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VIBRATION ANALYSIS BY DISCRETE SINGULAR CONVOLUTION

G. W. Wei

Department of Computational Science, National University of Singapore, Singapore 117543, *Singapore. E-mail: cscweigw@nus.edu.sg*

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This paper explores the utility of a discrete singular convolution algorithm for vibration analysis. A number of different realizations of singular convolution kernels are selected to illustrate the present algorithm. Vibration analysis of strings, rods, beams, diatomic molecules, membranes, waveguides and thin plates are utilized to test numerical accuracy and speed of convergence of the present approach. Numerical experiments indicate that the discrete singular convolution is a simple and reliable algorithm for vibration analysis.

1. INTRODUCTION

Historically, vibration analysis is regarded as a branch of mechanics initiated by the observation of mechanical oscillators such as a pendulum. The study of vibration is concerned with the oscillatory motion of physical objects with or without external forces. Vibration analysis is one of the most important factors in the engineering design, since certain resonances may lead to the failure of major structures such as bridges, buildings, or airplane wings. Recently, vibration analysis has emerged as a multidisciplinary research field. Apart from the engineering research activities, there are studies in chemistry concerned with chemical oscillations and molecular vibrations, in physics with waveguide modes, plasma vibrators, and earthquakes, in physiology with the periodic motion of the lungs, or heart, or impact of medicine intake, and vibration of eardrums, in environment science with periodic or quasi-periodic climatic motion, and pollution of noise by man-made devices. At a more fundamental level, mathematicians are interested in non-linear quasi-periodic dynamical systems and their relation to ergodics and chaos. Conventional methods for vibration analysis are based either on theory or on experiment. However, experiments can be extremely expensive and practical problems are either too difficult or simply impossible to accomplish by analytical methods. Therefore, numerical simulations play a more and more important role in modern vibration analysis.

The advent of digital computers has given tremendous impetus to all numerical methods for solving science and engineering problems, including vibration simulations. Although there has been a great deal of achievement in developing accurate, efficient and robust computational methods, finding numerical solutions for partial differential equations (PDEs) is still a challenge owing to the presence of possible singularities and/or homoclinic manifolds that induce sharp transitions in the solutions. These phenomena can be observed in many real systems such as black holes in astronomy, shock waves in compressible fluid

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flow, vortex sheets in incompressible flow associated with a high Reynolds number, and burst events in the turbulent boundary layer. The difficulties associated with these phenomena can often be characterized by sharp changes occurring in a very small spatial region which can strongly influence the global properties of the system. The presence of these phenomena can be extremely sensitive to numerical algorithms and can easily lead to numerically induced spatial and/or temporal chaos [1]. The conventional methods for these problems may be classified as either global methods [2–6] or local methods [7–13]. In global methods, unknown functions and their derivatives are expanded in terms of a finite basis set with each element having a global support. The expansion coefficients are often determined by the method of Tau, or Galerkin, or collocation, or others. In the Tau method, the residual for a truncated expansion is required to be orthogonal to a subset of basis functions used in the expansion, which, together with the boundary conditions, determines the expansion coefficients. In the global Galerkin method, a new set of basis functions is constructed by the superposition of the original basis functions. The requirement that the residual be orthogonal to the new set of basis functions, together with the boundary conditions, determines the expansion coefficients. In the global collocation approach, the residual vanishes at a subset of node points of the highest order basis function used in the expansion. The global collocation is also called pseudospectral method. Three most important local approaches are finite difference, finite volume and finite element methods. In finite difference methods, the solution is interpolated in terms of a set of grid values; the spatial derivatives are usually approximated by algebraic expressions involving nearest-neighbor grid points. In finite volume approaches, the emphasis is on a set of integro-differential equations and their associated surface and volume integrations. The values on the boundary of each "numerical molecule" are usually interpolated by low order schemes. The spatial derivatives are approximated in the same way as those used in the finite difference methods. Finite element methods form one of the most versatile class of numerical methods. Depending on the system under study, finite element methods can be formulated either in terms of the method of weight residuals or in terms of variational principles. Usually, PDEs are integrated by using a set of trial functions, each with a small region of support. The solution is represented by linear superpositions of these trial functions. Global methods are highly localized in their spectral space, but are unlocalized in the co-ordinate space. By contrast, local methods have high spatial localization, but are delocalized in their spectral space. Moreover, the use of global methods is usually restricted to structured grids, whereas, local methods can be implemented to block-structured grids and even unstructured grids. In general, global methods are much more accurate than local methods, while the major advantage of local methods is their flexibility for handling complex geometries and boundary conditions. In ordinary applications, it is relatively safe and efficient to use either a global method or a local one for numerically solving an ordinary differential equation or a partial differential equation. However, when a differential equation has singularities and/or homoclinic orbits, neither the global methods nor the local methods can be applied without numerical instabilities. The global methods lose their accuracy near the singularities due to local high-frequency components. The local methods have to be implemented in an adaptive manner, which greatly limits their accuracy and requires extremely small (spatial and/or time) mesh sizes. In many situations, the rate of convergence of a numerical method simply cannot match the divergent rate of the problem under study at a singularity. It is desired to have a method that has both spectral and spatial localization, and is thus locally smooth and asymptotically decaying in both spectral and co-ordinate spaces, and combines a global method's accuracy with a local method's flexibility. Wavelet theory and its applications have been expected to fulfill this task. Theory of wavelets developed in recent years has great impact in telecommunication and electronic

engineering and has found their applications in a variety of other science and engineering disciplines. Mathematically, wavelets are functions generated from a single function by dilation and translation. They form building blocks for some space, such as $L^2(R)$, whether as a frame or as an orthonormal basis. Such building blocks are computationally important when they have certain regularity and localization in both time and frequency domains. Physically, the wavelet transform is a mathematical technique that can be used to split a signal into different frequency bands or components so that each component can be studied with a resolution matched to its scale, thus providing excellent frequency and spatial resolution, and achieving computational efficiency. The possibility of using wavelet theory for computational physics and engineering has been extensively explored recently. However, before wavelet approaches can be practical use in competing with the existing global and local methods, a number of technical difficulties are to be overcome. In our view, the first difficulty is the implementation of boundary conditions in a multiresolution setting. The second difficulty is the requirement of sufficiently high wavelet regularity to provide sufficiently weak solutions. Moreover, there is a lack of general and systematic numerical algorithms for incorporating wavelets in an efficient manner.

