# New bounds for spherical two-distance sets and equiangular lines

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**Definition**

If $X = \{x_1, x_2, \cdots, x_N\} \subset \mathbb{S}^{n-1}$ (unit sphere in $\mathbb{R}^n$) and $\langle x_i, x_j \rangle = a$ or $b$ for all $i \neq j$, then we call $X$ is a spherical two-distance set.

Q : What is the maximum cardinality of a spherical two-distance set?

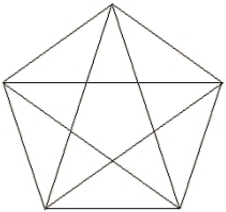

Figure: The maximum spherical two-distance set in $\mathbb{R}^2$: Pentagon.

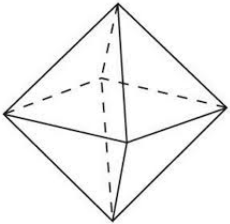

Figure: The maximum spherical two-distance set in $\mathbb{R}^3$: Octahedron.

# Known results

Let $g(n)$ denote the maximum size of spherical two-distance set in $\mathbb{R}^n$

1. Let $e_1, \ldots, e_{n+1}$ be the standard basis in $\mathbb{R}^{n+1}$. The points $e_i + e_j, i \neq j$ form a spherical two-distance set in the plane $x_1 + \cdots + x_{n+1} = 2$,

$$g(n) \geq \frac{n(n+1)}{2}, \quad n \geq 2. \tag{1}$$

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2. Delsarte, Goethals, and Seidel in 1977 proved so-called "harmonic" bound :

$$g(n) \leq \frac{n(n+3)}{2}. \tag{2}$$

They also showed that this bound is tight for $n = 2, 6, 22$.

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3. Therefore, $\frac{n(n+1)}{2} \leq g(n) \leq \frac{n(n+3)}{2}$.

# Musin's result

Musin used Delsarte's linear programming method to prove that

$$g(n) = \frac{n(n+1)}{2} \quad \text{if } 7 \leq n \leq 39, n \neq 22, 23$$

and $g(23) = 276$ or $277$.

# Barg and Yu 2014

We use the semidefinite programming (SDP) method showing that

$$g(n) = \frac{n(n+1)}{2}, \quad 7 \leq n \leq 93, n \neq 22, 46, 78.$$

In particular, $g(23) = 276$.

# Maximum spherical two-distance sets

**Theorem (Yu 2016+)**

$$g(n) = \frac{n(n+1)}{2}, \quad 7 \leq n \leq 417,$$

$n \neq 22, 46, 78, 118, 166, 222, 286, 358$ *which are* $(2k+1)^2 - 3$ *for* $k = 2, \cdots, 9$.

For $a + b \geq 0$, Oleg Musin proved the upper bounds are $\frac{n(n+1)}{2}$.
For $a + b < 0$, the upper bounds can be obtained from the bounds of equiangular lines in one higher dimension.

**Lemma**

*If there are at most* $\frac{n(n-1)}{2}$ *equiangular lines in* $\mathbb{R}^n$, *then maximum size of a spherical two-distance set in* $\mathbb{R}^{n-1}$ *is* $\frac{n(n-1)}{2}$.

**Musin Conjecture:**

$$g(n) = \frac{n(n+1)}{2} \quad \forall n \in \mathbb{N}, \text{except } n+3 = (2k+1)^2, k \in \mathbb{N}.$$

# Maximum spherical two-distance sets

**Theorem (Glazyrin and Yu 2016+)**

$$g(n) = \frac{n(n+1)}{2} \quad \forall n \in \mathbb{N}, \textit{except } n = (2k+1)^2 - 3, k \in \mathbb{N}.$$

**Definition**

A set of lines in $\mathbb{R}^n$ is called *equiangular* if the angle between each pair of lines is the same.

- An equiangular line set can be defined as an unit vectors set $S = \{x_i\}_{i=1}^{M}$ such that $|\langle x_i, x_j \rangle| = c, 1 \leq i < j \leq M$ for some $c > 0$.
- A equiangular line set can be defined as a spherical two-distance set with inner product value $c$ and $-c$.
- Question : What is the maximum cardinality of an equiangular line set in $\mathbb{R}^n$?

