ admit metrics of 3-positive Ricci curvature but not of 2-positive Ricci curvature. Examples of this type indicate that 3-positive Ricci curvature when $n \geq 4$ imposes much weaker restrictions on $\pi_1(M)$ than 2-positive Ricci curvature.

The surgery results of Theorem 2.1 do not allow any surgeries preserving 2-positive Ricci curvature (except connect sum). When $k > 2$ and $n > 3$, $q = n-1$-surgeries preserve the curvature condition. This suggests a difference between $k = 2$ and $k > 2$ in the rigidity of the fundamental group for manifolds with $k$-positive Ricci curvature.

**Conjecture 1:** If $M$ is a closed $n$-manifold that admits a metric with 2-positive Ricci curvature then its fundamental group, $\pi_1(M)$, is virtually free.

The first interesting case of this conjecture occurs when $n = 4$. When $n = 3$ the condition of positive scalar curvature is strictly weaker than 2-positive Ricci curvature. In the case of positive scalar curvature the conjecture can be answered in the affirmative. In fact, by the results of [S-Y1] and [G-L2] much more can be said about the topology of 3-manifolds of positive scalar curvature. We remark that the conjecture requires a positive curvature assumption. Requiring only 2-non-negative Ricci curvature is not sufficient. For example that manifold $S^2 \times T^2$, where $S^2$ is the round 2-sphere and $T^2$ is the flat 2-torus, has 2-non-negative Ricci curvature and fundamental group isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

The surgery statement of Theorem 2.1 shows that if $k = n-1$ then surgery is possible provided $q \geq 3$. This is the same condition as in the positive scalar curvature case. The surgery result is used by Gromov-Lawson [G-L1] to prove that if $n \geq 5$ then every compact simply-connected non-spin $n$-manifold carries a metric of positive scalar curvature and by Stolz [Sz] to prove that for $n \geq 5$ every compact simply-connected spin $n$-manifold with vanishing $\alpha$ invariant carries a metric of positive scalar curvature. Gromov-Lawson prove their result for non-spin manifolds using oriented bordism. Stolz proves his result for spin manifolds using spin bordism. Since $n-1$ positive Ricci curvature implies positive scalar curvature any necessary condition for positive scalar curvature is a necessary condition for $n-1$ positive Ricci curvature. In light of the surgery result for $n-1$ positive Ricci curvature exactly the same arguments apply to prove:

**Theorem 2.3.** *Let $n \geq 5$. Every compact simply-connected non-spin $n$-manifold carries a metric with $n-1$ positive Ricci curvature. Every compact simply-connected spin $n$-manifold with vanishing $\alpha$ invariant carries a metric with $n-1$ positive Ricci curvature.*

To prove the theorem, all that needs to be checked is that the generators of oriented bordism described in [G-L1] and the $\mathbb{HP}^2$ bundles used in [Sz] admit metrics with $n-1$ positive Ricci curvature. In particular for compact simply-connected $n$-manifold with $n \geq 5$ there is no distinction between positive scalar curvature and $n-1$ positive Ricci curvature.

**Question 1:** Is there a compact $n$-manifold, $n \geq 5$, that admits a metric of positive scalar curvature but does not admit a metric with $n-1$ positive Ricci curvature?

## 3. Fill radius and an approach to Conjecture 1

In this section we will describe an approach to Conjecture 1. In the Introduction we recalled that the Bonnet-Myers theorem on the fundamental group of a closed manifold with positive Ricci curvature is proved by deriving a diameter bound on such manifolds. The 2-positive Ricci curvature condition implies a positive lower bound on the sum of any two eigenvalues of Ricci. Accordingly we seek to derive a "two-dimensional diameter bound" for such manifolds. The notion we use for "two-dimensional diameter bound" is that of *fill radius* [G], [G-L2], [S-Y2].

