My research focuses on cluster theory and its application to other branches of mathematics, including algebraic geometry, symplectic geometry, representation theory, and mathematical physics.

(I) Cluster Theory and DT Transformation. Fomin and Zelevinsky [FZ02] introduced cluster algebras at the turn of the century. A cluster algebra is typically defined by a mutation equivalent family of quivers, and is generated by a distinguished set of generators called cluster variables, which are grouped into overlapping clusters.

Fock and Goncharov [FG09] introduced cluster varieties, the geometric counterparts of cluster algebras. Cluster varieties come in pairs called cluster ensembles, each consisting of a cluster $K_2$ variety $\mathcal{A}$ and a cluster Poisson variety $\mathcal{X}$. Many interesting spaces, such as Grassmannians, flag varieties, and Teichmüller spaces, turn out to be examples of, or closely related to, cluster varieties.

Each cluster ensemble comes with a discrete group $\mathcal{G}$ called the cluster modular group and it acts on both of the cluster varieties in the ensemble. Elements of the cluster modular group $\mathcal{G}$ are called cluster transformations. In many known examples, there is an important central element in the cluster modular group called the cluster Donaldson-Thomas (DT) transformation, which is a manifestation the DT invariant of a certain 3d Calabi-Yau category [KS08, Kel17, GS18].

A few results from my early research projects were on the construction of cluster DT transformations on various families of examples of cluster varieties, including Grassmannians [Wen16b], double Bruhat cells of semisimple Lie groups [Wen16a, Wen20], and double Bott-Samelson cells of Kac-Moody groups [SW19]. I will give a more detailed description of these constructions in Sec. 1.

These constructions combined with results from [GHKK18] and [GS18] imply the Fock-Goncharov cluster duality for the corresponding cluster ensembles. The cluster duality asserts the existence of a canonical basis in the coordinate rings of either cluster variety in the ensemble and the basis elements are parametrized by the integral tropical points of the other cluster variety in the ensemble. These cluster canonical bases are related to the Mirkovic-Vilonen basis from geometric representation theory [Kam10], the Lusztig-Kashiwara canonical bases from the theory of quantum groups [KKKO18], and framed BPS states from mathematical physics [GMN13].

(II) Applications of Cluster Theory. In addition to contributing to the development of cluster theory, I have also applied results from cluster theory to solve problems in combinatorics, algebraic geometry, symplectic geometry, and representation theory. Here is a brief summary list of results I have obtained with my collaborators in the past.

- a new geometric proof of the Zamolodchikov’s periodicity conjecture in the cases of $\Delta \times A_r$ for general Dynkin diagrams $\Delta$ [SW19];
- the construction of the cluster duals (mirror Landau-Ginzburg models) of Grassmannians and a cyclic sieving phenomenon of plane partitions [SW20];
- for any positive braid Legendrian link, the construction of a cluster Poisson structure on its moduli space of microlocal rank-1 sheaves [SW19] and a cluster $K_2$ structure on its augmentation variety [GSW20a]; consequently, a proof of the existence of Legendrian links admitting infinitely many fillings, as well as a sufficient condition on when positive braid Legendrian links admit infinitely many fillings [GSW20b].

I will go over these applications in more details in Sec. 2, 3, and 4 respectively.
1. Constructions of Cluster Donaldson-Thomas Transformations

A cluster DT transformation can be described combinatorially as a reddening sequence [Kel17]. Geometrically the cluster DT transformation is a birational automorphism that acts on both cluster varieties in the unfrozen cluster ensemble [GS18]. In particular, the cluster DT transformation, if exists, is a cluster transformation uniquely characterized by the following property, where \( \text{ord}_xf \) denotes the order of the variable \( x \) in a rational expression \( f \):

\[
\text{ord}_x DT^*(X_j) = -\delta_{ij}.
\]

1.1. Grassmannians. Consider the affine cone of Grassmannian \( \mathcal{G}_m(n) := \text{SL}_m \setminus \text{Mat}_{m\times n}^{\text{full-ranked}} \). Inside \( \mathcal{G}_m(n) \) we define \( \mathcal{G}_m^x(n) \) to be non-vanishing locus of the product of consecutive minors. A result of [Sco06] implies that \( \mathcal{G}_m^x(n) \) is a cluster \( K_2 \) variety.