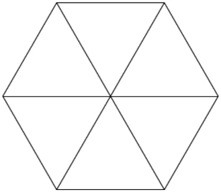

Figure: Maximum equiangular lines in $\mathbb{R}^2$: 3 lines through opposite vertices of a regular hexagon.

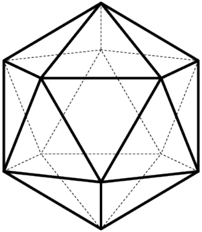

Figure: Maximum equiangular lines in $\mathbb{R}^3$: 6 lines through opposite vertices of the icosahedron.

# Known results

Let $M(n)$ denote the maximum size of an equiangular line set in $\mathbb{R}^n$

- Hanntjes found $M(n)$ for $n = 3$ and $4$ in $1948$.
- Van Lint and Seidel found the largest number of equiangular lines for $5 \leq n \leq 7$ in 1966.
- Lemmens and Seidel used linear-algebraic methods to determine $M(n)$ for most values of $n$ in the region $8 \leq n \leq 23$ in 1973.

| $n$ | $M(n)$ | $1/\alpha$ | | $n$ | $M(n)$ | $1/\alpha$ |
|---|---|---|---|---|---|---|
| 2 | 3 | 2 | | 17 | 48-50 | 5 |
| 3 | 6 | $\sqrt{5}$ | | 18 | 48-61 | 5 |
| 4 | 6 | 3; $\sqrt{5}$ | | 19 | 72-76 | 5 |
| 5 | 10 | 3 | | 20 | 90-96 | 5 |
| 6 | 16 | 3 | | 21 | 126 | 5 |
| $7 \leq n \leq 13$ | 28 | 3 | | 22 | 176 | 5 |
| 14 | 28-29 | 3; 5 | | 23 | 276 | 5 |
| 15 | 36 | 5 | | $24 \leq n \leq 42$ | $\geq$ 276 | 5 |
| 16 | 40-41 | 5 | | 43 | $\geq$ 344 | 7 |

Table: Known bounds on $M(n)$ in small dimensions

# Our results

**Theorem (Barg and Yu 2014)**

*We use the semidefinite programming (SDP) method to show that $M(n) = 276$ for $24 \le n \le 41$ and $M(43) = 344$.*

| $n$ | $M(n)$ | SDP bound | | $n$ | $M(n)$ | SDP bound |
|---|---|---|---|---|---|---|
| 3 | 6 | 6 | | 18 | 48-61 | 61 |
| 4 | 6 | 6 | | 19 | 72-76 | 76 |
| 5 | 10 | 10 | | 20 | 90-96 | 96 |
| 6 | 16 | 16 | | 21 | 126 | 126 |
| $7 \le n \le 13$ | 28 | 28 | | 22 | 176 | 176 |
| 14 | 28-29 | 30 | | 23 | 276 | 276 |
| 15 | 36 | 36 | | $24 \le n \le 41$ | 276 | 276 |
| 16 | 40-41 | 42 | | 42 | $\ge 276$ | 288 |
| 17 | 48-50 | 51 | | 43 | 344 | 344 |

Table: Bounds on $M(n)$ including new results

Remark : Recently, we can show no 76 equiangular lines in $\mathbb{R}^{19}$.

# Gegenbauer polynomials

Let $G_k^{(n)}(t), k = 0, 1, \ldots$ denote the Gegenbauer polynomials of degree $k$. They are defined recursively as follows: $G_0^{(n)} \equiv 1, G_1^{(n)}(t) = t$, and

$$G_k^{(n)}(t) = \frac{(2k+n-4)tG_{k-1}^{(n)}(t) - (k-1)G_{k-2}^{(n)}(t)}{k+n-3}, \quad k \geq 2.$$