Discrete singular convolution (DSC) has recently been proposed [14] as a potential approach for numerically solving some science and engineering problems [15], including linear and non-linear dynamics, Hilbert transform, processing of analytic signals, electromagnetics, and computed tomography. The DSC algorithm is found to provide a basis for a unification of many computational methods [16–18] such as methods of global, local, Galerkin, collocation and finite difference. In fact, the underlying mathematical structure of DSC is the theory of distributions and wavelet analysis. One of the distributions used in the aforementioned applications is the Dirac delta function which is a generalized function following from the fact that it is an integrable function inside a particular interval but itself needs not to have a value. Heaviside introduced both the unit step Heaviside function and the Dirac delta function as its derivative and referred to the latter as the *unit impulse*. Dirac, for the first time, explicitly discussed the properties of δ in his classic text on quantum mechanics; for this reason δ is often called Dirac delta function. However, delta distribution has a history which antedates both Heaviside and Dirac. It appeared in an explicit form as early as 1822, in Fourier's *Théorie Analytique de la Chaleur*. The work of Heaviside, and subsequently of Dirac, in the systematic but informal exploitation of the step function and delta function has made delta distribution familiar to physicists and engineers before Sobolev, Schwartz [19], Korevaar [20] and others put it into a rigorous mathematical form. General orthogonal series analysis of the delta distribution have been subsequently studied by Walter [21] and others [22–24]. The use of many delta sequences as probability density estimators was discussed by Walter and Blum [24] and others [23, 25, 26].

The purpose of the present paper is to explore the utility of the DSC algorithm for vibration analysis. This is illustrated by numerically solving a few classes of vibration problems. This paper is organized as follows. Section 2 is devoted to a brief review of the theory and implementation of the DSC algorithm. We only review the part of the theory that is relevant to the present applications. The reader is referred to reference [14] for more details. The numerical application of vibration analysis by using the DSC algorithm is presented in section 3. Mechanical vibrations of strings, rods, beams are discussed, followed by the numerical analysis of the quantum vibrations of I_2 molecules. To further test the present DSC algorithm, the vibration of a rectangular membrane is analyzed. Accurate frequency computation of waveguide modes, which is crucial for the design of electromagnetic devices, is carefully studied before a high-accuracy analysis of thin plate vibrations is given. This paper ends with a brief conclusion.

2. DISCRETE SINGULAR CONVOLUTION

It is convenient to discuss the theory of *singular convolution* in the context of distributions. Let us denote by T a distribution and $\eta(t)$ as an element of the space of test functions. A singular convolution can be expressed as

$$F(t) = (T * \eta)(t) = \int_{-\infty}^{\infty} T(t-x)\eta(x)\,\mathrm{d}x. \tag{1}$$

Here $T(t-x)$ is a singular kernel. Depending on the form of the kernel T, the singular convolution is the central issue for a wide range of science and engineering problems. For example, singular kernels of the *Hilbert type* have a general form of

$$T(x) = \frac{1}{x^n} \quad (n = 1, 2, \dots). \tag{2}$$

Here, kernel $T(x) = 1/x$ commonly occurs in electrodynamics, theory of linear response, signal processing, theory of analytic functions, and the Hilbert transform; $T(x) = 1/x^2$ is the kernel used in tomography. Singular kernels of the *Abel type*

$$T(x) = \frac{1}{x^\beta} \quad (0 < \beta < 1), \tag{3}$$

was introduced in the tautochrone problem. It has applications in the area of holography and interferometry with phase objects and is of practical importance in aerodynamics, heat and mass transfer, and plasma diagnostics. Other interesting examples are singular kernels of the *delta type*

$$T(x) = \delta^{(n)}(x) \quad (n = 0, 1, 2, \dots). \tag{4}$$

Here, kernel $T(x) = \delta(x)$ is important for interpolation of surfaces and curves (including atomic, molecular and biological potential energy surfaces); and $T(x) = \delta^{(n)}(x)$ $(n = 1, 2, \dots)$ are essential for numerically solving partial differential equations. However, since these kernels are singular, they cannot be directly digitized in computer. Hence, the singular convolution, (1), is of little numerical merit. To avoid the difficulty of using singular expressions directly in computer, sequences of approximations (T_α) of the distribution T can be constructed

$$\lim_{\alpha \to \alpha_0} T_\alpha(x) \to T(x), \tag{5}$$

where α_0 is a generalized limit. Obviously, in the case of $T(x) = \delta(x)$, the sequence, $T_\alpha(x)$, is a delta sequence. Moreover, with a sufficiently smooth approximation, it is useful to consider a *discrete singular convolution* (DSC)

$$F_\alpha(t) = \sum_k T_\alpha(t - x_k) f(x_k), \tag{6}$$

where $F_\alpha(t)$ is an approximation to $F(t)$ and $\{x_k\}$ is an approximate set of discrete points on which the DSC (6) is well defined. Note that, the original test function $\eta(x)$ has been replaced by $f(x)$. The mathematical property or requirement of $f(x)$ is determined by the

approximate kernel T_α. In general, the convolution is required being Lebesgue integrable. A sequence of approximation can be improved by a regularizer

$$\lim_{\alpha \to \infty} R_\sigma(x) = 1. \tag{7}$$

The regularizer is designed to increase the regularity of convolution kernels. For the delta sequence, it follows from Eq. (5) that

$$\int \lim_{\alpha \to \alpha_0} T_\alpha(x) R_\sigma(x)\, \mathrm{d}x = R_\sigma(0) = 1, \tag{8}$$

where $R_\sigma(0) = 1$ is the special requirement for a *delta regularizer*. A typical delta regularizer used in this work and elsewhere [27] is $\exp(-x^2/2\sigma^2)$. However, for certain class of problems, this regularization may not be necessary (see for example, the molecular vibration in the next section).

As a special example, Shannon's kernel $\sin \alpha x/\pi x$ is a delta sequence kernel

$$\lim_{\alpha \to \alpha_0} \left\langle \frac{\sin \alpha x}{\pi x}, \eta(x) \right\rangle = \eta(0). \tag{9}$$

The Shannon delta sequence kernel can be regarded as a series of the Fourier base approximations of the delta kernel. Similarly, Hermite function expansion of the delta distribution is given by

$$\delta_{n,\sigma}(x) = \exp(-x^2/2\sigma^2) \sum_{k=0}^{n} \left(\frac{-1}{4}\right)^k \frac{1}{\sqrt{\pi} k!} H_{2k}\left(\frac{x}{\sqrt{2\sigma^2}}\right), \tag{10}$$

where $H_{2k}(x/\sqrt{2}\sigma)$ is the usual Hermite polynomial. This delta sequence was studied by Schwartz [19], Korevaar [28], and Hoffman *et al.* [29]. Various cases involving many classic L^2 bases can be found in Walter and Blum's paper [24]. Other important examples include the Dirichlet kernel

$$\frac{\sin[(l + \frac{1}{2})(x - x')]}{2\pi \sin[\frac{1}{2}(x - x']},$$

the modified Dirichlet kernel

$$\frac{\sin[(l + \frac{1}{2})(x - x')]}{2\pi \tan[\frac{1}{2}(x - x')]}$$

and the de la Vallée Poussin kernel

$$\frac{1}{\pi\alpha} \frac{\cos[\alpha(x - x')] - \cos[2\alpha(x - x')]}{(x - x')^2}.$$