We construct the unfrozen cluster Poisson counterpart of \( \mathcal{G}_m^x(n) \) as the configuration space

\[
\text{Conf}_n^x(m) := \text{PGL}_m \left\{ \{l_1, \ldots, l_n\} \in (\mathbb{P}^{n-1})^n \mid l_i, \ldots, l_{i+m-1} \text{ are in general position with indices taken modulo } n \right\}.
\]

There is a natural projection map \( p : \mathcal{G}_m^x(n) \to \text{Conf}_n^x(m) \) defined by taking the linear span of each column vector. This projection map \( p \) also coincides with the cluster theoretical map \( p : \mathcal{A} \to \mathcal{X}^{\text{uf}} \).

We construct the cluster DT transformation as follows. Let \( \{l_1, \ldots, l_n\} \) be a point in \( \text{Conf}_n^x(m) \). Since the lines \( l_1, \ldots, l_{i+m-2} \) are assumed to be in general position, they span a hyperplane \( h_{[i,i+m-2]} \) in \( \mathbb{C}^m \), which is in turn a line in the dual space. We then define

\[
\text{DT} : \{l_1, \ldots, l_n\} \mapsto [h_{[2-m,n]}, h_{[3-m,1]}, \ldots, h_{[1-m,n-1]}].
\]

Due to the fact that we have quotiented out the \( \text{PGL}_m \) action in the definition of \( \text{Conf}_n^x(m) \), \( \text{Conf}_n^x(V) \) is canonically isomorphic to \( \text{Conf}_n^x(m) \) for any \( m \)-dimensional vector space \( V \). Therefore \([h_{[2-m,n]}, h_{[3-m,1]}, \ldots, h_{[1-m,n-1]}]\) defines a point in the configuration space \( \text{Conf}_n^x(m) \), and the map \( \text{DT} \) defined above is a birational automorphism on \( \text{Conf}_n^x(m) \). One can verify geometrically that \( \text{DT} \) is equal to the projectivization of Muller and Speyer’s right twist map on the Grassmannian [MS16, MS17]. By verifying the defining property (1) we prove the following result.

**Theorem 1** (W. [Wen16b]). The automorphism \( \text{DT} \) is the cluster DT transformation on \( \text{Conf}_n^x(m) \).

The proof of this theorem is built upon the work of Scoot [Sco06] on the cluster structures on Grassmannians and the work of Postnikov [Pos06] on plabic graphs. In the process of proving this theorem, we made an interesting discovery: let \( T_{\mathcal{X}^{\text{uf}}} \subset \text{Conf}_n^x(m) \) and \( T_{\mathcal{A}} \subset \mathcal{G}_m^x(n) \) be corresponding cluster charts arisen from the same plabic graph; then the restriction of the automorphism \( \text{DT} \) to \( T_{\mathcal{X}^{\text{uf}}} \), which is a birational map, can be computed using the following diagram, where

- \( \psi \) is Postnikov’s boundary measurement map [Pos06];
- \( s \) is any algebraic section, which selects a non-zero vector \( v_i \) inside each line \( l_i \);
- \( \alpha \) is the evaluation map of cluster \( K_2 \) coordinates (\( \alpha \) is rational because we need these coordinates to be non-vanishing in order to land inside \( T_{\mathcal{A}} \)).

\[
\begin{align*}
\mathcal{G}_m^x(n) & \xrightarrow{\alpha} T_{\mathcal{A}} \\
\text{Conf}_n^x(m) & \xrightarrow{s} T_{\mathcal{X}^{\text{uf}}}
\end{align*}
\]

1.2. Double Bruhat Cells. A double Bruhat cell is the intersection of two Bruhat cells associated with opposite Borel subgroups [FZ99]:

\[
G_{u,v} := B_+ u B_+ \cap B_- v B_-.
\]

Following a result of Berenstein-Fomin-Zelevinsky [BFZ05] we know that when \( G = G_{sc} \) is simply connected, the double Bruhat cell \( G_{sc}^{u,v} \) is a cluster \( K_2 \) variety. On the other hand, when \( G = G_{ad} \)
is adjoint, Fock and Goncharov [FG06] described a cluster Poisson variety structure on $G_{\text{ad}}^{u,v}$ using the amalgamation procedure. Motivated by Diagram (3), I hypothesized that a similar diagram could produce the cluster DT transformation of double Bruhat cells, which turned out to be true.

