1. (24 points) Compute each of the following limits or show that they do not exist. **Show your work.**

(a) $\lim_{x\to -3} \frac{x+3}{x^2+7x+12}$ subs $x=-3$, $\frac{0}{0}$

$$\overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to -3} \frac{1}{2x+7} \overset{DSP}{=} \frac{1}{-6+7} = 1$$

or $$\lim_{x\to -3} \frac{x+3}{(x+3)(x+4)} = \lim_{x\to -3} \frac{1}{x+4} = 1$$

$\boxed{1}$

(b) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$ subs $x=9$. $\frac{0}{0}$

$$\overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 9} \frac{\frac{1}{2}x^{-1/2}}{1} \overset{DSP}{=} \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

or $$\lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)}$$

$$= \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{6}$$

$\boxed{1/6}$

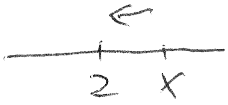
(c) $\lim_{x\to 0} \frac{x}{\sin 5x}$

$$\overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{1}{5\cos 5x} \overset{\frac{0}{0}}{\underset{}{\overset{DSP}{=}}} \frac{1}{5}$$

or $$\lim_{x\to 0} \left(\frac{5x}{\sin 5x} \cdot \frac{1}{5}\right) = 1 \cdot \frac{1}{5} = \frac{1}{5}$$

$\boxed{1/5}$

(d) $\lim_{x\to 2^+} \frac{|x-2|}{2-x}$



$x > 2 \Rightarrow x-2 > 0 \Rightarrow |x-2| = x-2$

$$= \lim_{x\to 2^+} \frac{x-2}{2-x}$$

$$= -1$$

$\boxed{-1}$

2. (24 points) Find the derivative of each of the following functions. **Do not simplify.**

(a) $f(x) = x^2\sqrt{\sin x}$

$(f \cdot g)' = f' g + f \cdot g'$, $(f^n)' = n f^{n-1} \cdot f'$

$f' = 2x \cdot \sqrt{\sin x} + x^2 \cdot \frac{1}{2}(\sin x)^{-1/2} \cdot \cos x$

(b) $g(x) = \dfrac{x^2+x+1}{\sqrt{x^2+1}}$

$\left(\frac{f}{g}\right)' = \frac{f' g - f \cdot g'}{g^2}$

$$g' = \frac{(2x+1)\sqrt{x^2+1} - (x^2+x+1) \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x}{x^2+1}$$

(c) $p(x) = \displaystyle\int_1^{\sqrt{x}} \sin^5 t \, dt$

$\sqrt{x}$ is the inside function

FTC I. $\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x)) \cdot b' - f(a(x))a$

$$f' = (\sin\sqrt{x})^5 \cdot (\sqrt{x})' - 0$$
$$= (\sin\sqrt{x})^5 \cdot \frac{1}{2} \frac{1}{\sqrt{x}}$$

3. (10 points) Let $f(x) = \dfrac{1}{3x}$. Use the *definition of the derivative* to compute $f'(2)$. **Show your work.**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{3(x+h)} - \frac{1}{3x}}{h} = \lim_{h \to 0} \frac{x - (x+h)}{3x(x+h) \cdot h}$$

$$= \lim_{h \to 0} \frac{-h}{3x(x+h) \cdot h} = \lim_{h \to 0} \frac{-1}{3x(x+h)} = \frac{-1}{3x^2}$$

$$f'(2) = \frac{-1}{3 \cdot 2^2} = -\frac{1}{12}$$

4. (14 points) Evaluate each of the following indefinite integrals. **Show your work.**

(a) $\int x^2\sqrt{1+10x^3}\,dx$ let $t = 1+10x^3$

$dt = 30x^2\,dx$

$= \int \sqrt{t}\cdot\frac{1}{30}\cdot dt$

$= \frac{1}{30}\int t^{1/2}\,dt = \frac{1}{30}\,\frac{2}{3}\,t^{3/2} + c$

$= \frac{1}{45}(1+10x^3)^{3/2} + C$

(b) $\int \frac{x}{\sqrt{x+1}}\,dx$ let $t = x+1$, $, x = t-1$

$dt = dx$

$= \int \frac{t-1}{\sqrt{t}}\,dt$

$= \int \sqrt{t}\,dt - \int \frac{1}{\sqrt{t}}\,dt = \frac{2}{3}t^{3/2} - 2t^{1/2} + c$

$= \frac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2} + C$

5. (14 points) Evaluate each of the following definite integrals. **Show your work.**

(a) $\int_0^1 (x^2+1)(3x-2)\,dx$

$= \int_0^1 (3x^3+3x-2x^2-2)\,dx$

$= \left[\frac{3}{4}x^4 + \frac{3}{2}x^2 - \frac{2}{3}x^3 - 2x\right]_0^1$

$= \left(\frac{3}{4} + \frac{3}{2} - \frac{2}{3} - 2\right) - 0 = -\frac{5}{12}$

(b) $\int_0^{\pi} \sin x\,\cos^2 x\,dx$ $t = \cos x$, $dt = -\sin x\,dx$

$= \int_1^{-1} -t^2\,dt$

$= \int_{-1}^{1} t^2\,dt = \frac{1}{3}t^3\Big|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right)$

