F, G, H, I Bases for Polynomial Rings and their Relations

Wenyuan Wu

Department of Mathematics, MSU.

joint work with Greg Reid (UWO, Canada)

AMS 2009, Washington DC, USA

January 7, 2009

F, G, H, I Bases in Polynomial Rings

G Basis: Gröbner Basis, foundation of computer algebra

H Basis: a weakened **G** Basis, a "bridge" between computer algebra in polynomial rings (infinite dim) and linear algebra (finite dim).

F Basis: Formally Integrable basis, contain all the integrability conditions of PDE.

I Basis: Involutive Basis, provides E-U theorem of local solutions of PDE in jet space.

Why we study these bases?

- Many applications in PDE and DAE:
  Determination of Approximate Symmetries of Differential Equations (Bonasia, Lemaire, Reid, Scott and Zhi). Determination of hidden constraints for mechanical systems (Zhou, Jeffrey, Reid). Numerical completion of PDE (Wu and Reid ISSAC'06).

- Apply the theories (techniques) of computer algebra to linear PDEs and the PDE theory can also be applied to polynomial rings.
  Example: Camera Pose Problem, to determine the orientation of a camera from reference points (Reid, Tang and Zhi ISSAC'03).

- Geometric method to find Hilbert Functions and Applications to Approximate Commutative Algebra (Scott, Reid, Wu and Zhi ApCoA'08)

Computational Methods for Polynomial Systems

The most important information of polynomial system is the geometric structure, such as dimension of zero sets, irreducible components ...
But sometimes we also concern about algebraic structures, such as ideals, the quotient rings and multiplicities.

- Methods to study the geometric structure (zero sets)?
  Symbolic methods: G basis, Triangular decomposition...
  Numerical methods: Newton method, Homotopy continuation method, Numerical Algebraic Geometry ...
- Methods to study the algebraic structure (ideals) ?
  Symbolic methods: Border Basis, **G** basis, ...
  Numerical methods: see the work by Bates, Peterson and Sommese et al; Leykin's talk about Numerical primary decomposition

Computation with Approx. Data

Because of the measurement errors (approx. data), some coefficients may be known with inaccuracy. The key operation in symbolic computation of polynomial ring is reduction (polynomial division). But this may be numerically unstable!

$(\sqrt{2}x^{16} + 3\sqrt{22}x^{15} + x + 3\sqrt{11}) \div (x + \sqrt{99}) = \sqrt{2}x^{15} + 1$ (Zeng's example)

A tiny perturbation cause huge error in the output:

$$(1.414213562\,x^{16} + x + 14.07124727\,x^{15} + 9.949874370) \div (x + 9.949874370)$$

$$= 1.41421356200000\,x^{15} - 0.00000000425020593942553\,x^{14} + 0.0000000422890151439119\,x^{13} - 0.000000420770387912951\,x^{12} + 0.00000418661249835003\,x^{11} - 0.0000416562683944546\,x^{10} + 0.000414474637247825\,x^{9} - 0.00412397057016718\,x^{8} + 0.0410329890787407\,x^{7} - 0.408273086359052\,x^{6} + 4.06226591792473\,x^{5} - 40.4190355408838\,x^{4} + 402.164325788359\,x^{3} - 4001.48451768992\,x^{2} + 39814.2682445147\,x - 396145.967166402$$

Main Ideas

How to explore the algebraic structure of polynomial systems with approx. coefficients?

- **F** and **I** Bases have tight connection with **H** and **G** Bases respectively.
- Using their geometrical definitions (dimension) to compute the **F** and **I**.
- Using the **F** and **I**, we can reduce the problems of polynomial systems to linear algebra. There are lots of techniques in this well-developed area to obtain numerically stable solutions.

G Basis

Let $F = \{f_1, ..., f_s\} \subset K[x_1, ..., x_n]$, the ideal generated by F is

$$I := \{f : \sum_{i=1}^{s} c_i f_i, c_i \in K[x_1, ..., x_n]\} \tag{1}$$

denoted by $I = \langle F \rangle$. F is a set of generators of I.