Let us state the concrete replacements for the parts of Diagram (3). First we observe that the unfrozen cluster Poisson variety associated with a double Bruhat cell $G_{\text{sc}}^{u,v}$ is the double quotient $T \backslash G_{\text{sc}}^{u,v} / T$. Note that there is a natural projection map $p : G_{\text{sc}}^{u,v} \to T \backslash G^{u,v} / T$, which again coincides with the cluster theoretical map $p : A \to A^{\text{ad}}$. Second, instead of cluster seeds arisen from plabic graphs, we consider cluster seeds arisen from reduced words of $(u, v) \in W \times W$. Lastly, we replace Postnikov’s boundary measurement map with Fock and Goncharov’s amalgamation map (which we still denote by $\psi$ on the right).

The resulting diagram is the one on the left, where $T_{X^{\text{ad}}} \subset T \backslash G^{u,v} / T$ and $T_A \subset G^{u,v}$ are two corresponding cluster charts arisen from the same reduced word of $(u, v)$. By verifying the defining property (1) we prove the following result.

**Theorem 2** (W. [Wen16a, Wen20]). The rational map $DT$ in Diagram (4) can be extended to a biregular automorphism on $T \backslash G^{u,v} / T$ and this biregular automorphism is the cluster DT transformation on $T \backslash G^{u,v} / T$.

By comparing (3) and (4) with Fomin and Zelevinsky’s separation formula [FZ07] we see that, up to a multiple of some monomial factor, Postnikov’s boundary measurement map and Fock-Goncharov’s amalgamation map are actually computing the $F$-polynomials of the cluster DT transformations. I believe that this is a hint to a more general combinatorial formula for the $F$-polynomials of cluster DT transformations, and is worth further investigation.

Similar to how the cluster DT transformation of Grassmannian is related to the right twist map, the cluster DT transformation of a double Bruhat cell is related to Fomin and Zelevinsky’s twist map [FZ99]. By comparing the action of DT and the descendant of $tw \circ t$ on $T \backslash G^{u,v} / T$, we conclude the following theorem.

**Theorem 3** (W. [Wen16a, Wen20]). $DT = tw \circ t$ as biregular automorphisms on $T \backslash G^{u,v} / T$. Here $t$ is a particular anti-involution on $G$ [FZ99].

### 1.3. Double Bott-Samelson Cells

L. Shen and I introduced double Bott-Samelson (BS) cells in [SW19] as a generalization of double Bruhat cells. Let $G$ be a Kac-Peterson group and let $B_\pm$ be a pair of opposite Borel subgroups. Recall that the $G$-orbits in $(G/B_+) \times (G/B_-)$ are parametrized by the Weyl group $W$. For a pair of positive braids $(b, d)$ associated to $G$, we choose positive braid words $i = (i_1, \ldots, i_l)$ for $b$ and $j = (j_1, \ldots, j_m)$ for $d$. Then the double Bott-Samelson cell $\text{Conf}_d(B)$ is defined to be the moduli space (stack) of the $G$-orbits of the following flag configurations, where

$$
\begin{align*}
B_0 & \xrightarrow{s_{i_1}} B_1 \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_l}} B^i \\
B_0 & \xrightarrow{s_{j_1}} B_1 \xrightarrow{s_{j_2}} \cdots \xrightarrow{s_{j_m}} B^j
\end{align*}
$$

- $B^i \xrightarrow{s_i} B^j$ means $(B^i, B^j)$ is in the $G$-orbit of $(G/B_+) \times (G/B_-)$ associated with $s_i \in W$;
- $B_i \xrightarrow{s_i} B_j$ means $(B_i, B_j)$ is in the $G$-orbit of $(G/B_-) \times (G/B_-)$ associated with $s_i \in W$;
- $B_i \xrightarrow{s_i} B^j$ means $B_i$ and $B^j$ are opposite Borel subgroups.

We further prove that for all possible choices of the braid words $i$ and $j$, the resulting spaces $\text{Conf}_d(B)$ are canonically isomorphic, and therefore $\text{Conf}_d(B)$ is well-defined.

We also introduced a decorated version of the double BS cell. Consider the pair of maximal unipotent subgroups $U_\pm := [B_\pm, B_\pm]$. We define their corresponding decorated flag varieties to be
$\mathcal{A}_\pm := G/U_\pm$ and call elements of $\mathcal{A}_\pm$ decorated flags. For either of the decorated flag varieties there is a natural projection map $\pi : \mathcal{A}_\pm \to \mathcal{B}_\pm$. We say that a decorated flag $A$ is over an undecorated flag $B$ if $\pi(A) = B$. We define the decorated double BS cell $\text{Conf}^b_d(A)$ to be the moduli space of $G$-orbits of flag configurations similar to those of $\text{Conf}^d_b(B)$ but with $B^0$ and $B_m$ replaced by two decorated flags $A^0$ and $A_m$ over them.