For sequences of both the delta type and the Hilbert type, an interpolating (or quasi-interpolating) algorithm sampling at *Nyquist frequency*, $\alpha = \pi/\Delta$, has great advantage over a non-interpolating discretization. Hence the Shannon's kernel is discretized and regularized [27] as

$$\frac{\sin[\alpha(x - x')]}{\pi(x - x')} \to \frac{\sin(\pi/\Delta)(x - x_k)}{(\pi/\Delta)(x - x_k)} \mathrm{e}^{-(x - x_k)^2/2\sigma^2}. \tag{11}$$

The wavelet connection of this expression is discussed elsewhere [27] and it is referred to as a quasi-wavelet scaling function (QWSF). Not only does the interpolating (or quasi-interpolating) nature guarantee the highest accuracy on the set of grid points, but also it provides the highest possible computational efficiency off a grid. This is because the *Nyquist interval* given by $[-\pi/\Delta, \pi/\Delta]$ is the largest possible sampling interval that is free of alias whenever the L^2 function $f(x)$ under study satisfies the *Nyquist condition*

$$\operatorname{supp} \hat{f}(k) \subset \left\{ -\frac{\pi}{\Delta}, \frac{\pi}{\Delta} \right\}. \tag{12}$$

This fact can be formally given by Shannon's sampling theorem

$$f(x) = \sum_{k=-\infty}^{\infty} f(x_k) \frac{\sin(\pi/\Delta)(x - x_k)}{(\pi/\Delta)(x - x_k)}. \tag{13}$$

The significance of Shannon's sampling theorem is that by a discrete, but infinite set of sampling data $\{ f(x_k) \}$ one can actually recover a band-limited L^2 function on a real line. Shannon's sampling theorem has great impact on information theory, signal and image processing because the Fourier transform of Shannon's kernel is an ideal low-pass filter for signals band-limited to $[-\pi/\Delta, \pi/\Delta]$.

The uniform, Nyquist-rate, interpolating discretization and the regularization are used for the Dirichlet kernel:

$$\frac{\sin[(l + \frac{1}{2})(x - x')]}{2\pi \sin[\frac{1}{2}(x - x')]} \to \frac{\sin((\pi/\Delta)(x - x_k))}{(2M + 1)\sin((\pi/\Delta)(x - x_k)/(2M + 1))} \exp\left(-\frac{(x - x_k)^2}{2\sigma^2} \right). \tag{14}$$

This is referred to as regularized Dirichlet kernel (RDK). In comparison to Shannon's kernel, the Dirichlet kernel has one more parameter M which can be optimized to achieve better results in computations. Usually, we set a sufficiently large M for various numerical applications. Obviously, the Dirichlet kernel converts to Shannon's kernel at the limit of $M \to \infty$. These uniform interpolating discretization and the regularization will also be used for the modified Dirichlet kernel

$$\frac{\sin[(l + \frac{1}{2})(x - x')]}{2\pi \tan[\frac{1}{2}(x - x')]} \to \frac{\sin((\pi/\Delta)(x - x_k))}{(2M + 1)\tan((\pi/\Delta)(x - x_k)/(2M + 1))} \exp\left(-\frac{(x - x_k)^2}{2\sigma^2} \right) \tag{15}$$

and for the de la Vallée Poussin kernel

$$\frac{1}{\pi\alpha} \frac{\cos[\alpha(x - x')] - \cos[2\alpha(x - x')]}{(x - x')^2} \to \frac{2}{3} \frac{\cos(\pi/\bar{\Delta})(x - x_k) - \cos(2\pi/\bar{\Delta})(x - x_k)}{[(\pi/\bar{\Delta})(x - x_k)]^2}$$

$$\times \exp\left(-\frac{(x - x_k)^2}{2\sigma^2} \right), \tag{16}$$

where $\bar{\Delta} = \frac{3}{2}\Delta$. We refer to the right-hand sides of equations (15) and (16) as regularized modified Dirichlet kernel (RMDK) and regularized de la Vallée Poussin kernel (RDLVPK) respectively.

Since π/Δ is proportional to the highest frequency which can be reached in the Fourier representation, the Δ should be very small for a given problem involving very oscillatory

functions or very high frequency components. Finally, a regularized Lagrange kernel (RLK) [30]

$$\prod_{i \neq k}^{2M} \frac{x - x_i}{x_k - x_i} \exp\left(- \frac{(x - x_k)^2}{2\sigma^2} \right), \tag{17}$$

will also be employed in some calculations.

We use a symmetrically (or antisymmetrically) truncated, translation invariant convolution kernel

$$f^{(n)}(x) \approx \sum_{k=-W}^{W} \delta_{\alpha,\sigma}^{(n)}(x - x_k) f(x_k) \quad (n = 0, 1, 2, \dots), \tag{18}$$

where $2W + 1$ is the computational bandwidth, or effective kernel support, which is usually smaller than the whole computational domain, $[a, b]$. Here $\delta_{\alpha,\sigma}^{(n)}(x - x_k)$ is a collective symbol for the nth derivative of any of the right-hand side of equations (11), (14)–(17).

Consider a linear operator $\mathscr{L}$ having a differential part $\mathscr{D}$ and a function part F:

$$\mathscr{L} = \mathscr{D} + F. \tag{19}$$

In the DSC approach, it is convenient to choose a grid representation for the co-ordinate so that the function part F of the operator is diagonal. Hence, its discretization is simply given by a direct interpolation on the grid

$$F(x) \to F(x_k)\delta_{m,k}. \tag{20}$$

The differential part of the operator on the co-ordinate grid is then represented by functional derivatives

$$\mathscr{D} = \sum_n d_n(x) \frac{\mathrm{d}^n}{\mathrm{d}x^n} \to \sum_n d_n(x_m) \delta_{\alpha,\sigma}^{(n)}(x_m - x_k), \tag{21}$$

where $d_n(x)$ is a coefficient and $\delta_{\alpha,\sigma}^{(n)}(x_m - x_k)$ is *analytically* given by

$$\delta_{\alpha,\sigma}^{(n)}(x_m - x_k) = \left[\left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^n \delta_{\alpha,\sigma}(x - x_k) \right]_{x = x_m}. \tag{22}$$

Thus, the full DSC-matrix representation for the operator, $\mathscr{L}$, is given by

$$\mathscr{L}(x_m - x_k) = \sum_n d_n(x_m) \delta_{\alpha,\sigma}^{(n)}(x_m - x_k) + F(x_m)\delta_{m,k}. \tag{23}$$

This treatment can be easily extended to higher dimensions as the generalized functions can be easily extended to higher dimensions.