Define a matrix $Y_k^n(u,v,t), k \geq 0$

$$(Y_k^n(u,v,t))_{ij} = u^i v^j ((1-u^2)(1-v^2))^{k/2} G_k^{(n-1)}\Big(\frac{t-uv}{\sqrt{(1-u^2)(1-v^2)}}\Big)$$

and a matrix $S_k^n(u,v,t)$ by setting

$$S_k^n(u,v,t) = \frac{1}{6}\sum_{\sigma \in S_3} Y_k^n(\sigma(u,v,t)), \quad (3)$$

I.J. Schoenberg (1942) proved that if $C$ is the finite set in $\mathbb{S}^{n-1}$, then

$$\sum_{(x,y)\in C^2} G_k^n(\langle x,y\rangle) \geq 0.$$

$$\sum_{(x,y,z)\in C^3} S_k^n(x\cdot y, x\cdot z, y\cdot z) \succeq 0.$$

# Semidefinite programming

$$\min c^T x$$
$$\text{subject to} \quad F_0 + \sum_{i=1}^{m} F_i x_i \succeq 0$$

where $c, x \in \mathbb{R}^m$ and $F_i$ is an $n$ by $n$ symmetric matrix $\forall i$. The sign "$\succeq$" means that the matrix is positive semidefinite.

CVX (MATLAB toolkit) can solve an SDP in a second.

**Theorem (Gerzon, absolute bounds)**

*If there are $M$ equiangular lines in $\mathbb{R}^n$, then $M \leq \frac{n(n+1)}{2}$.*

Gerzon bounds are known to be attained only for $n = 2, 3, 7$, and $23$.

**Theorem (Neumann)**

*If there are $M$ equiangular lines in $\mathbb{R}^n$ with angle $\arccos \alpha$ and $M > 2n$, then $1/\alpha$ is an odd integer.*

**Theorem (Lemmens and Seidel)**

*$M_{1/3}(n) = 2(n-1)$ for $n \geq 16$, where $M_\alpha(n)$ is the maximum size of an equiangular line set when the value of the angle is $\arccos \alpha$.*

**Theorem (Relative bounds)**

$$M_\alpha(n) \leq \frac{n(1-\alpha^2)}{1-n\alpha^2} \tag{4}$$

*valid for all $\alpha$ such that the denominator is positive.*

# SDP bound

**Theorem**

*Let $\mathcal{C}$ be an equiangular line set with inner product values either $a$ or $-a$. Let $p$ be a positive integer. The cardinality $|\mathcal{C}|$ is bounded above by the solution of the following semi-definite programming problem :*

$$1 + \frac{1}{3}\max(x_1 + x_2) \tag{5}$$

*subject to*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(x_1 + x_2) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(x_3 + x_4 + x_5 + x_6) \succeq 0 \tag{6}$$

$$\begin{aligned} & S_k^n(1,1,1) + S_k^n(a,a,1)x_1 + S_k^n(-a,-a,1)x_2 + S_k^n(a,a,a)x_3 \\ & + S_k^n(a,a,-a)x_4 + S_k^n(a,-a,-a)x_5 + S_k^n(-a,-a,-a)x_6 \succeq 0 \end{aligned} \tag{7}$$