To equip the decorated double BS cells with cluster structures, we follow H. Williams’ generalization of simply connected form and adjoint form to Kac-Moody groups [Wil13] and denote the corresponding decorated double BS cells as $\text{Conf}^b_d(A_{sc})$ and $\text{Conf}^b_d(A_{ad})$ respectively.

We prove the following theorem for double BS cells.

**Theorem 4** (Shen-W. [SW19]). $\text{Conf}^b_d(A_{sc})$ is a smooth affine variety. $\text{Conf}^b_d(A_{ad})$ is a cluster Kac-Moody variety and $\text{Conf}^b_d(A_{ad})$ is a cluster Poisson variety; together they form a cluster ensemble. $\text{Conf}^b_d(B)$ is the unfrozen cluster Poisson variety associated with this cluster ensemble.

Besides the double BS cells, we also studied some interesting maps between them. First we have the reflection maps, which are biregular isomorphisms that go between double BS cells as follows:

$$
\text{Conf}^b_{s,d}(B) \xrightarrow{\text{left reflections}} \text{Conf}^{s,b}_d(B), \quad \text{Conf}^b_{d,s}(B) \xrightarrow{\text{right reflections}} \text{Conf}^{b,s}_d(B).
$$

From the transposition map on the group $G$ we obtain another biregular isomorphism

$$
\text{Conf}^b_d(B) \xleftarrow{\tau} \text{Conf}^{b^o}_d(B),
$$

where $b^o$ and $d^o$ denote the opposite positive braids of $b$ and $d$ respectively. Using these maps we construct the cluster DT transformation as follows.

**Theorem 5** (Shen-W. [SW19]). The cluster DT transformation on $\text{Conf}^b_d(B)$ is the composition

$$
(6) \quad \text{DT} : \text{Conf}^b_d(B) \xrightarrow{\text{left reflections}} \text{Conf}^{s,b}_d(B) \xrightarrow{\text{right reflections}} \text{Conf}^{b,s}_d(B) \xrightarrow{\tau} \text{Conf}^b_d(B).
$$

By studying the action of cluster DT transformations on double BS cells, we obtain the following result on the periodicity of cluster DT transformations.

**Theorem 6** (Shen-W. [SW19]). Let $G$ be a semisimple Lie group and let $(b,d)$ be a pair of positive braids associated with $G$. Let $w_0$ be the longest element in the corresponding Weyl group. If $(d^o b)^m = w_0^{2n}$ as positive braids, then the cluster DT transformation of $\text{Conf}^b_d(B)$ (see (6) for definition) satisfies $\text{DT}^{2(m+n)} = \text{id}$.

Theorem 6 gives rise to many examples of periodic cluster DT transformations that have not been observed before. For example, consider the semisimple Lie group $\text{PGL}_{2m+1}$. Consider the cyclic permutation element $w = (1,2,\ldots,m,2m+1,2m,\ldots,m+2)$ in the Weyl group $S_{2m+1}$. We observe that $w^m = w_0$. Let $b := w^n \in B_r$ for some positive integer $n$; then $b^{2m} = w_0^{2n}$. By Theorem 6 we can conclude that on $\text{Conf}^b_e(B)$, $\text{DT}^{2(2m+n)} = \text{id}$.

2. **A New Geometric Proof of Zamolodchikov’s Periodicity Conjecture**

When the defining quiver of a cluster variety is $\Delta \boxtimes \Delta'$ for two bipartite Dynkin diagrams $\Delta$ and $\Delta'$, there is an interesting cluster transformation called the “Zamolodchikov transformation”, which is related to the Zamolodchikov’s periodicity conjecture on the corresponding $Y$-systems [FZ03, Kel13]. Define $\tau_+$ to be the sequence of mutations that take place at the black vertices of the product quiver and define $\tau_-$ to be the sequence of mutations that take place at the white vertices of the product quiver. Then the mutation sequence $\tau_- \circ \tau_+$ defines a cluster transformation
Za, which we call the Zamolodchikov transformation. Zamolodchikov’s periodicity conjecture can be stated in terms of the Zamolodchikov transformation as

\[ Z^a_h h' = \text{id}, \]

where \( h \) and \( h' \) are the Coxeter numbers associated to the Dynkin diagrams \( \Delta \) and \( \Delta' \) respectively. By using a categorical method, Keller [Kel13] proves this periodicity conjecture in the most general setting for \( \Delta \bowtie \Delta' \).