Definition 1 *[G_{tdeg} **Basis**] Let $G := \{g_1, ..., g_s\} \subset K[x_1, ..., x_n]$ is a G basis of ideal $I = \langle G \rangle$ with total degree order if for all nonzero $p \in I$, the highest degree term of p is divisible by the highest degree term of some g_i.*

H Basis

Definition 2 ***[H Basis]*** *$F = \{f_1, ..., f_s\} \subset K[x_1, ..., x_n]$ is an H-basis of ideal $I = \langle F \rangle$ if for all non zero $p \in I$, $\exists h_1, ..., h_s$:*

$$p = \sum_{i=1}^{s} h_i f_i$$

and $deg(h_i) + deg(f_i) \leqslant deg(p), i = 1, ..., s.$

Example : easy to check that

$$\{y^2 - x, xy - x^2\} \tag{2}$$

is an H Basis of $I = \langle y^2 - x, xy - x^2 \rangle$, and the corr. Gröbner basis with total degree order $(y > x)$ is:

$$\{y^2 - x, xy - x^2, x^3 - x^2\} \tag{3}$$

Obviously, any such G_{tdeg} is an H Basis.

Involutive Bases

Definition 3 ***[Involutive System]*** *A system of linear homogeneous PDE* $\mathsf{R} = 0$ *with constant coefficients is involutive if* $\dim \pi\mathbf{D}(\mathsf{R}) = \dim \mathsf{R}$ *and the symbol of* R *is involutive.*

Note: Consider all the derivatives (monomials) as formal variables in jet space. So R is a linear system. And $\dim \mathsf{R}$ means the dimension of its null-space. The symbol of R is sub-matrix with respect to the highest order derivatives.

Definition 4 ***[Formally integrable System]*** *A q-th order system of linear homogeneous PDE R is formally integrable if for any $k \geqslant 0$:*

$$\dim\left(\mathbf{D}R\right) = \dim \pi^{k}\left(\mathbf{D}^{k+1}R\right). \tag{4}$$

Note: **D** is prolongation (differentiate) of PDE or equivalently each polynomial is multiplied by $\{x_1, ..., x_n\}$; π is projection of jet varieties or equivalently elimination of the highest degree terms in polynomial systems.

Proposition 1 *Involutive implies formally integrable. (Pommaret 1979, Kuranishni-Cartan Prolongation Theorem 1957)*

Embedding of F and I Bases in Polynomial Rings

F and **I** Bases are derived from formal theory of PDE and are embedded in polynomial rings by the bijection:

$$\phi : x_i \leftrightarrow \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq n \tag{5}$$

$K[x_1, ..., x_n] \simeq$ Linear homogenous PDEs with constant coeffs.
Example: $xy - y^2 + 5\,x - 2 \mapsto u_{xy} - u_{yy} + 5\,u_x - 2\,u$
$x \cdot (xy - y^2 + 5\,x - 2) \mapsto \frac{\partial}{\partial x}(u_{xy} - u_{yy} + 5\,u_x - 2\,u)$

Relation between H Bases and F Bases

Theorem 1 *If H is an **H** Basis, then $D^0 H$ is formally integrable. If F is formally integrable, then F_{red} is an **H** Basis.*

D^0: differentiate all lower order eqns up to the order of this system.