$$3 + G_k^{(n)}(a)x_1 + G_k^{(n)}(-a)x_2 \geq 0, \tag{8}$$

*where $k = 0, 1, \cdots, p$ and $x_j \geq 0$, $j = 1, \cdots, 6$.*

# SDP bound table

| $n$ | 1/5 | 1/7 | 1/9 | 1/11 | 1/13 | 1/15 | max | Gerzon | angle |
|---|---|---|---|---|---|---|---|---|---|
| 22 | 176 | 39 | 29 | 26 | 25 | 24 | 176 | 253 | 1/5 |
| 23 | 276 | 42 | 31 | 28 | 26 | 25 | 276 | 276 | 1/5 |
| 24 | 276 | 46 | 33 | 29 | 27 | 26 | 276 | 300 | 1/5 |
| 25 | 276 | 50 | 35 | 31 | 29 | 28 | 276 | 325 | 1/5 |
| 26 | 276 | 54 | 37 | 32 | 30 | 29 | 276 | 351 | 1/5 |
| 27 | 276 | 58 | 40 | 34 | 31 | 30 | 276 | 378 | 1/5 |
| 28 | 276 | 64 | 42 | 36 | 33 | 31 | 276 | 406 | 1/5 |
| 29 | 276 | 69 | 44 | 37 | 34 | 33 | 276 | 435 | 1/5 |
| 30 | 276 | 75 | 47 | 39 | 36 | 34 | 276 | 465 | 1/5 |
| 31 | 276 | 82 | 49 | 41 | 37 | 35 | 276 | 496 | 1/5 |
| 32 | 276 | 90 | 52 | 43 | 39 | 37 | 276 | 528 | 1/5 |
| 33 | 276 | 99 | 55 | 45 | 40 | 38 | 276 | 561 | 1/5 |
| 34 | 276 | 108 | 57 | 46 | 42 | 39 | 276 | 595 | 1/5 |
| 35 | 276 | 120 | 60 | 48 | 43 | 41 | 276 | 630 | 1/5 |
| 36 | 276 | 132 | 64 | 50 | 45 | 42 | 276 | 666 | 1/5 |
| 37 | 276 | 148 | 67 | 52 | 47 | 44 | 276 | 703 | 1/5 |
| 38 | 276 | 165 | 70 | 54 | 48 | 45 | 276 | 741 | 1/5 |
| 39 | 276 | 187 | 74 | 57 | 50 | 46 | 276 | 780 | 1/5 |
| 40 | 276 | 213 | 78 | 59 | 52 | 48 | 276 | 820 | 1/5 |
| 41 | 276 | 246 | 82 | 61 | 53 | 49 | 276 | 861 | 1/5 |
| 42 | 276 | 288 | 86 | 63 | 55 | 51 | 288 | 903 | 1/7 |
| 43 | 276 | 344 | 90 | 66 | 57 | 52 | 344 | 946 | 1/7 |
| 44 | 276 | 422 | 95 | 68 | 59 | 54 | 422 | 990 | 1/7 |
| 45 | 276 | 540 | 100 | 71 | 60 | 56 | 540 | 1035 | 1/7 |
| 46 | 276 | 736 | 105 | 73 | 62 | 57 | 736 | 1081 | 1/7 |
| 47 | 276 | 1128 | 110 | 76 | 64 | 59 | 1128 | 1128 | 1/7 |
| 48 | 276 | 1128 | 116 | 78 | 66 | 60 | 1128 | 1176 | 1/7 |
| 49 | 276 | 1128 | 122 | 81 | 68 | 62 | 1128 | 1225 | 1/7 |
| 50 | 276 | 1128 | 129 | 84 | 70 | 64 | 1128 | 1275 | 1/7 |
| 51 | 276 | 1128 | 136 | 87 | 72 | 65 | 1128 | 1326 | 1/7 |
| 52 | 276 | 1128 | 143 | 90 | 74 | 67 | 1128 | 1378 | 1/7 |
| 53 | 276 | 1128 | 151 | 93 | 76 | 69 | 1128 | 1431 | 1/7 |
| 54 | 276 | 1128 | 160 | 96 | 78 | 70 | 1128 | 1485 | 1/7 |
| 55 | 276 | 1128 | 169 | 100 | 81 | 72 | 1128 | 1540 | 1/7 |
| 56 | 276 | 1128 | 179 | 103 | 83 | 74 | 1128 | 1596 | 1/7 |
| 57 | 276 | 1128 | 190 | 106 | 85 | 76 | 1128 | 1653 | 1/7 |
| 58 | 276 | 1128 | 201 | 110 | 87 | 77 | 1128 | 1711 | 1/7 |
| 59 | 276 | 1128 | 214 | 114 | 90 | 79 | 1128 | 1770 | 1/7 |
| 60 | 276 | 1128 | 228 | 118 | 92 | 81 | 1128 | 1830 | 1/7 |
| 61 | 279 | 1128 | 244 | 122 | 94 | 83 | 1128 | 1891 | 1/7 |
| 62 | 290 | 1128 | 261 | 126 | 97 | 85 | 1128 | 1953 | 1/7 |
| 63 | 301 | 1128 | 280 | 130 | 99 | 87 | 1128 | 2016 | 1/7 |
| 64 | 313 | 1128 | 301 | 134 | 102 | 89 | 1128 | 2080 | 1/7 |