In the cases of \( \Delta \bowtie A_r \) with an arbitrary Dynkin diagram \( \Delta \), we discover a new geometric proof of Zamolodchikov’s periodicity conjecture. Fix a bipartite coloring on \( \Delta \) and a semisimple group \( G \) of type \( \Delta \). Define

\[ b := \prod_{\text{black vertices } i} s_i, \quad w := \prod_{\text{white vertices } j} s_j, \]

\[ p = \overbrace{wbw\ldots}^{r+1 \text{ factors}}, \quad \text{and} \quad q = \overbrace{wbw\ldots}^{r+1 \text{ factors}}. \]

Then the double BS cell \( \text{Conf}_{p,q}(\mathcal{B}) \) is of type \( \Delta \bowtie A_r \).

We show that a point in \( \text{Conf}_{p,q}(\mathcal{B}) \) can be represented by a configuration of the form on the right, where \( c := bw \) is a Coxeter element in the Weyl group. By analyzing the action of the Zamolodchikov transformation \( Z^a \), we prove the following.

**Theorem 7** (Shen-W. [SW19]). The Zamolodchikov transformation acts on \( \text{Conf}_{p,q}(\mathcal{B}) \) by rotating the above circle one step in the counterclockwise direction. Consequently, we have \( Z^a_h h^{r+1} = \text{id} \).

### 3. Cluster Duality of Grassmannians and Cyclic Sieving of Plane Partitions

Recall from Section 1 that \( \mathcal{G}_{r,a}(n) \) is the affine cone of \( \text{Gr}_{r,a}(n) \) with respect to the Plücker embedding. By the Borel-Weil theorem, we know that \( \mathcal{O}(\mathcal{G}_{r,a}(n)) \cong \bigoplus_{c=0}^\infty V_{\omega_a} \), where \( \omega_a \) is the \( a \)th fundamental weight of \( \text{GL}_n \) and \( V_{\lambda} \) denotes the irreducible \( \text{GL}_n \)-representation associated with a dominant integral weight \( \lambda \).

Let \( D_i \subset \mathcal{G}_{r,a}(n) \) be the vanishing locus of the Plücker coordinate \( \Delta_{i,i+1,\ldots,i+a-1} \) (indices taken modulo \( n \)). Define \( D := \bigcup_i D_i \). As we have mentioned in Section 1, it follows from a result of Scott [Sco06] that \( \mathcal{G}_{r,a}^\times(n) \), the complement of \( D \), is a cluster \( K_2 \) variety. We hope to apply Fock-Goncharov’s cluster duality to obtain a basis for the representations \( V_{\omega_a} \) by intersecting the cluster canonical basis of \( \mathcal{O}(\mathcal{G}_{r,a}^\times(n)) \) with the linear subspace \( V_{\omega_a} \subset \bigoplus_{c=0}^\infty V_{\omega_a} \cong \mathcal{O}(\mathcal{G}_{r,a}(n)) \subset \mathcal{O}(\mathcal{G}_{r,a}^\times(n)) \). We start by constructing the cluster dual of Grassmannian as the following moduli space of lines with decorations:

\[ \text{Conf}_{n}^\times(a) := \text{PGL}_a \left\{ \begin{array}{l}
\text{1-dimensional subspaces } l_1, \ldots, l_n \subset \mathbb{C}^a \\
\text{and linear isomorphisms } \phi_i : l_i \rightarrow l_{i-1}
\end{array} \right\} \]
By verifying the GHKK sufficient condition of the cluster duality conjecture, we prove the following theorem.

**Theorem 8** (Shen-W. [SW20]). \( (\mathcal{G}_{r_{\alpha}^n}(n), \text{Conf}_{\alpha}^n(a)) \) form a cluster ensemble, on which the Fock-Goncharov cluster duality holds. Consequently, \( \mathcal{O} (\mathcal{G}_{r_{\alpha}^n}(n)) \) admits a cluster canonical basis

\[
\Theta = \{ \theta_q \mid q \in \text{Conf}_{\alpha}^n(a) (\mathbb{Z}^t) \}.
\]

To obtain a basis for \( V_{\omega_a} \), we further investigate the cluster canonical basis elements for extendability to \( D \) and their transformation under the action of the maximal torus \( T \subset \text{GL}_n \). Based on the separation formula of cluster canonical basis elements [FZ07, GHKK18], we obtain the following list of dualities between \( \mathcal{G}_{r_{\alpha}^n}(n) \) and \( \text{Conf}_{\alpha}^n(a) \).