Let $\gamma$ be a smooth simple closed curve in $M$ which bounds a disk in $M$. Set $N_r(\gamma) = \{x \in M : d(x, \gamma) \leq r\}$. We define the *fill radius of* $\gamma$ to be:

$$\text{fillrad}(\gamma) = \sup\{r : \text{dist}(\gamma, \partial M) > r \text{ and } \gamma \text{ does not bound a disc in } N_r(\gamma)\}$$

We say a Riemannian manifold $(M, g)$ has its *fill radius bounded by* $C$ if every smooth simple closed curve $\gamma$ which bounds a disk in $M$ satisfies,

$$\text{fillrad}(\gamma) \leq C.$$

Clearly if the diameter of $(M, g)$ is bounded so is its fill radius. In particular, if for all $p \in M$ each eigenvalue $\lambda_i, i = 1, \ldots, n$ of $\text{Ric}(p)$ satisfies $\lambda_i \geq \alpha$, where $\alpha$ is a positive constant, then there is a constant $C = C(\alpha)$ such that the fill radius of $M$ is bounded by $C$.

In [G-L2] and [S-Y2] versions of the following result on positive scalar curvature and fill radius are proved.

**Theorem 3.1.** *Let $(M, g)$ be a complete Riemannian three dimensional manifold with bounded geometry and with positive scalar curvature $S$ that satisfies $S \geq \alpha > 0$, for a constant $\alpha$. Then if $\gamma$ is a smooth simple closed curve in $M$ which bounds a disk in $M$:*

$$\text{fillrad}(\gamma) \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\alpha}}.$$

The statements of the results and the details of the proofs in [G-L2] and [S-Y2] differ but the essential ideas of the proof are the same. Initially observe that if $\gamma$ is a simple closed curve in $M$ that bounds a disc $\Sigma$ (more generally, $\Sigma$ can be taken to be a Riemann surface with boundary) then:

$$\text{fillrad}(\gamma) \leq \sup_{x\in\Sigma} d(x,\gamma),$$

where $d(-,-)$ is the distance in $M$. Clearly,

$$\sup_{x\in\Sigma} d(x,\gamma) \leq \sup_{x\in\Sigma} d_\Sigma(x,\gamma),$$

where $d_\Sigma(-,-)$ denotes the distance on $\Sigma$ in the induced metric. Thus the fill radius of $\gamma$ can be bounded above by an upper bound on the diameter of $\Sigma$ in the induced metric. For arbitrary $\Sigma$ spanning $\gamma$ such a bound is, of course, impossible. However in [G-L2] and [S-Y2], $\Sigma$ is taken to be an area minimizer among discs spanning $\gamma$ (i.e., a solution of the Plateau problem). In particular, $\Sigma$ is a stable, minimal immersion. After perturbing $\gamma$ inward along $\Sigma$ the minimal surface is strictly stable for normal variations vanishing on the boundary. In [S-Y] the second variation of area for a minimal surface $\Sigma$ in a three manifold $M$ is given. Let $f\nu$ be a compactly supported normal variation, where $f \in C_0^\infty(\Sigma)$ and $\nu$ is a unit normal. Then the second variation formula can be written:

$$\frac{d^2|\Sigma|}{dt^2}_{|t=0} = \int_\Sigma |\nabla f|^2 + f^2(K - S - \frac{1}{2}||A||^2)da \tag{3.1}$$

Here $|\Sigma| = \text{Area}(\Sigma)$, $S$ is the scalar curvature of $M$, $K$ is the Gauss curvature of $\Sigma$ and $A$ is the second fundamental form. It follows then that if $\Sigma$ is a strictly stable minimal surface the second order linear elliptic operator:

$$L(f) = \Delta f + f(K - S) \tag{3.2}$$

is a positive operator on $f \in C_0^\infty(\Sigma)$. Denote the first eigenfunction of (3.2) for the Dirichlet problem by $k$. Then $k$ is positive on $\Sigma$ and vanishes on the boundary. It is the positivity of (3.2) that is used in both [G-L2] and [S-Y2] to derive a diameter bound on $\Sigma$. In [S-Y2] the argument proceeds as follows. The argument in [G-L2] is somewhat different.

