Name (Print Clearly): Student Number:

MTH132 Section 1 & 12, Test 1

Feb. 2, 2009 Instructor: Dr. W. Wu

Instructions: Answer the following questions in the space provided. There is more than adequate space provided to answer each question. The total time allowed for this quiz is **50** minutes.

1 [5 pts each]. Find the following limits.

(a) $\lim_{x\to 0}\frac{\sqrt{x^2+2}-\cos x}{x^2+x-1}$

$\overset{DSP}{=} \frac{\sqrt{0+2}-\cos 0}{0+0-1} = \frac{\sqrt{2}-1}{-1} = 1-\sqrt{2}$

(b) $\lim_{x\to 3}\frac{x^2-x-6}{x^2-3x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 3}\frac{(x+2)(x-3)}{x(x-3)} = \lim_{x\to 3}\frac{x+2}{x} \overset{DSP}{=} \frac{3+2}{3} = 5/3$

(c) $\lim_{x\to \infty}\frac{x^2-x-6}{x^2-3x}$

$= \lim_{x\to\infty}\frac{(x^2-x-6)/x^2}{(x^2-3x)/x^2} = \lim_{x\to\infty}\frac{1-1/x-6/x^2}{1-3/x} = \frac{1-0-0}{1-0} = 1$

(d) $\lim_{x\to 3^-}\frac{x^2-6}{x^2-3x}$ $\left(\frac{3}{0}\right)$

$x-3<0,\ x>0,\ x^2-6>0$

$= \lim_{x\to 3^-}\frac{x^2-6}{x(x-3)} = -\infty$

$\frac{+}{-} = -$

(e) $\lim_{x\to 0} x\cot(3x)$

$$= \lim_{x\to 0} x \cdot \frac{\cos 3x}{\sin 3x} = \lim_{x\to 0} \frac{3x}{\sin 3x} \cdot \frac{\cos 3x}{3}$$

$$= \lim_{x\to 0} \frac{3x}{\sin 3x} \cdot \lim_{x\to 0} \frac{\cos 3x}{3} = 1 \cdot \frac{\cos 0}{3} = \frac{1}{3}$$

2 [5 pts each]. If $f(x) = \sqrt{2x}$, please evaluate the limit of the form $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$,

(a) when $x = 2$

$$f(x+h) = \sqrt{2(x+h)} = \sqrt{4+2h}, \quad f(x) = \sqrt{2x} = 2$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{4+2h}-2}{h} = \lim_{h\to 0} \frac{(\sqrt{4+2h}-2)(\sqrt{4+2h}+2)}{h\,(\sqrt{4+2h}+2)}$$

$$= \lim_{h\to 0} \frac{2h}{h(\sqrt{4+2h}+2)} = \lim_{h\to 0} \frac{2}{\sqrt{4+2h}+2} \overset{DSP}{=} \frac{2}{\sqrt{4}+2} = \frac{1}{2}$$

(b) when $x = x_0$.

$$f(x+h) = \sqrt{2x_0+2h}, \quad f(x) = \sqrt{2x_0}$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2x_0+2h}-\sqrt{2x_0}}{h} = \lim_{h\to 0} \frac{(\sqrt{2x_0+2h}-\sqrt{2x_0})(\sqrt{2x_0+2h}+\sqrt{2x_0})}{h\cdot(\sqrt{2x_0+2h}+\sqrt{2x_0})}$$

$$= \lim_{h\to 0} \frac{2h}{h(\sqrt{2x_0+2h}+\sqrt{2x_0})} = \frac{2}{\sqrt{2x_0}+\sqrt{2x_0}} = \frac{1}{\sqrt{2x_0}}$$

3 [10 pts]. Please use the **intermediate value theorem** to show that the equation $\cos(2x) - 2\sin x = 0$ has a solution in $[0, \pi/2]$. $\cos 2x = 2\sin x \Leftrightarrow \cos 2x - 2\sin x = 0$

① $f(x) = \cos(2x) - 2\sin x$, $\cos 2x$ and $\sin x$ are continuous so $f(x)$ is continuous on $[0, \pi/2]$

② $f(0) = \cos 0 - 2\sin 0 = 1$

$f(\pi/2) = \cos\pi - 2\cdot\sin\pi/2 = -1 - 2 = -3$

③ $f(0) > 0 > f(\pi/2)$. by IVT there exists $c \in [0, \pi/2]$ s.t. $f(c) = 0$. So c is a root of this equation

4 [5 pts each]. Suppose $y = \dfrac{x^2 - 5}{x - 2}$.

4 Pts (a) Use polynomial division to decompose this rational function into a polynomial and a remainder.

$$\begin{array}{r|l} & x + 2 \\ x-2 & x^2 - 5 \\ & -\left(x^2 - 2x\right) \\ \hline & 2x - 5 \\ & -\left(2x - 4\right) \\ \hline & -1 \end{array}$$

so $\dfrac{x^2-3}{x-2} = (x+2) - \dfrac{1}{x-2}$

6 Pts (b) Find the vertical asymptote and oblique asymptote of this function.

$\lim\limits_{x\to\infty} \dfrac{-1}{x-2} = 0$, $\lim\limits_{x\to-\infty} \dfrac{-1}{x-2} = 0$, so $y \approx x+2$ as $x \to \pm\infty$

This implies $y = x+2$ is an oblique asymptote.

$\lim\limits_{x\to 2^+} \dfrac{-1}{x-2} = -\infty$, $\lim\limits_{x\to 2^-} \dfrac{-1}{x-2} = +\infty$, so $y \approx \dfrac{-1}{x-2}$ as $x \to 2$

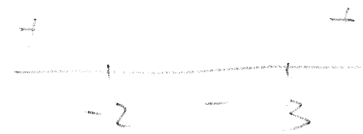
and $x = 2$ is a vertical asymptote.

5 [5 pts each]. At which points are each of the following functions continuous?

(a) $y = \dfrac{x^2 - 4}{\sqrt{x^2 - x - 6}}$

find its domain:

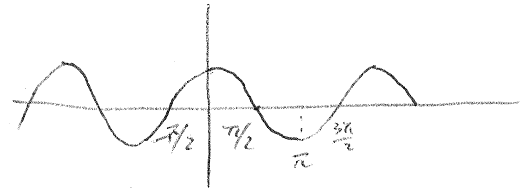
$x^2 - x - 6 > 0 \Rightarrow (x+2)(x-3) > 0$



$\Rightarrow x > 3$ or $x < -2$.

this function is continuous on $(-\infty, -2) \cup (3, \infty)$

(b) $y = \dfrac{x+2}{\cos x}$



find its domain :

$\cos x \neq 0, \Rightarrow x \neq \frac{\pi}{2} + n\pi, \quad n \in \{0, \pm 1, \pm 2, \ldots\} = \mathbb{Z}$

this function is continuous on $\mathbb{R} / \{\frac{\pi}{2} + n\pi : n \in \mathbb{Z}\}$

(all real numbers except $\frac{\pi}{2} + n\pi$)

6 [5 pts]. Use the Sandwich Theorem to find $\lim\limits_{x\to\infty} \dfrac{\cos(x^2)}{x}$.

$-1 \leq \cos(x^2) \leq 1$, $x \to \infty$, so $x > 0$.

$\dfrac{-1}{x} \leq \dfrac{\cos(x^2)}{x} \leq \dfrac{1}{x}$

$\lim\limits_{x\to\infty} \dfrac{-1}{x} = 0$, $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$.

by the Sandwich Th $\lim\limits_{x\to\infty} \dfrac{\cos(x^2)}{x} = 0$