\[
\begin{align*}
\text{decorated grassmannian } \mathcal{G}_{r_{\alpha}^n}(n); & \quad \text{decorated configuration space } \text{Conf}_{\alpha}^n(a); \quad \text{twisted monodromy } P; \quad \text{potential function } \mathcal{W} = \sum_i \theta_i; \quad \text{weight map } M : \text{Conf}_{\alpha}^n(a) \rightarrow \mathbb{T}^\vee; \quad \text{cyclic rotation } R. \\
\text{free recalling } G_n\text{-action}; & \quad \text{boundary divisor } D = \bigcup_i D_i; \quad \text{action by a maximal torus } T \subset \text{GL}_n; \quad \text{twisted cyclic rotation } C_\alpha. \quad &
\end{align*}
\]

We show that the maps \( P, \mathcal{W}, \) and \( M \) are all positive (in the sense that their pull-backs of cluster coordinates are subtraction-free) and hence their tropicalizations, denoted as \( P^t, \mathcal{W}^t, \) and \( M^t \) respectively, are function-on the space of integral tropical points \( \text{Conf}_{\alpha}^n(a) (\mathbb{Z}^t) \). It follows that \( (\text{Conf}_{\alpha}^n(a), \mathcal{W}) \) is equivalent to the mirror Landau-Ginzburg model of the Grassmannian considered by Eguchi-Hori-Xiong [EHX97], Marsh-Rietsch [MR20] and Rietsch-Williams [RW19].

Utilizing the tropical functions \( P^t \) and \( \mathcal{W}^t \) we select from \( \Theta \) a basis for \( V_{\omega_a} \) and prove that it is also compatible with the weight space decomposition.

**Theorem 9** (Shen-W. [SW20]). Let \( b := n - a \) and define

\[
\Theta(a, b, c) := \{ \theta_q \in \Theta \mid q \in \text{Conf}_{\alpha}^n(a) (\mathbb{Z}^t), \mathcal{W}^t(q) \geq 0, P^t(q) = c \}.
\]

Then \( \Theta(a, b, c) = V_{\omega_a} \cap \Theta \) and \( \Theta(a, b, c) \) is a basis of \( V_{\omega_a} \). In addition, \( \Theta(a, b, c) \cap V_{\omega_a}(\mu) = \{ \theta_q \in \Theta(a, b, c) \mid M^t(q) = \mu \} \) and \( \Theta(a, b, c) \cap V_{\omega_a}(\mu) \) is a basis of the weight space \( V_{\omega_a}(\mu) \).

The cluster canonical basis we obtain for the irreducible \( \text{GL}_n \)-representation \( V_{\omega_a} \) admits a natural parametrization by plane partitions, arrays of non-negative integers that are non-increasing along rows from left to right and along columns from top to bottom. Let \( P(a, b, c) \) be the set of plane partitions of size \( a \times b \) with entries no bigger than \( c \). This parametrization is obtained by relating plane partitions of size \( a \times b \) to a lattice realization of the space of integral tropical points \( \text{Conf}_{\alpha}^n(a) (\mathbb{Z}^t) \).

The cluster canonical basis \( \Theta \) has the property that it is preserved under the action of the cluster modular group. It is known that the Zamolodchikov transformation \( Z_a \) of Grassmannian \( \mathcal{G}_{r_{\alpha}^n}(n) \) (which is of type \( A_{n-a-1} \square A_{a-1} \)) acts by cyclically rotating the columns of a matrix representative in \( \mathcal{G}_{r_{\alpha}^n}(n) \cong \text{SL}_{n_a} \setminus \text{Mat}_{a \times n}^{\text{full-ranked}} [GSV10] \). By proving the invariance of \( \mathcal{W} \) and the twisted monodromy \( P \) under the \( Z_a \) action, we deduce that \( \Theta(a, b, c) \) is preserved under the action of \( Z_a \).