Let $\Sigma$ be a strictly stable minimal surface with boundary $\partial\Sigma$. Fix $x \in \Sigma$ and consider the family $\mathcal{F}$ of curves on $\Sigma$ from $x$ to $\partial\Sigma$. For $\sigma$ in $\mathcal{F}$ consider the functional $F(\sigma) = \int_\sigma k(\sigma(t))|\sigma'(t)|dt$, where $k$ is the eigenfunction of (3.2 ). Minimize $F$ over $\mathcal{F}$. A smooth minima can be found by the direct method (as with geodesics). Denote the minimizer by $\tau$. Since $F$ is independent of arclength we can suppose that $\tau$ is parameterized by arclength and has length $\ell$. Since $\tau$ is a minimizer it is stable under normal variations fixing the end points. Let $\mu$ be a unit normal vector field along $\tau$ in $\Sigma$. The normal vector field $X = \psi\mu$, where $\psi$ is a function of the arclength parameter $s$ that vanishes at its endpoints, is an admissible variational vector field. The second variation of the functional $F$ at $\tau$ determines a quadratic form $I$ given by:

$$I(\psi,\psi) = - \int_0^\ell \left( (\psi'' + k^{-1}\psi'\frac{dk}{ds} + \psi(K + k^{-1}\Delta k + k^{-1}\frac{d^2k}{ds^2}))\psi + \frac{2}{k^2}(\nabla k \cdot \nu)^2\psi^2 \right) kds \tag{3.3}$$

where $\Delta$ is the Laplace-Beltrami operator on $\Sigma$ and $K$ is the Gauss curvature of $\Sigma$. (This expression uses that $\psi$ vanishes at its endpoints. In the general case there is a boundary term.) Set

$$L_0(\psi) = -[\psi'' + k^{-1}\psi'\frac{dk}{ds} + \psi(K + k^{-1}\Delta k + k^{-1}\frac{d^2k}{ds^2})]. \tag{3.4}$$

Then,

$$I(\psi,\psi) = \int_0^\ell \big(L_0(\psi)\psi - \frac{2}{k^2}(\nabla k\cdot\nu)^2\psi^2\big)kds \tag{3.5}$$

On a minimizer $I(\psi,\psi)\geq 0$ and hence, since the term $\frac{2}{k^2}(\nabla k\cdot\nu)^2\psi^2\geq 0$,

$$\int_0^\ell L_0(\psi)\psi kds \geq 0 \tag{3.6}$$

for all functions $\psi$ on $[0,\ell]$ vanishing at the endpoints.

Using that $k$ is the first eigenfunction of $L$ (3.4) becomes:

$$L_0(\psi) = -[\psi'' + k^{-1}\psi'\frac{dk}{ds} + \psi(k^{-1}L(k) + S + k^{-1}\frac{d^2k}{ds^2})]. \tag{3.7}$$

Choose a function $g\in C^\infty([0,\ell])$ that is a first eigenfunction of $L_0$ on $[0,\ell]$ for the Dirichlet problem. Then $g>0$ on $(0,\ell)$, $g$ vanishes at the endpoints and $L_0(g)\geq 0$. Hence,

$$g'' + k^{-1}g'k' + g(k^{-1}L(k) + S + k^{-1}k'')\leq 0.$$

Since $k^{-1}L(k)>0$, this implies,

$$g^{-1}g'' + g^{-1}k^{-1}g'k' + S + k^{-1}k''\leq 0, \tag{3.8}$$

on $[0,\ell]$. Let $\varphi$ be any smooth function on $[0,\ell]$ vanishing at the endpoints and multiply (3.8) by $\varphi^2$ to give:

$$\int_0^\ell (g^{-1}g''\varphi^2 + g^{-1}k^{-1}g'k'\varphi^2 + k^{-1}k''\varphi^2 + S\varphi^2)ds\leq 0$$

Integrate by parts to give,

$$\begin{aligned}\int_0^\ell \Big(\tfrac12[(g^{-1}g')^2 \quad &+\quad (k^{-1}k')^2]\varphi^2 + \tfrac12(\frac{d}{dt}\ln(gk))^2\varphi^2 + S\varphi^2\Big)ds\\ &\leq\quad 2\int_0^\ell \varphi\varphi'(\frac{d}{dt}\ln(gk))ds\end{aligned} \tag{3.9}$$