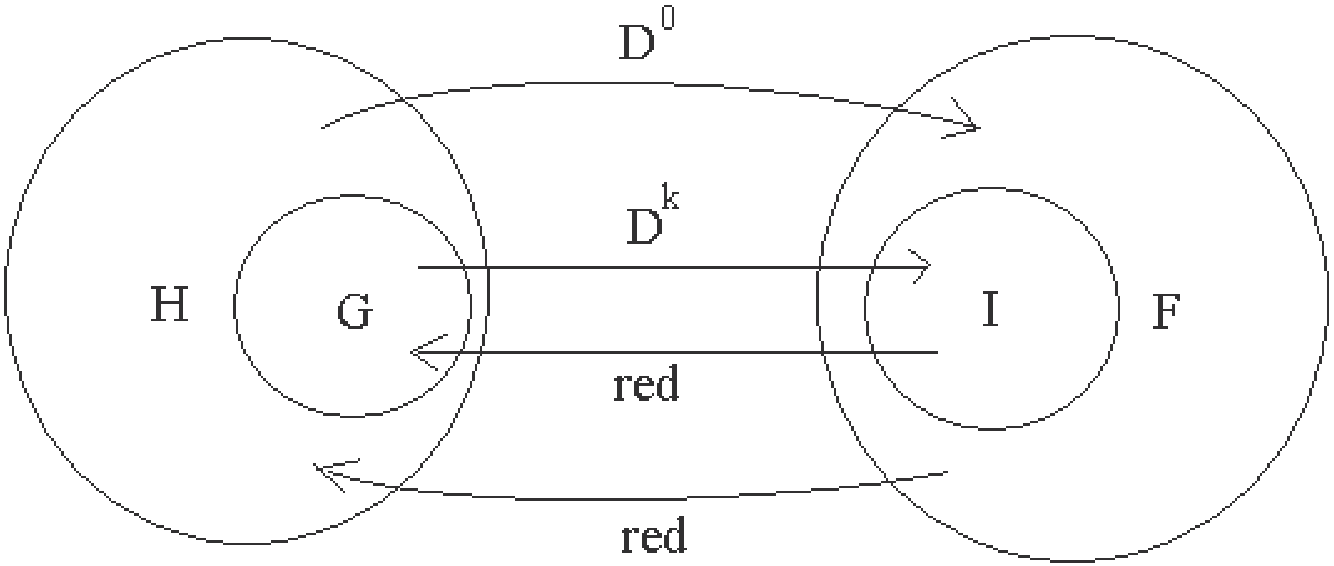
Note: F Basis $\not\Rightarrow$ H Basis.
Example: $\langle x^2 - x, x^2 - y, x^2 - xy, xy - y^2 \rangle$
is an I (Involutive) Basis and an F (Formally integrable) Basis.
However $x - y \in \langle x^2 - x, x^2 - y, x^2 - xy, xy - y^2 \rangle$. Because $x - y$ has degree 1 and the input polynomials have degree 2, it cannot satisfy the Criterion of an H Basis.

Relation between G Bases and I Bases

Theorem 2 *If G is an* ***G*** *Basis, then there exists a finite number k, s.t. $D^k G$ is involutive. If I is an involutive basis, then there exists an ordering s.t. I_{red} is a* ***G*** *Basis.*

These results can be generalized to modules over polynomial rings.

Roughly: the relations between **F, G, H, I** Bases in 0-dim



- red: to reduce the basis with total degree order (this projection could be done by using SVD).
- D^k : prolong the system k times.
- D^0: differentiate all lower order equations up to the order of this system.

Special case: Complete Intersection

In Möller and Sauer [**?**], there is simple H Basis criterion and the determination of the Hilbert function for 0 dimensional polynomial systems. In this case, it is easy to find F,G,H and I bases.

Proposition 2 *Suppose* $F = \{f_1, ..., f_n\} \subseteq K[x_1, ..., x_n]$ *with* $d_i = deg(f_i)$, $q = max\{d_i\}$ *and let* $F^H := \{f_1^H, ..., f_n^H\}$ *be the leading homogenous part of* F*.*
If F^H *only has the point* $(0, ..., 0)$ *as a common zero then:*

1. F *is an* H *Basis.*
2. *for* $d \geq \hat{d} = \sum d_i - n + 1$, $\dim NullSpace(Sym_d) = 0$
3. *Let* $R = \phi(F)$, $D^{\hat{d}-q}R$ *is involutive.*

Random (generic) square polynomial systems

Proposition 3 *A generic polynomial system with* n *equations and* n *variables only has the point* $(0, ..., 0)$ *as a common zero of its leading homogeneous part* F^H*.*

Remark: A generic square polynomial system will be involutive after a certain number of prolongation steps (no projections!)

Future Work

- a system with some special structure will degenerate to the generic case under a small perturbation!
- challenging problems: how to discover whether there exist some nearby system with special structures? How to reconstruct these structures ?
- our idea: introduce approximate involutive basis by using SVD (See our ApCoA'08 paper for 0-dim case and positive dim systems with 2 variables)

Main obstacle is the huge size of the matrices when # var is big!