In the present study we limit our attention to the quasi-wavelet scaling function (QWSF), regularized Dirichlet kernel (RDK), regularized modified Dirichlet kernel (RMDK), regularized de la Vallée Poussin kernel (RDLVPK) and regularized Lagrange kernel (RLK). Nevertheless, various other delta sequence kernels can be similarly employed [14]. It is noted that the QWSF is parameter free, which is convenient for applications. The $2M + 1$ parameter for the RDK and RMDK is chosen as 188 and $2M = 80$ is used for the RLK for all calculations. We note that as long as the M value chosen is sufficiently large the

numerical results are not sensitive to the specific values used. It is noted that the choice of M does not depend on the computational grid.

3. NUMERICAL APPLICATIONS

In this section, we consider problems of various physical origins, including vibrations of strings, rods, beams, molecules, membranes, waveguides and plates, to test the present algorithm and to illustrate its application. Details of these applications are described in the following six subsections.

3.1. VIBRATION OF STRINGS AND RODS

3.1.1. *Lateral vibration of spring*

Let us consider a string of line density v that is taut initially with a tension T. We are assuming that the deflection of the string does not affect the tension of the string, so that the tension remains constant at all times. The displacement $y(x, t)$ of the taut string in the normal direction is governed[†]

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{24}$$

where $c = (T/v)^{1/2} \geqslant 0$.

3.1.2. *Longitudinal vibration of rods*

The motion $u(x, t)$ of a rod in the longitudinal direction is governed by

$$\frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x}\right) = v \frac{\partial^2 u}{\partial t^2}, \tag{25}$$

where v is the line density of the rod, A is the cross-section and E is the modulus of the elasticity. Equation (25) does not admit an analytical solution in general. However, for a rod of constant properties along its length, one obtains

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{26}$$

where $c = (E/\rho)^{1/2} \geqslant 0$ is the sound speed in the rod, and $\rho = v/A$ is the density of the rod. Both equations (24) and (25) are a one-dimensional wave equation which also describes the propagation of electromagnetic waves in homogeneous media and torsional vibration of rods induced by shear modulus [31]. As a test calculation, we consider a string or rod of finite length a and denote the common equation as

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}, \quad x \in [0, a], \tag{27}$$

[†]by Introduced by D'Alembert in his memoitr to the Berlin Academy in 1750.

TABLE 1

Errors for string and rod vibrations

		QWSF		RDK	
n	Exact	$N = 22$	$N = 32$	$N = 22$	$N = 32$
1	0·01	4·47(− 10)	7·98(− 17)	4·57(− 10)	1·31(− 15)
2	0·04	4·05(− 10)	2·29(− 16)	4·14(− 10)	1·22(− 15)
3	0·09	3·38(− 10)	1·94(− 16)	3·46(− 10)	1·64(− 15)
4	0·16	2·52(− 10)	5·83(− 16)	2·58(− 10)	1·53(− 15)
5	0·25	1·54(− 10)	3·33(− 16)	1·57(− 10)	1·75(− 15)
6	0·36	5·16(− 11)	5·00(− 16)	5·28(− 11)	1·22(− 15)
7	0·49	4·24(− 11)	6·11(− 16)	4·36(− 11)	2·16(− 15)
8	0·64	1·09(− 10)	3·33(− 16)	1·13(− 10)	1·78(− 15)
9	0·81	2·93(− 10)	8·88(− 16)	2·92(− 10)	1·44(− 15)
10	1·00	5·19(− 09)	1·33(− 15)	5·06(− 09)	7·77(− 16)

with boundary conditions

$$w(0, t) = w(a, t) = 0. \tag{28}$$

By the method of separation of variables, equation (28) can be solved as an eigenvalue problem for the string or rod vibration with the exact eigenvalue given by $(n\pi/a)^2$.

To demonstrate the high accuracy of the DSC algorithm for string and rod vibrations, we choose the QWSF and RDK for the grid representation. Direct diagonalization by using a standard eigenvalue solver is performed to generate eigenvalues and eigenfunctions. In the closed interval of $[0, 10\pi]$, two sets of grids, $N = 22$ and 32, are employed associated with $\sigma/\Delta = 3{\cdot}7$ and 4·0 respectively. Our results are extremely accurate as indicated by the absolute errors listed in Table 1.

3.2. VIBRATION OF BEAMS

The natural vibration of a beam of uniform flexural rigidity, subjected to either tensile or compressive axial forces whose magnitude is below the value of the so-called Euler elastic buckling critic, can be formulated as an eigenvalue problem [32]

$$\frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} + \frac{S}{EI}\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = k^2 \frac{mA}{EI} w(x), \quad x \in [0, a], \tag{29}$$

with simply supported boundary condition

$$w(0) = \frac{\mathrm{d}^2 w(0)}{\mathrm{d}x^2} = 0, \quad w(a) = \frac{\mathrm{d}^2 w(a)}{\mathrm{d}x^2} = 0, \tag{30}$$

where a positive S represents the magnitude of a compressive force and a negative S represents the magnitude of a tensile force, m and A are the mass density of the material and the cross-sectional area of the beam respectively.

TABLE 2

Errors for beam eigenvalues ($S/EI = 0$)

n	Exact $k_n^2\, mA/EI$	QWSF		RDK	
		$N = 22$	$N = 32$	$N = 22$	$N = 32$
1	0·0001	2·85(− 09)	2·80(− 14)	2·92(− 00)	1·21(− 14)
2	0·0016	2·56(− 09)	1·87(− 14)	2·63(− 09)	3·97(− 15)
3	0·0081	2·11(− 09)	2·35(− 14)	2·17(− 09)	3·93(− 15)
4	0·0256	1·53(− 09)	2·06(− 14)	1·56(− 09)	1·29(− 15)
5	0·0625	8·41(− 10)	2·92(− 14)	8·62(− 10)	5·36(− 15)
6	0·1296	1·14(− 10)	2·07(− 14)	1·16(− 10)	8·38(− 15)
7	0·2405	5·96(− 10)	2·52(− 14)	6·11(− 10)	9·02(− 15)
8	0·4096	1·17(− 09)	1·61(− 14)	1·20(− 09)	8·55(− 15)
9	0·6561	3·47(− 09)	2·01(− 14)	3·47(− 09)	6·55(− 15)
10	1·0000	5·73(− 08)	2·01(− 14)	5·58(− 08)	5·22(− 15)

This problem is analytically soluble and its exact solutions have the form

$$w_n(x) = B_n \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, \ldots, \tag{31}$$

where B_n is an arbitrary constant, which satisfies all of the boundary conditions in equation (30). A set of eigenvalues are given by

$$k_n^2 \frac{mA}{EI} = \left(\frac{n\pi}{a}\right)^4 - \frac{S}{EI}\left(\frac{n\pi}{a}\right)^2. \tag{32}$$

To ensure the statement of an eigenvalue problem, the compressive force is required to be bounded from above

$$S < EI\left(\frac{n\pi}{a}\right)^2. \tag{33}$$

To demonstrate the high accuracy of the DSC algorithm for beam eigenvalue problems, we discretize the grid by using the QWSF and RDK. In the closed interval of $[0, 10\pi]$, two sets of grids, $N = 22$ and 32, are employed in associated with $\sigma/\Delta = 3{\cdot}7$ and 4·0 respectively. Table 2 lists the absolute errors when no load is imposed ($S/EI = 0$). The errors in the cases of compressive force ($S/EI = 0{\cdot}01$) and the tensile force ($S/EI = -1$) are given in Tables 3 and 4 respectively. We note that all errors are extremely small even if only 22 grid points are deployed in the large interval of 10π. In fact, when $N = 32$, all of the first 10 eigenvalues are of machine precision.