# SDP bound table (Cont.)

| $n$ | 1/5 | 1/7 | 1/9 | 1/11 | 1/13 | 1/15 | max | Gerzon | angle |
|---|---|---|---|---|---|---|---|---|---|
| 65 | 326 | 1128 | 325 | 139 | 105 | 91 | 1128 | 2145 | 1/7 |
| 66 | 339 | 1128 | 352 | 144 | 107 | 92 | 1128 | 2211 | 1/7 |
| 67 | 353 | 1128 | 382 | 148 | 110 | 94 | 1128 | 2278 | 1/7 |
| 68 | 367 | 1128 | 418 | 153 | 113 | 97 | 1128 | 2346 | 1/7 |
| 69 | 382 | 1128 | 460 | 159 | 115 | 99 | 1128 | 2415 | 1/7 |
| 70 | 398 | 1128 | 509 | 164 | 118 | 101 | 1128 | 2485 | 1/7 |
| 71 | 416 | 1128 | 568 | 170 | 121 | 103 | 1128 | 2556 | 1/7 |
| 72 | 434 | 1128 | 640 | 176 | 124 | 105 | 1128 | 2628 | 1/7 |
| 73 | 453 | 1128 | 730 | 182 | 127 | 107 | 1128 | 2701 | 1/7 |
| 74 | 473 | 1128 | 845 | 188 | 130 | 109 | 1128 | 2775 | 1/7 |
| 75 | 494 | 1128 | 1000 | 195 | 134 | 112 | 1128 | 2850 | 1/7 |
| 76 | 517 | 1128 | 1216 | 202 | 137 | 114 | 1216 | 2926 | 1/9 |
| 77 | 542 | 1128 | 1540 | 210 | 140 | 116 | 1540 | 3003 | 1/9 |
| 78 | 568 | 1128 | 2080 | 217 | 144 | 118 | 2080 | 3081 | 1/9 |
| 79 | 596 | 1128 | 3160 | 225 | 147 | 121 | 3160 | 3160 | 1/9 |
| 80 | 626 | 1128 | 3160 | 234 | 151 | 123 | 3160 | 3240 | 1/9 |
| 81 | 658 | 1128 | 3160 | 243 | 154 | 126 | 3160 | 3321 | 1/9 |
| 82 | 693 | 1128 | 3160 | 252 | 158 | 128 | 3160 | 3403 | 1/9 |
| 83 | 731 | 1128 | 3160 | 262 | 162 | 130 | 3160 | 3486 | 1/9 |
| 84 | 772 | 1128 | 3160 | 272 | 166 | 133 | 3160 | 3570 | 1/9 |
| 85 | 816 | 1128 | 3160 | 283 | 170 | 136 | 3160 | 3655 | 1/9 |
| 86 | 866 | 1128 | 3160 | 294 | 174 | 138 | 3160 | 3741 | 1/9 |
| 87 | 920 | 1128 | 3160 | 307 | 178 | 141 | 3160 | 3828 | 1/9 |
| 88 | 979 | 1128 | 3160 | 320 | 182 | 143 | 3160 | 3916 | 1/9 |
| 89 | 1046 | 1128 | 3160 | 333 | 186 | 146 | 3160 | 4005 | 1/9 |
| 90 | 1120 | 1128 | 3160 | 348 | 191 | 149 | 3160 | 4095 | 1/9 |
| 91 | 1203 | 1128 | 3160 | 364 | 196 | 152 | 3160 | 4186 | 1/9 |
| 92 | 1298 | 1128 | 3160 | 380 | 200 | 154 | 3160 | 4278 | 1/9 |
| 93 | 1406 | 1128 | 3160 | 398 | 205 | 157 | 3160 | 4371 | 1/9 |
| 94 | 1515 | 1128 | 3160 | 417 | 210 | 160 | 3160 | 4465 | 1/9 |
| 95 | 1556 | 1128 | 3160 | 438 | 215 | 163 | 3160 | 4560 | 1/9 |
| 96 | 1599 | 1128 | 3160 | 460 | 220 | 166 | 3160 | 4656 | 1/9 |
| 97 | 1644 | 1128 | 3160 | 485 | 226 | 169 | 3160 | 4753 | 1/9 |
| 98 | 1691 | 1128 | 3160 | 511 | 231 | 172 | 3160 | 4851 | 1/9 |
| 99 | 1739 | 1128 | 3160 | 540 | 237 | 176 | 3160 | 4950 | 1/9 |
| 100 | 1790 | 1128 | 3160 | 571 | 243 | 179 | 3160 | 5050 | 1/9 |
| 101 | 1842 | 1128 | 3160 | 606 | 249 | 182 | 3160 | 5151 | 1/9 |
| 102 | 1897 | 1128 | 3160 | 644 | 255 | 185 | 3160 | 5253 | 1/9 |
| 103 | 1954 | 1128 | 3160 | 686 | 262 | 189 | 3160 | 5356 | 1/9 |
| 104 | 2014 | 1128 | 3160 | 734 | 268 | 192 | 3160 | 5460 | 1/9 |
| 105 | 2077 | 1128 | 3160 | 787 | 275 | 196 | 3160 | 5565 | 1/9 |
| 106 | 2142 | 1128 | 3160 | 848 | 282 | 199 | 3160 | 5671 | 1/9 |
| 107 | 2211 | 1128 | 3160 | 917 | 289 | 203 | 3160 | 5778 | 1/9 |