By translating the \( Z_a \)-action from \( \Theta(a, b, c) \) to \( P(a, b, c) \) via the parametrization map \( \Theta(a, b, c) \cong P(a, b, c) \), we obtain a toggling sequence \( \eta \) of order \( n \) on \( P(a, b, c) \). For any \( \pi \in P(a, b, c) \) let us define \( |\pi| \) to be the number of fixed points in \( P(a, b, c) \) under the action of \( \eta^n \) is equal to the evaluation of \( M(a, b, c) \) at \( q = \zeta^d \), where \( \zeta \) is a primitive \( n \)th root of unity.
4. Positive Braid Legendrian Links

A Legendrian link $\Lambda$ is a smooth closed 1-manifold embedded in $\mathbb{R}^3_{xyz}$ along which the standard contact 1-form $\alpha = dz - ydx$ vanishes. The study of Legendrian links centers around the construction of various Legendrian invariants for distinguishing Legendrian isotopy classes of Legendrian links.

Denote $\text{Br}_n^+$ the semigroup of positive braids in the Artin braid group of $n$ strands. For any $\beta \in \text{Br}_n^+$, we draw its braid diagram on $\mathbb{R}^2_{xz}$ from left to right and then close it with cusps on both ends. The resulting diagram is automatically the front projection of a Legendrian link $\Lambda_\beta$, which we call the positive braid Legendrian link associated with $\beta$. The picture on the left below is an example of a positive braid Legendrian link $\Lambda_\beta$ for $\beta = s_1^3s_2s_1^3s_2^2 \in \text{Br}_3^+$.

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\text{(7)}
\end{array}
\end{array}
\end{equation}

Given a positive braid word $i$ of a positive braid $\beta \in \text{Br}_n^+$ we can draw a quiver $Q_i$ according to the wiring diagram convention of Fomin-Zelevinsky [FZ99] or the amalgamation procedure of Fock-Goncharov [FG06]. It is not hard to see that such a quiver is also a defining quiver for the cluster structure on the double Bott-Samelson cell $\text{Conf}_{\beta}^e(\mathcal{B}_{\text{PGL}_n})$. The picture on the right above is one such quiver for $\Lambda_\beta$ in (7); note that it is of affine type $\tilde{E}_6$.

Denote $\mathcal{M}_1(\Lambda)$ the moduli space (stack) of isomorphism classes of rank-1 objects in the category of microlocal sheaves supported at $\pi_F(\Lambda)$. It is known that $\mathcal{M}_1(\Lambda)$ is an invariant of Legendrian links [GKS12, STZ17]. During our study of double BS cells, L. Shen and I follow an idea of Shende-Treumann-Zaslow [STWZ19] also found a cluster Poisson structure on $\mathcal{M}_1(\Lambda)$ using Postnikov’s plabic graphs. We conjecture that in the cases of $\mathcal{M}_1(\Lambda_\beta)$, the two cluster Poisson structures are indeed identical.

In [SW19] we develop a concrete algorithm to compute the number of $\mathbb{F}_q$-points in any double BS cells based on a stratification by Tits codistance. As a corollary of Theorem 11, we obtain a rational Legendrian invariant

$$g_\beta(q) := \# \left| \mathcal{M}_1(\Lambda_\beta)(\mathbb{F}_q) \right| \equiv \# \left| \text{Conf}_{\beta}^e(\mathcal{B}_{\text{PGL}_n})(\mathbb{F}_q) \right|.$$  

This counting of $\mathbb{F}_q$-points is known to be related to the HOMFLY homology of the link [STZ17].

For example, this rational Legendrian invariant for $\Lambda_\beta$ in (7) is $g_\beta(q) = \frac{q^8 - q^7 + q^6 - q^4 + q^2 - q + 1}{q-1}$.

Another interesting geometric space to consider is the augmentation variety of a positive braid Legendrian link. Recall that the Chekanov-Eliashberg (CE) dga $\mathcal{A}(\Lambda)$ of a Legendrian link $\Lambda$ is a $\mathbb{Z}_2$-algebra freely generated by the set of Reeb chords and marked points on $\Lambda$, with a differential defined by a certain counting of immersed disks [Che02]. Let $\mathbb{F}$ be an algebraically closed field of characteristic 2. An augmentation of $\mathcal{A}(\Lambda)$ is a dga homomorphism $\mathcal{A}(\Lambda) \to \mathbb{F}$ (where $\mathbb{F}$ is equipped with a trivial differential). The augmentation variety $\text{Aug}(\Lambda)$ is defined to be the moduli space of augmentations of the CE dga $\mathcal{A}(\Lambda)$.

In the cases of positive braid Legendrian links, generators of $\mathcal{A}(\Lambda_\beta)$ are concentrated at degrees 0 and 1; therefore $\text{Aug}(\Lambda_\beta)$ is by definition $\text{Spec} \ H_0(\mathcal{A}(\Lambda_\beta), \mathbb{F})$, which is an affine variety. H. Gao, L. Shen, and I prove the following result about the augmentation varieties of positive braid Legendrian links.