$= \frac{2}{3}$

6. (12 points) Find the equation of the tangent line to the curve $x^3 + y^3 = 9xy$ at the point (4, 2). **Show your work.**

use implicit differentiation:

$$3x^2 + 3y^2 \cdot y' = 9(y + xy')$$

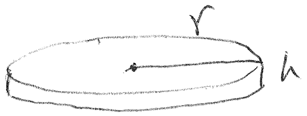
$$3y^2 \cdot y' - 9xy' = 9y - 3x^2$$

$$y' = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$$

at (4, 2) $y' = \frac{6 - 16}{4 - 12} = \frac{-10}{-8} = \frac{5}{4}$ = slope of the tangent

tangent line: $y - 2 = \frac{5}{4}(x - 4)$ $\quad (y - y_0 = m(x - x_0))$

7. (16 points) 100 m^3 of oil is spilled when a tanker collides with a tuna boat. The resulting oil slick forms a right circular cylinder on the surface of the water. If the thickness (h) of the slick is *decreasing* at a rate of 0.001 m/sec, how fast is the radius (r) increasing when the slick is 0.01 m thick? *Note:* $V = \pi r^2 h$



$$\left.\begin{array}{l} V = \pi r^2 h = 100 \\ h = 0.01 \end{array}\right\} \Rightarrow \pi r^2 = 10^4 \Rightarrow r = \frac{100}{\sqrt{\pi}}$$

given info : $\frac{dh}{dt} = -0.001$.

unknown : $\frac{dr}{dt} = ?$

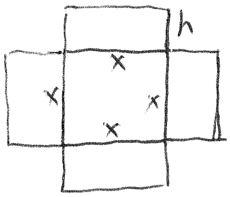
V is a constant

$$\therefore \frac{dV}{dt} = \pi\left(2r\frac{dr}{dt} \cdot h + r^2\frac{dh}{dt}\right) = 0 .$$

$$2\, \frac{100}{\sqrt{\pi}} \cdot 0.01 \cdot \frac{dr}{dt} + \frac{10^4}{\pi} \cdot (-0.001) = 0$$

$$\therefore \frac{dr}{dt} = \frac{10}{\pi} \cdot \frac{\sqrt{\pi}}{2} = \frac{5}{\sqrt{\pi}}$$

8. (16 points) A rectangular box with volume 18 ft^3 is to be built with a **square** base and *NO top.* The material used for the bottom panel costs \$2 per ft^2 while the material used for the side panels costs \$1.50 per ft^2. Find the minimum cost of such a box. **Justify your answer using the methods of calculus.**



h : height of box.

x : length of box (square base)

total cost $= \$2 \cdot x^2 + 4 \cdot x \cdot h \cdot \$1.5 = 2x^2 + 6xh$ (dolla

Volume $= 18 = x^2 \cdot h \Rightarrow h = \frac{18}{x^2}$

total cost $= 2x^2 + \frac{108}{x} = C(x) \Rightarrow$ minimize $C(x)$, $x > 0$

$C' = 4x - \frac{108}{x^2} = 0 \Rightarrow x^3 = 27 \Rightarrow x = 3$, $h = \frac{18}{3^2} = 2$.

total cost $= C(3) = 2 \cdot 9 + \frac{108}{3} = 54$.

− +

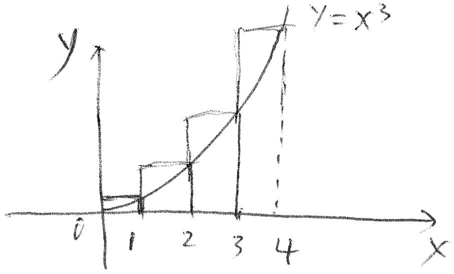
sign of C' 3

(3, 54) is the local min.

also the global min.

9. (8 points) Set up a Riemann sum approximation to the integral below by partitioning the interval [0, 4] into 4 subintervals of equal length and using the right end point x_k of each subinterval to calculate the height of the corresponding rectangle. *DO NOT EVALUATE THE SUM.*

$$\int_0^4 x^3\,dx$$



$\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$

$x_k = a + k\Delta x = 0 + k \cdot 1 = k$

$C_k = x_k = k$. (use the right end point)

$f(C_k) = k^3$ $(f(x) = x^3)$

$S = \sum_{k=1}^{4} f(C_k) \cdot \Delta x = \sum_{k=1}^{4} k^3$

10. (16 points) Let $f(x) = x^2 - 4x + 3$. The graph of $y = f(x)$ is shown below.

(a) Compute $\int_0^5 f(x)\,dx$.

$$\int_0^5 (x^2 - 4x + 3)\,dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_0^5 = \frac{20}{3}$$

(b) Find the total **area** of the shaded region.