# SDP bound table (Cont.)

| $n$ | 1/5 | 1/7 | 1/9 | 1/11 | 1/13 | 1/15 | max | Gerzon | angle |
|---|---|---|---|---|---|---|---|---|---|
| 108 | 2282 | 1128 | 3160 | 997 | 297 | 206 | 3160 | 5886 | 1/9 |
| 109 | 2358 | 1128 | 3160 | 1090 | 305 | 210 | 3160 | 5995 | 1/9 |
| 110 | 2437 | 1128 | 3160 | 1200 | 313 | 214 | 3160 | 6105 | 1/9 |
| 111 | 2521 | 1128 | 3160 | 1332 | 321 | 218 | 3160 | 6216 | 1/9 |
| 112 | 2609 | 1128 | 3160 | 1493 | 330 | 222 | 3160 | 6328 | 1/9 |
| 113 | 2702 | 1128 | 3160 | 1695 | 339 | 226 | 3160 | 6441 | 1/9 |
| 114 | 2800 | 1128 | 3160 | 1954 | 348 | 230 | 3160 | 6555 | 1/9 |
| 115 | 2904 | 1128 | 3160 | 2300 | 357 | 234 | 3160 | 6670 | 1/9 |
| 116 | 3015 | 1128 | 3160 | 2784 | 367 | 238 | 3160 | 6786 | 1/9 |
| 117 | 3132 | 1128 | 3160 | 3510 | 378 | 242 | 3510 | 6903 | 1/11 |
| 118 | 3257 | 1128 | 3160 | 4720 | 388 | 247 | 4720 | 7021 | 1/11 |
| 119 | 3390 | 1128 | 3160 | 7140 | 399 | 251 | 7140 | 7140 | 1/11 |
| 120 | 3532 | 1128 | 3160 | 7140 | 411 | 256 | 7140 | 7260 | 1/11 |
| 121 | 3684 | 1128 | 3160 | 7140 | 423 | 260 | 7140 | 7381 | 1/11 |
| 122 | 3848 | 1128 | 3160 | 7140 | 436 | 265 | 7140 | 7503 | 1/11 |
| 123 | 4024 | 1128 | 3160 | 7140 | 449 | 270 | 7140 | 7626 | 1/11 |
| 124 | 4214 | 1128 | 3160 | 7140 | 462 | 275 | 7140 | 7750 | 1/11 |
| 125 | 4419 | 1128 | 3160 | 7140 | 477 | 280 | 7140 | 7875 | 1/11 |
| 126 | 4643 | 1128 | 3160 | 7140 | 492 | 285 | 7140 | 8001 | 1/11 |
| 127 | 4887 | 1128 | 3160 | 7140 | 508 | 290 | 7140 | 8128 | 1/11 |
| 128 | 5153 | 1128 | 3160 | 7140 | 524 | 295 | 7140 | 8256 | 1/11 |
| 129 | 5447 | 1128 | 3160 | 7140 | 541 | 301 | 7140 | 8385 | 1/11 |
| 130 | 5770 | 1128 | 3160 | 7140 | 560 | 306 | 7140 | 8515 | 1/11 |
| 131 | 6130 | 1128 | 3160 | 7140 | 579 | 312 | 7140 | 8646 | 1/11 |
| 132 | 6531 | 1130 | 3160 | 7140 | 599 | 317 | 7140 | 8778 | 1/11 |
| 133 | 6982 | 1158 | 3160 | 7140 | 620 | 323 | 7140 | 8911 | 1/11 |
| 134 | 7493 | 1187 | 3160 | 7140 | 643 | 329 | 7493 | 9045 | 1/5 |
| 135 | 8075 | 1218 | 3160 | 7140 | 667 | 336 | 8075 | 9180 | 1/5 |
| 136 | 8747 | 1249 | 3160 | 7140 | 692 | 342 | 8747 | 9316 | 1/5 |
| *137 | 9528 | 1282 | 3160 | 7140 | 719 | 348 | 9528 | 9453 | 1/5 |
| *138 | 10450 | 1315 | 3160 | 7140 | 747 | 355 | 10450 | 9591 | 1/5 |
| *139 | 11553 | 1350 | 3160 | 7140 | 778 | 362 | 11553 | 9730 | 1/5 |