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Theorem 12 (Gao-Shen-W. [GSW20a]). For any positive braid Legendrian link $\Lambda_\beta$, $H_0(\mathcal{A}(\Lambda_\beta), \mathbb{F})$ is a cluster algebra and $\text{Aug}(\Lambda_\beta)$ is a cluster $K_2$ variety (both in characteristic 2). The cluster type of both is defined by the quiver $Q_i$ for any choice of positive braid word $i$ of $\beta$.

In addition to equipping $\text{Aug}(\Lambda_\beta)$ with a cluster $K_2$ structure, we also study the interaction between the cluster structure and exact Lagrangian fillings of $\Lambda_\beta$. We focus on the special family, which we call admissible fillings, consisting of compositions of saddle cobordisms, cyclic rotations, braid moves, and minimum cobordisms. We decorate each admissible filling $L$ with a collection of oriented marked curves $P$. Following a result of Ekholm-Honda-Kálmán [EHK16] we obtain a functorial morphism of algebraic varieties

$$\Phi_L : \text{Aug}(\emptyset, P) \to \text{Aug}(\Lambda_\beta).$$

Theorem 13 (Gao-Shen-W. [GSW20a]). For any admissible filling $L$ of $\Lambda_\beta$, the morphism $\Phi_L$ is an open embedding of an algebraic torus into $\text{Aug}(\Lambda_\beta)$, and the image of $\Phi_L$ coincides with a cluster chart on $\text{Aug}(\Lambda_\beta)$. Moreover, if $\text{img}(\Phi_L) \neq \text{img}(\Phi_{L'})$ for two admissible fillings $L$ and $L'$ of $\Lambda_\beta$, then $L$ and $L'$ are not Hamiltonian isotopic.

Theorems 11 and 13 announce an unfrozen cluster Poisson structure on $M_1(\Lambda_\beta)$ and a cluster $K_2$ structure on $\text{Aug}(\Lambda_\beta)$ respectively. We believe that a certain enrichment of $M_1(\Lambda_\beta)$ will form a cluster ensemble with $\text{Aug}(\Lambda_\beta)$. Meanwhile, we conjecture that the cluster theoretical map $p : \text{Aug}(\Lambda_\beta) \to M_1(\Lambda_\beta)$ is a manifestation of the augmentation-sheaf correspondence [NRSSZ15], which is an example of the more general Nadler-Zaslow correspondence [NZ09]. These questions are worth further research.

Theorem 13 allows us to employ the cluster $K_2$ structure on $\text{Aug}(\Lambda_\beta)$ to distinguish non-Hamiltonian isotopic admissible fillings. In particular, since the square of the cluster DT transformation is the full cyclic rotation on $\Lambda_\beta$, we can use an aperiodic DT to generate infinitely many non-Hamiltonian isotopic admissible fillings, proving a conjecture on the existence of Legendrian links with infinitely many fillings; independent proofs of such conjecture are also given by Casals-Gao [CG20], Casals-Zaslow [CZ20], and Casals-Ng [CN].

Theorem 14 (Gao-Shen-W. [GSW20b]). Let $i$ be a positive braid word for $\beta$. If the quiver $Q_i$ is acyclic and of infinite type, then $\Lambda_\beta$ admits infinitely many non-Hamiltonian isotopic exact Lagrangian fillings.

For example, since the $\tilde{E}_6$ quiver in (7) is acyclic and of infinite type, the positive braid Legendrian link $\Lambda_\beta$ admits infinitely many fillings.

Among the positive braid Legendrian links there are several special ones which we call standard links of finite type. They are the closures of the following positive braids:

$$A_r : s_{r+1}^1 \in Br^+_{>0}, \quad D_r : s_{r-2}^1\cdot2s_1^2s_2, \quad E_{r=6,7,8} : s_{r-3}^1\cdot2s_1^3s_2.$$ 

Note that these positive braid links are closely related to the classification of simple surface singularities.

Building upon Theorem 14 and making use of admissible concordance, we prove the following stronger statement.

Theorem 15 (Gao-Shen-W. [GSW20b]). If $\Lambda_\beta$ is not Legendrian isotopic to a split union of unknots and connect sums of standard links of finite type, then $\Lambda_\beta$ admits infinitely many non-Hamiltonian isotopic exact Lagrangian fillings.

REFERENCES