Note that,

$$\begin{aligned}|2\varphi\varphi'(\frac{d}{dt}\ln(gk))| \quad&\leq\quad \tfrac43(\varphi')^2 + \tfrac34\varphi^2(\frac{d}{dt}\ln(gk))^2\\ &\leq\quad \tfrac43(\varphi')^2 + \tfrac12\varphi^2((g^{-1}g')^2 + (k^{-1}k')^2) + \tfrac12\varphi^2(\frac{d}{dt}\ln(gk))^2,\end{aligned} \tag{3.10}$$

where the second inequality follows from,

$$\tfrac14(\frac{d}{dt}\ln(gk))^2\leq\tfrac12((g^{-1}g')^2 + (k^{-1}k')^2)$$

Combining (3.9) and (3.10) we have:

$$\tfrac12\alpha\int_0^\ell \varphi^2ds\leq\tfrac43\int_0^\ell(\varphi')^2ds, \tag{3.11}$$

where $0 < \alpha \leq S$. Thus,

$$\int_0^\ell (-\varphi'' - \tfrac{3}{8}\alpha\varphi)\varphi ds \geq 0,$$

and so the operator:

$$-\frac{d^2}{ds^2} - \frac{3}{8}\alpha$$

has nonnegative first eigenvalue on $[0, \ell]$. Hence,

$$\ell \leq \sqrt{\frac{8}{3}}\frac{\pi}{\sqrt{\alpha}}$$

This inequality holds for any $x \in \Sigma$. Therefore,

$$\sup_{x \in \Sigma} \mathrm{dist}_\Sigma(x, \partial\Sigma) \leq \ell.$$

From this the theorem follows.

A fill radius bound has strong geometric implications. To illustrate this we describe two results. The first, due to [S-Y1], [G-L2], concerns closed three manifolds with positive scalar curvature. According to Milnor [Mi] any closed three manifold $M$ has a connect sum decomposition:

$$M = S_1 \# \ldots \# S_k \# (S^2 \times S^1) \# \ldots \# (S^2 \times S^1) \# K_1 \# \ldots \# K_j$$

where the $S_i$ are spherical space forms (this uses the solution of the Poincare conjecture) and the $K_i$ are $K(\pi, 1)$ manifolds (a $K(\pi, 1)$ manifold is a closed manifold with contractible universal cover and fundamental group isomorphic to a group $\pi$.) Using the fill radius bound it can be shown that if, in addition, $M$ has positive scalar curvature then no $K(\pi, 1)$ summands occur in this direct sum decomposition (see [G-L2 ] for a proof). In particular, this implies that if $M$ is a closed three manifold with positive scalar curvature then the fundamental group of $M$ is virtually free. This verifies Conjecture 1 in the three dimensional case since 2-positive Ricci curvature implies 3-positive Ricci curvature (positive scalar curvature).

The second result is due to Gromov-Lawson [G-L2]:

**Theorem 3.2.** *Let $(M, g)$ be a closed $n$-manifold with fill radius bounded above by $\beta$. Then there is a distance decreasing map $\phi : M \to \Lambda$ onto a metric graph, such that, for each $p \in \Lambda$,*

$$\mathrm{diameter}(\phi^{-1}(p)) \leq C(\beta).$$

*Proof.* The theorem follows from the proof of Corollary 10.11 in [G-L2] . Also see [G] Appendix 1. □

The theorem implies that closed $n$-manifolds with fill radius bounded above and large diameter are "long" and "thin", exactly like compact 3-manifolds with positive scalar curvature and large diameter.

We have already observed that in three dimensions 2-positive Ricci curvature implies 3-positive Ricci curvature (positive scalar curvature). Therefore closed three manifolds with 2-positive Ricci curvature bounded away from zero by $\alpha$ satisfy a fill radius bound depending on $\alpha$. This partly motivates the following conjecture:

**Conjecture 2:** If $M$ is a closed four manifold that admits a metric with 2-positive Ricci curvature bounded below by $\alpha > 0$ then $M$ satisfies a fill radius bound depending on $\alpha$.

There is another "motivation" for this conjecture: Let $(\Sigma, \partial\Sigma)$ be a stable minimally immersed surface with boundary in a Riemannian four-manifold $(M, g)$. Using an averaging technique, the Gauss equation and an appropriate choice of variational vector field, the second variation formula for area and stability can be used to show that for any smooth function of compact support $f \in C_0^\infty(\Sigma)$ we have the inequalities:

$$\int_\Sigma \Big( |\nabla f|^2 + f^2 \big(K - K_\nu - \frac{1}{2}(\mathrm{Ric}_{11} + \mathrm{Ric}_{22})\big)\Big) e^{\sigma} da \geq 0 \tag{3.12}$$

and

$$\int_\Sigma \Big( |\nabla f|^2 + f^2 \big(K + K_\nu - \frac{1}{2}(\mathrm{Ric}_{11} + \mathrm{Ric}_{22})\big)\Big) e^{-\sigma} da \geq 0. \tag{3.13}$$