# Our results

**Theorem (Barg and Yu)**

*We use the semidefinite programming method to show that* $M(n) = 276$ *for* $24 \leq n \leq 41$ *and* $M(43) = 344$ *and we get tighter upper bounds for* $M(n)$ *when* $n \leq 136$.

# Observation

| $angle$ | $dim$ | $bounds$ |
|---|---|---|
| $\frac{1}{5}$ | $23-59$ | $276=\frac{23\cdot 24}{2}$ |
| $\frac{1}{7}$ | $47-131$ | $1128=\frac{47\cdot 48}{2}$ |
| $\frac{1}{9}$ | $79-227$ | $3160=\frac{79\cdot 80}{2}$ |
| $\frac{1}{11}$ | $119-349$ | $7140=\frac{119\cdot 120}{2}$ |
| $\dots$ | | |

**Theorem (Yu 2016+)**

*We prove* $M_a(n) \leq \frac{1}{2}(\frac{1}{a^2}-1)(\frac{1}{a^2}-2)$ *for all* $a \in \mathbb{N}$ *and for all* $\frac{1}{a^2}-2 \leq n \leq \frac{3}{a^2}-16$

# Relaxation semidefinite programming problems

**Theorem**

*Let $\mathcal{C}$ be an equiangular line set with inner product values either $a$ or $-a$. Let $p$ be a positive integer. The cardinality $|\mathcal{C}|$ is bounded above by the solution of the following semi-definite programming problem :*

$$1+\frac{1}{3}\max(x_1+x_2) \tag{9}$$

*subject to*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\frac{1}{3}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(x_1+x_2)+\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(x_3+x_4+x_5+x_6)\succeq 0 \tag{10}$$

$$S_k^n(1,1,1)+S_k^n(a,a,1)x_1+S_k^n(-a,-a,1)x_2+S_k^n(a,a,a)x_3 \\ +S_k^n(a,a,-a)x_4+S_k^n(a,-a,-a)x_5+S_k^n(-a,-a,-a)x_6\succeq 0 \tag{11}$$

$$3+G_k^{(n)}(a)x_1+G_k^{(n)}(-a)x_2\geq 0, \tag{12}$$

*where $k=0,1,\cdots,p$ and $x_j\geq 0$, $j=1,\cdots,6$.*

We find that above constraints only $S_1^n\succeq 0$, $S_3^n\succeq 0$ and (9) are crucial.