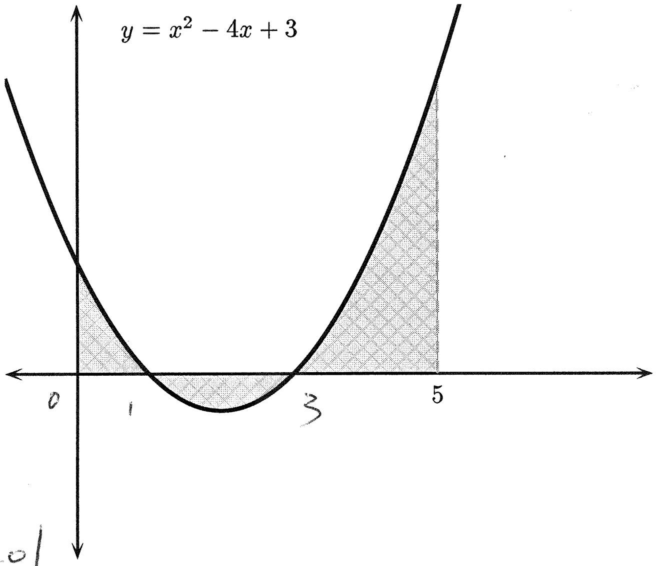
$$x^2 - 4x + 3 = 0 \Rightarrow x = 1, 3$$

$$\int_0^1 f(x)\,dx = 4/3$$

$$\int_1^3 f(x)\,dx = -4/3$$

$$\int_3^5 f(x)\,dx = \frac{20}{3}$$

$$\text{Total area} = \left|\frac{4}{3}\right| + \left|-\frac{4}{3}\right| + \left|\frac{20}{3}\right| = \frac{28}{3}$$



11. (18 points) Let $f(x) = \frac{x^2}{(x-3)^2}$. Answer the questions below. **Show all reasoning using the methods of calculus.**

Note: $f'(x) = \frac{-6x}{(x-3)^3}$ and $f''(x) = \frac{12x+18}{(x-3)^4}$.

(a) Find all points where f is not continuous.

when $x = 3$, $f(x)$ is undefined

∴ f is not continuous at 3.

(b) Find the intervals where f is increasing and the intervals where f is decreasing.

$f' = \frac{-6x}{(x-3)^3} \Rightarrow$ critical pts: $x = 0, 3$

sign of f': $- \quad 0 \quad + \quad 3 \quad -$

increasing: $(0, 3)$

decreasing: $(-\infty, 0) \cup (3, \infty)$

(c) Find the intervals where f is concave up and the intervals where f is concave down.

$f'' = \frac{12x+18}{(x-3)^4} \Rightarrow x = -\frac{3}{2}, 3$.

sign of f'': $- \quad -3/2 \quad + \quad 3 \quad +$

concave up: $(-3/2, 3) \cup (3, \infty)$

concave down: $(-\infty, -3/2)$

(d) Find all local extrema.

local minimum at $x = 0$. ("−" to "+")

when $x = 3$, $f(x)$ is undefined.

(e) Find all inflection points.

$x = -3/2$ (f'' changes sign)

(f) Find the equation(s) of all asymptotes.

vertical asymptote $x = 3$, ($\lim_{x\to 3} f(x) = \infty$)

horizontal asymptote $y = 1$.

($\lim_{x\to\infty} f(x) = \underset{\substack{\uparrow \\ L_1}}{1}$, $\lim_{x\to-\infty} f(x) = \underset{\substack{\uparrow \\ L_2}}{1}$)

if $L_1 \neq L_2$, then $y = L_1$, $y = L_2$ are horizontal asymptotes.

12. (16 points) Solve the initial value problem below.

$$\frac{dy}{dx} = 3\sin 2x + 6, \quad y(0) = 1$$

$$y = \int (3\sin 2x + 6)\,dx = 3\int \sin 2x\,dx + 6\int dx$$

$$= -\frac{3}{2}\cos 2x + 6x + C$$

$$y(0) = -\frac{3}{2}\cos 0 + 0 + C = -\frac{3}{2} + C = 1 \Rightarrow C = \frac{5}{2}$$

$$\therefore y = -\frac{3}{2}\cos 2x + 6x + 5/2$$

13. (12 points) Let $f(x) = 3x^2 + 5x - 9$.

(a) Explain why f satisfies the hypotheses of the Mean Value Theorem over the interval [0, 3].

$f(x) = 3x^2 + 5x - 9$ is a polynomial which is continuous and differentiable over $\mathbb{R}$, and $f' = 6x + 5$

so f is continuous on $[0, 3]$. and differentiable on $(0, 3$

(b) Find a point $c \in (0, 3)$ such that the slope of the tangent line at $(c, f(c))$ is equal to the slope of the line containing the points $(0, -9)$ and $(3, 23)$.

slope of the line containing $(0, -9)$. $(3, 23)$

is $\frac{23-(-9)}{3-0} = \frac{32}{3}$ $\left(\frac{f(b)-f(a)}{b-a} = f'(c) \right)$

$f'(c) = 6c + 5 = \frac{32}{3} \Rightarrow c = \frac{17}{18}$

$\frac{17}{18} \in (0, 3)$.