3.3. VIBRATIONS OF I_2 MOLECULES

To illustrate the use of the DSC algorithm for molecular vibrations, we consider a benchmark problem: the Morse potential for the I_2 molecule vibration. The Schrödinger

TABLE 3

Errors for beam eigenvalues ($S/EI = 0{\cdot}01$)

n	Exact $k_n^2\, mA/EI$	QWSF		RDK	
		$N = 22$	$N = 32$	$N = 22$	$N = 32$
1	0·0000	2·84(− 09)	3·81(− 14)	2·91(− 09)	1·05(− 14)
2	0·0012	2·56(− 09)	2·11(− 14)	2·62(− 09)	5·95(− 15)
3	0·0072	2·11(− 09)	2·28(− 14)	2·16(− 09)	7·37(− 15)
4	0·0240	1·52(− 09)	1·89(− 14)	1·56(− 09)	3·36(− 15)
5	0·0600	8·40(− 10)	2·50(− 14)	8·60(− 10)	3·09(− 15)
6	0·1260	1·13(− 10)	1·66(− 14)	1·15(− 10)	3·19(− 15)
7	0·2352	5·96(− 10)	2·37(− 14)	6·11(− 10)	7·24(− 15)
8	0·4032	1·17(− 09)	1·96(− 14)	1·20(− 09)	6·77(− 15)
9	0·6480	3·48(− 09)	2·07(− 14)	3·47(− 09)	9·77(− 15)
10	0·9900	5·72(− 08)	1·81(− 14)	5·58(− 08)	3·77(− 15)

TABLE 4

Errors for beam eigenvalues ($S/EI = -1{\cdot}00$)

n	Exact $k_n^2\, mA/EI$	QWSF		RDK	
		$N = 22$	$N = 32$	$N = 22$	$N = 32$
1	0·0101	3·30(− 09)	2·45(− 14)	3·38(− 09)	1·20(− 14)
2	0·0461	2·97(− 09)	1·25(− 14)	3·05(− 09)	2·16(− 15)
3	0·0981	2·46(− 09)	2·25(− 14)	2·51(− 09)	8·66(− 15)
4	0·1856	1·79(− 09)	1·03(− 14)	1·83(− 09)	3·16(− 15)
5	0·3125	1·01(− 09)	2·15(− 14)	1·04(− 09)	1·24(− 14)
6	0·4896	1·88(− 10)	2·27(− 14)	1·92(− 10)	1·58(− 14)
7	0·7301	6·10(− 10)	2·69(− 14)	6·25(− 10)	1·91(− 14)
8	1·0496	1·25(− 09)	2·53(− 14)	1·28(− 09)	5·33(− 15)
9	1·4661	3·74(− 09)	1·27(− 14)	3·73(− 09)	6·66(− 16)
10	2·0000	6·25(− 08)	1·02(− 14)	6·09(− 08)	1·33(− 15)

equation for the problem in the co-ordinate representation is given by

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \right] \Phi_k(x) = E_k \Phi_k(x), \tag{34}$$

where Φ_k and E_k are the kth eigenfunction and eigenvalue respectively, and $\hbar$ is the Planck constant divided by 2π. Here the Morse potential for the I_2 molecule is given by

$$V(x) = D[\mathrm{e}^{-2\alpha x} - 2\mathrm{e}^{-\alpha x} + 1], \tag{35}$$

where $D = 0{\cdot}0224$ a.u., $\alpha = 0{\cdot}9374$ a.u., and the reduced mass for the Schrödinger equation is $\mu = 119406$ a.u.. The anharmonic character of Morse potential allows dissociation, here it is one of the most popular potentials for modelling the spectroscopy and dynamics of the I_2 molecule. The Schrödinger equation of the I_2 Morse system is actually soluble. The analytical results for the eigenfunctions are the well-known generalized Laguerre

polynomials [33]

$$\Phi_k = N_k z^{p/2} \mathrm{e}^{-z/2} L_k^P(z), \tag{36}$$

where $z = \beta \mathrm{e}^{-\alpha x}$, $p = \beta - 2k - 1$, $\beta = 156{\cdot}047612535$. Here, N_k is the normalization constant and is given by [33]

$$N_k = \left[\frac{\Gamma(p)}{\alpha} \sum_{\gamma=0}^{k} (-1)^{\gamma} \binom{-p}{\gamma} \right]^{-1/2}. \tag{37}$$

The analytical expression of the eigenvalues of the I_2 molecule is [33]

$$E_k = \kappa \left[k + \frac{1}{2} - \frac{1}{\beta} \left(k + \frac{1}{2} \right)^2 \right], \tag{38}$$

where $\kappa = 5{\cdot}741837286 \times 10^{-4}$ a.u. is calculated according to the physical property of I_2. Since κ is very small, the density of state of this system is obviously very high. Thus, it often serves as a standard problem for testing new numerical algorithms for nearly degenerated systems.

In a recent study, Braun *et al.* [34] has used this system to test their efficient Chebyshev–Lanczos method. They achieve a remarkably high accuracy which ranges from 7 to 9 digits using 128 grid points. To obtain a desired number of eigenvalues and eigenfunctions, we directly diagonalize the DSC-Hamiltonian matrix. In these calculations, the regularization is unnecessary because the potential function is bounded from below. Therefore, we set $\mathrm{e}^{-x^2/2\sigma^2} = 1$ in all kernels. In computations utilizing the QWSF, RDK, RMDK the number of grid points used is 64 ($N = 64$) which corresponds to the grid spacing of 0·043077 ($\Delta = 0{\cdot}043077$). For a comparison, we have listed our calculation and those of Braun *et al.* [34] in Table 5. As seen from the Table 5, the present 64 grid point results are from 200 to 1000 times more accurate than those of Braun *et al.*, obtained by using 128 grid points. Our test indicates that it requires about 1·5 time as many grid points ($N = 96$) for the RDLVPK to achieve the same level of accuracy as those obtained by using three other kernels.