Here $K$ is the Gauss curvature on $\Sigma$ (in the induced metric), $K_\nu$ is the curvature of the normal bundle, Ric is the Ricci curvature on $M$, $\{e_1, e_2\}$ is an orthonormal frame on $\Sigma$ and $\sigma$ is a function on $\Sigma$ that satisfies $\Delta\sigma = K_\nu$. Suppose that $(M, g)$ has 2-positive Ricci curvature bounded below by $\alpha > 0$. Then,

$$\mathrm{Ric}_{11} + \mathrm{Ric}_{22} \geq \alpha.$$

In the case that the normal bundle is flat or, more generally, that the function $\sigma$ has suitably small oscillation, equations (3.12) and (3.13) imply:

$$\int_\Sigma |\nabla f|^2 + f^2 \big(K - \frac{1}{2}(\mathrm{Ric}_{11} + \mathrm{Ric}_{22})\big) da \geq 0 \tag{3.14}$$

Then (3.14) can be used in the above argument from [S-Y2] to prove a diameter bound on $\Sigma$ in the induced metric. This implies a fill radius bound and hence Conjecture 2. However, it can be shown that this line of reasoning does not, in general, hold, the obstruction being the normal curvature. This does not mean that a stable minimally immersed surface in a Riemannian four-manifold $(M, g)$ with 2-positive Ricci curvature bounded below does not satisfy a diameter bound. Rather that the above reasoning does not apply. It remains an interesting, if unexploited, fact that the 2-positive Ricci curvature occurs in an averaged version of the second variation formula.

In the next section we will show that Conjecture 2 implies Conjecture 1 (in the four dimensional case).

## 4. The fundamental group and fill radius bounds

The main theorem on the relation between fill radius and the fundamental group can be stated:

**Theorem 4.1.** *Let $M$ be a closed Riemannian $n$-manifold. Suppose that the universal cover $\pi : \tilde{M} \to M$ is given the Riemannian metric $\tilde{g}$ such that $\pi$ is a local isometry. If $(\tilde{M}, \tilde{g})$ has bounded fill radius then the fundamental group of $M$ is virtually free.*

Note, in particular, if a curvature condition implies a fill radius bound then a closed Riemannian manifold, satisfying this curvature condition, satisfies the hypotheses of the theorem. The proof of this result is somewhat technical and is done in [R-W]. In this survey we describe the main ideas and give a complete proof under the additional hypothesis that $\pi_1(M)$ is torsion free.

One of the main ideas in the proof of Theorem 4.1 concerns the number of ends of a group.

**Definition 4.1.** Given a group $G$ we define the number of ends, $e(G)$, of $G$ to be the number of geometric ends of $\tilde{K}$, where $\tilde{K} \to K$ is a regular covering of the finite simplicial complex $K$ by the simplicial complex $\tilde{K}$ and $G$ is the group of covering transformation.

In particular, if $G$ is the fundamental group of a closed manifold $M$ then the number of ends of $G$ is the number of ends of the universal cover $\tilde{M}$ of $M$. It is clear that a group $G$ with no ends is virtually trivial and hence finite. It can be shown [E] that a group with two ends is virtually infinite cyclic. A group with three ends, in fact, has infinitely many ends. This can be seen by using the Deck transformations on the universal cover. Thus a group $G$ can have 0,1,2 or infinitely many ends.

Our first result is on the fill radius and the number of ends of subgroups of the fundamental group. We assume that $\pi_1(M)$ is torsion free though the result holds without this assumption [R-W]. We begin with a lemma.

**Lemma 4.2.** *Let $M$ be a closed manifold. Suppose that $N \to M$ is a covering of $M$ such that $N$ has fundamental group $G$ that is finitely generated and has exactly one end. Let $\gamma$ be a simple closed curve in $N$ that represents an infinite order generator $[\gamma]$ of $G$. Let $\tilde{M} \to N$ be the universal cover and let $\tilde{\gamma}$ be the lift of $\gamma$ to $\tilde{M}$. Then the two ends of $\tilde{\gamma}$ lie in the same end of $\tilde{M}$.*

*Proof.* There is a finite simplicial complex $K$ with regular covering $\tilde{K}$ such that $G$ acts as the group of covering transformations. There is an imbedding $\imath : K \to N$ that induces an epimorphism of fundamental groups. In particular, the generators of $G$ all lie in $K$. Then there is an imbedding $\tilde{\imath} : \tilde{K} \to \tilde{M}$. If $B \subset \tilde{M}$ is compact then $\tilde{\imath}^{-1}(B) \subset \tilde{K}$ is compact.