# Symbolic semidefinite programming problems

**Theorem**

*The solution of following optimization problem is* $\frac{1}{2}(\frac{1}{a^2}-2)(\frac{1}{a^2}-1)$.

$$\max(1+A)$$

*subject to*

$$A+\frac{2a^4(3a+1)}{(6a^2-1)(a+1)^3}B+\frac{2a^4(3a-1)}{(6a^2-1)(a-1)^3}C\geq 0 \quad (13)$$

$$A+\frac{a}{1+a}B+\frac{a}{a-1}C\geq 0 \quad (14)$$

$$A(A-1)\leq B+C \quad (15)$$

**proof**: We choose suitable $t$, where $t=\frac{-16a^6}{(6a^2-1)(a+1)^2(a-1)^2}$ such that $t\frac{a}{1+a}+\frac{2a^4(3a+1)}{(6a^2-1)(a+1)^3}=t\frac{a}{a-1}+\frac{2a^4(3a-1)}{(6a^2-1)(a-1)^3}$

If we calculate $t$(14) + (13), we will get

$$(t+1)A+(t\frac{a}{1+a}+\frac{2a^4(3a+1)}{(6a^2-1)(a+1)^3})(B+C)\geq 0$$

$$\Rightarrow -\frac{10a^6+13a^4-8a^2+1}{(6a^2-1)(a-1)^2(a+1)^2}A-\frac{2a^4(5a^2-1)}{(6a^2-1)(a-1)^2(a+1)^2}(B+C)\geq 0$$

notice that $\frac{a^4(5a^2-1)}{(6a^2-1)(a-1)^2(a+1)^2} \geq 0$ and we plug in (15).

$$\Rightarrow -\frac{10a^6+13a^4-8a^2+1}{(6a^2-1)(a-1)^2(a+1)^2}A - \frac{2a^4(5a^2-1)}{(6a^2-1)(a-1)^2(a+1)^2}A(A-1) \geq 0$$

$$\Rightarrow -\frac{10a^6+13a^4-8a^2+1}{(6a^2-1)(a-1)^2(a+1)^2} \geq \frac{2a^4(5a^2-1)}{(6a^2-1)(a-1)^2(a+1)^2}(A-1)$$

$$\Rightarrow \frac{1-3a^2-2a^4}{2a^4} \geq A-1$$

$$\Rightarrow A \leq \frac{1-3a^2}{2a^4}$$

Then, it is not hard to see that $A+1 \leq \frac{1}{2}(\frac{1}{a^2}-2)(\frac{1}{a^2}-1)$.

# New bounds for equiangular lines

**Theorem (Glazyrin and Yu 2016+)**

$M_{\frac{1}{a}}(n) \leq n(\frac{2}{3}a^2 + \frac{4}{7}) + 2$, *for all* $a \geq 3$ *and for all* $n \in \mathbb{N}$.

**Theorem**

*If* $n \geq 359$, *then*

$$M(n) \leq \frac{(a^2-2)(a^2-1)}{2},$$

*where* $a$ *is the unique positive odd integer such that* $a^2 - 2 \leq n \leq (a+2)^2 - 3$.

Notice that if $n = a^2 - 2$, then we just obtain Gerzon bound $M(n) \leq \frac{n(n+1)}{2}$. For other cases, we can prove that $M(n) \leq \frac{n(n-1)}{2}$.

# Maximum spherical two-distance sets

**Theorem (Glazyrin and Yu 2016+)**

$$g(n) = \frac{n(n+1)}{2} \quad \forall n \in \mathbb{N}, \textit{except } n = (2k+1)^2 - 3, k \in \mathbb{N}.$$

# Future work

1. $M(14) = 28$ or 29. $M(16) = 40$ or 41. Can we determine them?
2. The constructions and upper bounds for complex equiangular lines.
3. The maximum spherical 3-distance sets in $\mathbb{R}^n$. (only $n = 2, 3, 8$ and $22$ are known)
4. How much do we know the Maximum Separation Codes on sphere ? If we have $M$ points on $\mathbb{S}^{n-1}$, then $\max|\langle x_i, x_j \rangle| \geq \sqrt{\frac{M-n}{n(M-1)}}$. "$=$" attained iff equiangular tight frames.


Dr. BANNAI,
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