3.4. VIBRATION OF MEMBRANES

A membrane is a thin elastic sheet that is initially taut with a uniform tension p per unit area through its boundary. It is further assumed that the applied deformation is much less than the initial stretch due to the uniform tension and can be regarded as a constant. The small transverse vibrations of a tightly stretched membrane are governed by the two-dimensional wave equation of the form [35]

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \tag{39}$$

where $u(x, y, t)$ denotes the tranverse displacement of the membrane from its equilibrium position and $c = (ph/\rho)^{1/2} \geqslant 0$ is the sound velocity in the membrane. Here h is the thickness and ρ the density of the membrane. The membrane is supposed to cover the rectangle $[0, a] \times [0, b]$ with edge fixed; thus $u(x, y, t) = 0$ for $x = 0$, $x = a$, $y = 0$, $y = b$. This Sturm–Liouville problem can be solved by the method of separation of variables and has

TABLE 5

Comparison of errors for the I_2 Morse oscillator

k	Exact	Braun *et al.* ($N = 128$)	QWSF ($N = 64$)	RDK ($N = 64$)	RMDK ($N = 64$)	RDLVPK ($N = 96$)
0	0·8529966236266942(− 03)	− 0.10(− 10)	− 0·14(− 13)	− 0·14(− 13)	− 0·14(− 13)	− 0·14(− 13)
1	0·1412462184629706(− 02)	− 0·30(− 10)	− 0·42(− 13)	− 0·42(− 13)	− 0·42(− 13)	− 0·42(− 13)
2	0·1964568661834224(− 02)	− 0·50(− 10)	− 0·70(− 13)	− 0·70(− 13)	− 0·70(− 13)	− 0·70(− 13)
3	0·2509316055240247(− 02)	− 0·70(− 10)	− 0·97(− 13)	− 0·97(− 13)	− 0·97(− 13)	− 0·97(− 13)
4	0·3046704364847777(− 02)	− 0·89(− 10)	− 0·12(− 12)	− 0·12(− 12)	− 0·12(− 12)	− 0·12(− 12)
5	0·3576733590656813(− 02)	− 0·11(− 09)	− 0·15(− 12)	− 0·15(− 12)	− 0·15(− 12)	− 0·15(− 12)
6	0·4099403732667354(− 02)	− 0·13(− 09)	− 0·18(− 12)	− 0·18(− 12)	− 0·18(− 12)	− 0·18(− 12)
7	0·4614714790879402(− 02)	− 0·15(− 09)	− 0·20(− 12)	− 0·20(− 12)	− 0·20(− 12)	− 0·20(− 12)
8	0·5122666765292955(− 02)	− 0·16(− 09)	− 0·23(− 12)	− 0·23(− 12)	− 0·23(− 12)	− 0·23(− 12)
9	0·5623259655908014(− 02)	− 0·18(− 09)	− 0·25(− 12)	− 0·28(− 12)	− 0·25(− 12)	− 0·25(− 12)
10	0·6116493462724579(− 02)	− 0·20(− 09)	− 0·28(− 12)	− 0·28(− 12)	− 0·28(− 12)	− 0·28(− 12)
11	0·6602368185742650(− 02)	− 0·22(− 09)	− 0·30(− 12)	− 0·30(− 12)	− 0·30(− 12)	− 0·30(− 12)
12	0·7080883824962227(− 02)	− 0·23(− 09)	− 0·33(− 12)	− 0·33(− 12)	− 0·33(− 12)	− 0·33(− 12)
13	0·7552040380383310(− 02)	− 0·25(− 09)	− 0·35(− 12)	− 0·35(− 12)	− 0·35(− 12)	− 0·35(− 12)
14	0·8015837852005899(− 02)	− 0·27(− 09)	− 0·37(− 12)	− 0·37(− 12)	− 0·37(− 12)	− 0·37(− 12)
15	0·8472276239829993(− 02)	− 0·28(− 09)	− 0·39(− 12)	− 0·39(− 12)	− 0·39(− 12)	− 0·40(− 12)
16	0·8921355543855595(− 02)	− 0·30(− 09)	− 0·41(− 12)	− 0·41(− 12)	− 0·41(− 12)	− 0·41(− 12)
17	0·9363075764082702(− 02)	− 0·32(− 09)	− 0·42(− 12)	− 0·42(− 12)	− 0·42(− 12)	− 0·43(− 12)
18	0·9797436900511314(− 02)	− 0·33(− 09)	− 0·42(− 12)	− 0·42(− 12)	− 0·42(− 12)	− 0·43(− 12)
19	0·1022443895314143(− 01)	− 0·35(− 09)	− 0·39(− 12)	− 0·39(− 12)	− 0·39(− 12)	− 0·39(− 12)
20	0·1064408192197306(− 01)	− 0·36(− 09)	− 0·29(− 12)	− 0·30(− 12)	− 0·29(− 12)	− 0·28(− 12)
21	0·1105636580700619(− 01)	− 0·38(− 09)	− 0·13(− 12)	− 0·13(− 12)	− 0·13(− 12)	− 0·10(− 13)
22	0·1146129060824082(− 01)	− 0·39(− 09)	− 0·14(− 11)	− 0·14(− 11)	− 0·14(− 11)	− 0·61(− 12)
23	0·1185885632567697(− 01)	− 0·41(− 09)	− 0·29(− 11)	− 0·29(− 11)	− 0·29(− 11)	− 0·19(− 11)
24	0·1224906295931461(− 01)	− 0·42(− 09)	− 0·17(− 11)	− 0·17(− 11)	− 0·18(− 11)	− 0·46(− 11)

TABLE 6

Errors for membrane vibrations

Nth	Exact $\times 10^2$	QWSF	RLK	Nth	Exact $\times 10^2$	QWSF	RLK
5	10	2·7(− 13)	8·5(− 15)	55	82	1·8(− 10)	3·0(− 12)
10	17	2·6(− 13)	1·2(− 14)	60	85	3·5(− 13)	6·5(− 14)
15	25	1·0(− 14)	4·3(− 14)	65	97	1·4(− 13)	1·5(− 13)
20	32	7·9(− 12)	4·3(− 13)	70	101	1·7(− 14)	4·2(− 11)
25	40	1·8(− 14)	1·3(− 14)	75	106	5·7(− 14)	1·3(− 13)
30	45	1·8(− 14)	1·3(− 14)	80	116	2·4(− 13)	4·8(− 12)
35	52	2·6(− 12)	1·1(− 13)	85	122	9·5(− 14)	1·9(− 10)
40	61	5·5(− 12)	1·7(− 13)	90	128	5·3(− 13)	3·7(− 11)
45	65	1·4(− 12)	1·2(− 10)	95	136	1·5(− 12)	1·7(− 12)
50	73	6·6(− 13)	3·7(− 12)	100	145	1·2(− 11)	4·5(− 11)

a general solution of the form

$$u_{mn}(x, y, t) = \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{b}(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t), \tag{40}$$

where the frequency is given by

$$\omega_{mn} = c\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}. \tag{41}$$

Here the constants A_{mn} and B_{mn} can be chosen to fit various initial conditions by using the superposition principle and the methods of double Fourier series.