Let $\gamma$ be a simple closed curve in $N$ that represents an infinite order generator $[\gamma]$ of $G$. After a homotopy the lift $\tilde{\gamma}$ can be assumed to lie in $\tilde{K}$. Since $G$ has exactly one end, any two points on $\tilde{\gamma}$, not in $\tilde{\imath}^{-1}(B)$, can be joined by a curve $\alpha$ in $\tilde{K} \setminus \tilde{\imath}^{-1}(B)$. The curve $\tilde{\imath}(\alpha)$ then lies in $\tilde{M} \setminus B$ and joins points on $\tilde{\gamma}$ not in $B$. Since this is true for any compact set $B$ the conclusion follows. □

**Proposition 4.3.** *Let $M$ be a closed Riemannian manifold with torsion free fundamental group. Suppose that the universal cover $\tilde{M}$ of $M$ has the property that the fill radius of every null homotopic simple closed curve is uniformly bounded above. Then no finitely generated subgroup of $\pi_1(M)$ has exactly one end.*

*Proof.* Assume, by way of contradiction, that a finitely generated subgroup $G$ of $\pi_1(M)$ has exactly one end. Let $N$ be a covering of $M$ with fundamental group $\pi_1(N)$ isomorphic to $G$. Since $\pi_1(M)$ is torsion free every element of $G$ is of infinite order. Let $g \in G$ be a generator.

Denote by $\gamma$ a closed minimal geodesic in $N$ that represents $g$. Let $p : \tilde{M} \to N$ be the universal cover and let $\tilde{\gamma}$ be the geodesic line that is a lift to $\tilde{M}$ of $\gamma$. Let

$x \in \tilde{\gamma}$ and $B_R(x) \subset \tilde{M}$ be the metric ball of radius $R$, center $x$. Then because $G$ has exactly one end by Lemma 4.2 both ends of $\tilde{\gamma}$ in $\tilde{M} \setminus B_R(x)$ lie in the same end of $\tilde{M}$. The geodesic line $\tilde{\gamma}$ consist of two geodesic rays $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ beginning at $x$. For $i = 1, 2$, choose a point $p_i \in \tilde{M} \setminus B_R(x)$ along $\tilde{\gamma}_i$ and denote the segment of $\tilde{\gamma}_i$ from $x$ to $p_i$ by $\tau_i$. Since $p_1$ and $p_2$ lie in the same end there is a curve $\beta \subset \tilde{M} \setminus B_R(x)$ joining $p_1$ and $p_2$. Denote the closed curve $\tau_1 \cup \beta \cup \tau_2$ by $\eta$. Since $\tilde{M}$ is simply connected $\eta$ is null homotopic and has fill radius greater than $\frac{R}{2}$. For sufficiently large $R$ this contradicts the fill radius bound. □

To illustrate the use of Proposition 4.3 we prove the following special case of Theorem 4.1.

**Theorem 4.4.** *Let $M$ be a closed Riemannian manifold with torsion free fundamental group. Suppose that the universal cover $\tilde{M}$ of $M$ has uniformly bounded fill radius. Then $\pi_1(M)$ is a free group of finite rank.*

We will need the following theorem of Stallings [St]: *If $G$ is a torsion-free, finitely generated group with infinitely many ends then $G$ is a non-trivial free product.*

*Proof.* Applying Stallings' theorem to $G = \pi_1(M)$, we have $G \simeq G_1 * G_2$, where each $G_i$ is finitely generated (by Grushko's Theorem, see [Ma]). Each $G_i$ has either two or infinitely many ends (by Theorem 4.3). Then apply Stallings theorem to each $G_i$ with infinitely many ends and iterate. By Grushko's Theorem, this process terminates after finitely many steps resulting in $G \simeq G_1 * \cdots * G_k$, where each $G_i$ is finitely generated and has two ends. Since a torsion-free, finitely generated group with two ends is infinite cyclic, we conclude that $G = \pi_1(M)$ is a free group of finite rank. □

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