To simplify the problem further, we take the computational domain as $[0, 10\pi] \times [0, 10\pi]$. Both the QWSF and the RLK are deployed at a 32^2 mesh associated $\sigma = 3{\cdot}8\Delta$ and $2{\cdot}8\Delta$ respectively. Selected eigenvalues and errors of the present numerical calculation are listed in Table 6, obtained by the direct diagonalization of the DSC-matrices.

3.5. WAVEGUIDE MODES

The study of waveguide is based on the theory of electromagnetics. The precision of output frequency is particularly important in the waveguide design. Let us consider the problem of finding the eigenvalues that determine the parameters of waveguide modes, resonant frequencies of resonators, and many other physical parameters. To simplify the present discussion, we limit our attention to a rectangular waveguide, with propagation in the z direction. For the transverse magnetic (TM) mode, the four-field components E_x, E_y, H_x and H_y can be expressed in terms of E_z. In turn, E_z can be written as

$$E_z(x, y, z, t) = E(x, y)e^{i(\omega t - \alpha z)}, \tag{42}$$

where E satisfies

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + k^2 E = 0 \tag{43}$$

TABLE 7

Errors for waveguide modes

*N*th	Exact	QWSF	RDK	RDL
5	0·025625	4·1(− 14)	1·1(− 14)	1·6(− 14)
10	0·222500	5·1(− 14)	1·6(− 15)	3·8(− 14)
15	0·300625	6·8(− 14)	1·1(− 14)	1·6(− 14)
20	0·400625	3·9(− 14)	3·0(− 14)	6·7(− 14)
25	0·500000	4·5(− 14)	1·4(− 13)	1·5(− 14)
30	0·572500	3·5(− 14)	3·9(− 14)	7·3(− 14)
35	0·652500	7·4(− 15)	1·1(− 14)	4·5(− 14)
40	0·750625	2·0(− 13)	4·6(− 14)	2·3(− 13)
45	0·825625	1·8(− 14)	7·4(− 14)	7·6(− 14)
50	0·922500	2·4(− 13)	2·2(− 14)	2·1(− 13)
55	1·015625	3·2(− 14)	1·1(− 15)	2·0(− 13)
60	1·062500	1·5(− 13)	1·9(− 13)	5·4(− 13)
65	1·200625	1·1(− 13)	2·5(− 13)	6·7(− 14)
70	1·255625	7·7(− 14)	1·7(− 13)	1·3(− 14)
75	1·355625	2·6(− 14)	3·2(− 14)	1·4(− 13)
80	1·425625	1·3(− 13)	1·0(− 14)	1·2(− 13)
85	1·515625	1·0(− 13)	2·2(− 13)	6·5(− 14)
90	1·600625	4·7(− 14)	4·4(− 13)	6·0(− 12)
95	1·690000	1·1(− 12)	9·9(− 13)	6·0(− 11)
100	1·765625	2·9(− 14)	4·1(− 13)	5·1(− 13)

and vanishes on the boundary ($E = 0$). The eigenvalue k^2 determines the phase parameter α through

$$k^2 = \omega^2 v\varepsilon - \alpha^2, \tag{44}$$

where ε and v are dielectric constant and magnetic permeability respectively. To simplify the problem further, we take the computational domain as $[0, 10\pi] \times [0, 8\pi]$. The QWSF, RDK and the RLK are deployed at a 33×33 mesh with $\sigma = 3{\cdot}8\Delta$ for the former two kernels and $2{\cdot}8\Delta$ for the RLK. Results are obtained by a direct diagonalization of the DSC-matrix with a standard eigenvalue solver. Selected eigenvalues and errors of the present numerical calculation are listed in Table 7. The present results are accurate to at least 11 significant figures.

3.6. VIBRATION OF PLATES

Plate analysis is an important component of engineering design, associated with applications in many engineering fields, such as civil, mechanical, aerospace, etc. Apart from a few analytical soluble cases, there is no general solution for plate vibrations. Numerical simulation is one of the major approaches for plate analysis. Various numerical methods have been used for plate computations. In the present study, we consider only an isotropic plate with homogeneous boundary conditions for which analytical solutions are available for comparisons.

Let us consider the vibration of an isotropic plate with an undeformed middle surface having a governing equation [36]

$$\nabla^4 W + \frac{I}{D} \nabla^2 W - \frac{\rho\omega^2}{D} W = 0, \tag{45}$$

TABLE 8

Errors for plate vibrations

*N*th	Exact $\times 10^4$	Error	*N*th	Exact $\times 10^4$	Error	*N*th	Exact $\times 10^4$	Error
1	4	1·74(− 12)	34	2704	6·81(− 14)	67	9604	5·68(− 12)
2	25	1·54(− 12)	35	2704	3·64(− 12)	68	10000	1·34(− 13)
3	25	2·16(− 13)	36	2809	1·71(− 12)	69	10000	4·03(− 12)
4	64	1·79(− 13)	37	2809	1·05(− 13)	70	10201	1·80(− 12)
5	100	9·33(− 13)	38	3364	7·50(− 13)	71	10201	3·93(− 13)
6	100	2·72(− 14)	39	3364	5·65(− 13)	72	10816	2·76(− 13)
7	169	2·15(− 14)	40	3721	9·04(− 13)	73	10816	1·66(− 12)
8	169	4·40(− 14)	41	3721	2·12(− 12)	74	11236	1·00(− 11)
9	289	2·10(− 13)	42	4225	6·92(− 12)	75	11236	1·73(− 12)
10	289	1·82(− 13)	43	4225	1·01(− 12)	76	11881	1·34(− 12)
11	324	6·18(− 14)	44	4225	4·08(− 13)	77	11881	8·58(− 12)
12	400	1·92(− 12)	45	4225	1·76(− 13)	78	12769	4·24(− 13)
13	400	1·16(− 12)	46	4624	1·82(− 11)	79	12769	2·81(− 11)
14	625	8·41(− 13)	47	4624	3·09(− 14)	80	13456	2·85(− 11)
15	625	1·02(− 12)	48	5184	1·28(− 12)	81	13456	3·32(− 13)
16	676	2·61(− 13)	49	5329	7·15(− 12)	82	13689	2·91(− 13)
17	676	1·42(− 12)	50	5329	9·11(− 13)	83	13689	1·58(− 12)
18	841	4·11(− 13)	51	5476	1·19(− 12)	84	14884	1·08(− 11)
19	841	8·13(− 14)	42	5476	5·46(− 12)	85	14884	7·62(− 13)
20	1024	2·06(− 13)	53	6400	2·87(− 13)	86	15625	4·86(− 13)
21	1156	5·50(− 13)	54	6400	4·26(− 13)	87	15625	6·93(− 13)
22	1156	8·38(− 14)	55	6724	1·87(− 14)	88	15625	1·20(− 13)
23	1369	1·12(− 13)	56	6724	8·53(− 13)	89	15625	7·29(− 12)
24	1369	3·31(− 13)	57	7225	1·02(− 12)	90	16384	1·48(− 12)
25	1600	3·85(− 13)	58	7225	6·59(− 14)	91	16900	1·39(− 12)
26	1600	7·56(− 13)	59	7225	2·76(− 14)	92	16900	3·42(− 13)
27	1681	6·91(− 14)	60	7225	2·23(− 14)	93	16900	1·41(− 13)
28	1681	9·48(− 14)	61	7921	8·89(− 12)	94	16900	1·40(− 13)
29	2025	1·09(− 12)	62	7921	2·35(− 12)	95	18496	3·16(− 12)
30	2025	3·04(− 13)	63	8100	1·28(− 12)	96	18496	9·65(− 12)
31	2500	2·36(− 13)	64	8100	9·13(− 12)	97	18769	1·45(− 13)
32	2500	6·56(− 14)	65	9409	4·65(− 12)	98	18769	3·19(− 13)
33	2500	1·59(− 11)	66	9408	6·71(− 12)	99	21025	2·71(− 12)
						100	21025	1·69(− 13)

where W is solely a function of the spatial co-ordinates, I the inplane force intensity, D the flexural rigidity, ρ mass density per unit area, and ω the sinusoidal time response frequency. According to the accepted convention of the theory of elasticity, the normal force I is positive in equation (45) if the plate is in compression and negative of the plates in tension.

The problem of all-side simply supported rectangular plates in the absence of inplane forces ($I = 0$) is easily soluble. The simply supported boundary conditions are given by

$$W = 0, \qquad M_x = -D\left[\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2}\right] = 0 \quad (\text{for } x = 0, a),$$

$$W = 0, \qquad M_y = -D\left[\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2}\right] = 0 \quad (\text{for } y = 0, b), \tag{46}$$

where v is the Poisson's ratio, and a and b are the lengths of the rectangular plate. The analytical solution of this case is actually independent of the Poisson ratio and is given by [36]

$$W_{nm} = A_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}, \quad n, m = 1, 2, \ldots, \tag{47}$$

where A_{nm} is an amplitude coefficient determined from the initial condition of the problem and n and m are integers. The frequency is given by

$$\omega_{nm} = \sqrt{\frac{D}{\rho}} \left[\left(\frac{n\pi}{a} \right)^2 + \left(\frac{m\pi}{b} \right)^2 \right]. \tag{48}$$

We choose the RDK to discretize a square domain of $[0, 10\pi] \times [0, 10\pi]$ with 33 grid points associated with $\sigma = 4\Delta$ for each dimension. The first 100 exact eigenvalues and absolute errors of the present numerical calculation are listed in Table 8. With less than seven points per wavelength, some of the errors are as small as 10^{-14}. It is evident that the present approach is extremely accurate for plate vibration analysis.

4. CONCLUSION

This paper explores the utility of a discrete singular convolution (DSC) algorithm [14] for vibration analysis. Various realizations of the DSC kernels, including quasi-wavelet scaling function (QWSF), regularized Dirichlet kernel (RDK), regularized modified Dirichlet kernel (RMDK), regularized de la Vallée Poussin kernel (RDLVPK) and regularized Lagrange kernel (RLK), are tested for the present computations. A number of test problems, such as vibrations of strings, rods, beams, molecules, membranes, waveguides and plates, are examined to illustrate the present approach.

In the first example, we consider the vibration of strings and rods with fixed edges. Since this problem is governed by the one-dimensional wave equation which admits an analytical solution, the results of the DSC approach can be objectively evaluated. By using two sets of grid points ($N = 22$ and 32), the DSC algorithm can achieve the machine precision for the first 10 eigenmodes. Both the QWSF and RDK perform extremely well for this problem.

The second example is the free and forced vibrations of elastic Kirchhoff beams. Both compressive and tensile forces are considered in the present treatment. This example is also very valuable because it is analytically soluble. The performances of the QWSF and RDK are excellent; only 22 grid points in a large interval of 10π is required to achieve the accuracy of eight significant figures or better for the first 10 eigenvalues. An increase of 10^5 times in accuracy is achieved when the grid is slightly refined by a factor of 1·5 ($N = 32$).

The third example deals with a Morse oscillator representing the I_2 molecular vibration. The corresponding Schrödinger equation is analytically soluble. The DSC algorithm performs extremely well for this model. The first 25 eigenvalues are accurate to 12 significant figures when using only 64 grid points for the QWSF, RDK and RMDK, which is 200 to 1000 times better than those of an efficient Chebyshev–Lanczos method [34], recently obtained by using 128 grid points. The RDLVPK requires 1·5 times more grid points (96 points) to achieve the same level of accuracy as that of other kernels. This result, however, is still about 100 to 1000 times more accurate than those of Braun *et al.* [34] obtained by using 128 grid points.

The fourth example is concerned with the vibration of the membrane. The problem can be modelled with the two-dimensional wave equation which admits a general solution. The

QWSF and RLK are selected to discretize the $[0, 10\pi] \times [0, 10\pi]$ domain with 32^2 mesh points. All of the first 100 vibration modes are accurate to at least 11 significant figures.

The fifth example analyzed is waveguide mode computations. By appropriate physical consideration the full Maxwell equation reduces to an eigenvalue problem for transverse magnetic mode. We compute the resonance mode in a rectangular domain of $[0, 10\pi] \times [0, 8\pi]$ by using three different convolution kernels (QWSF, RDK and RLK). By using a reasonably small grid, we obtain results with errors at least as small as 10^{-11} for the first 100 eigenvalues.

The last problem tackled is the thin plate vibration analysis. A uniform rectangular plate is employed to test the present approach. To simplify the presentation, a simply supported boundary condition is used and the resulting biharmonic equation is analytically solvable. We utilize the RDK to estimate the first 100 eigenvalues (other DSC kernels provide similar results). By using a 33^2 mesh in a $[0, 10\pi] \times [0, 10\pi]$ domain, we obtain results with errors as small as 10^{-14} for many eigenvalues.

Our results from these six test problems indicate that the present DSC algorithm is a reliable and robust approach for the numerical analysis of vibrations. Numerical results for more complicated vibration problems, including non-linear vibrations, complex geometry and mixed boundary conditions, which usually do not have analytical solutions, will be presented elsewhere.

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