Name: ____

Limits and Limit laws

1. Evaluate the following limits. Be sure that you show your work, so that you know how to show your work.

a) $\lim_{h\to 0}\frac{3(2+h)^2-12}{h}$

$= \lim_{h\to 0}\frac{3(4+4h+h^2)-12}{h}$

$= \lim_{h\to 0}\frac{12h+3h^2}{h} = \lim_{h\to 0} 12+3h = 12$

b) $\lim_{x\to -3}\frac{x^2-x-12}{x^2+8x+15}$

$= \lim_{x\to -3}\frac{(x-4)\cancel{(x+3)}}{\cancel{(x+3)}(x+5)} = \frac{-7}{2}$

2. Evaluate the following limits. Be sure that you show your work, so that you know how to show your work.

a) $\lim_{x\to 2^+}\frac{|x-2|}{x^2-4}$ $= \lim_{x\to 2^+}\frac{\cancel{x-2}}{\cancel{(x-2)}(x+2)} = \frac{1}{4}$

b) $\lim_{x\to 2}\frac{|x-2|}{x^2-4}$

$= D.N.E.$

$\lim_{x\to 2^-}\frac{|x-2|}{x^2-4} = \lim_{x\to 2^-}\frac{-\cancel{(x-2)}}{\cancel{(x-2)}(x+2)} = -\frac{1}{4}$

$\lim_{x\to 2^+}\frac{|x-2|}{x^2-4} = \frac{1}{4}$

3. Write an inequality that would be useful to find $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$ using the Squeeze Theorem.

$-1 \le \sin\frac{1}{x} \le 1$

$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$

$\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$, the squeeze theorem implies $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

4. Please *explain*, in words, what the following statement means, and what it tells you about $\lim_{x\to 0} f(x)$.

- $5 - x^2 \le f(x) \le 5 + \sin^2 3x$

$\lim_{x\to 0}(5-x^2) = 5$ $\quad \lim_{x\to 0}(5+\sin^2 3x) = 5$ by the squeez theorem, we have

$\lim_{x\to 0} f(x) = 0$

5. Calculate the following limits.

a) $\lim_{x\to 0} \frac{\sin 4x}{\sin 6x} = \lim_{x\to 0} \frac{\sin 4x}{\sin 6x} \cdot \frac{4x}{4x} \cdot \frac{6x}{6x}$

$= \lim_{x\to 0} \underbrace{\frac{\sin 4x}{4x}}_{\to 1} \cdot \underbrace{\frac{6x}{\sin 6x}}_{\to 1} \cdot \frac{4x}{6x}$

$= \frac{4}{6}$

b)* $\lim_{t\to 0} \frac{\tan 6t}{\sin 2t} = \lim_{t\to 0} \frac{\sin 6t}{\sin 2t} \cdot \frac{1}{\cos 6t} \cdot \frac{6t}{6t} \cdot \frac{2t}{2t}$

$= \lim_{t\to 0} \underbrace{\frac{\sin 6t}{6t}}_{\to 1} \cdot \underbrace{\frac{2t}{\sin 2t}}_{\to 1} \cdot \underbrace{\frac{1}{\cos 6t}}_{\to 1} \cdot \frac{6\cancel{t}}{2\cancel{t}}$

$= 3$

c)* $\lim_{x\to 1} \frac{\sin(x-1)}{x^2+x-2} = \lim_{x\to 1} \frac{\sin(x-1)}{x^2+x-2} \cdot \frac{x-1}{x-1}$

$= \lim_{x\to 1} \underbrace{\frac{\sin(x-1)}{x-1}}_{\to 1} \cdot \frac{x-1}{x^2+x-2} = \lim_{x\to 1} \frac{\cancel{x-1}}{(x+2)\cancel{(x-1)}} = \frac{1}{3}$

d) $\lim_{x\to 0} \frac{\sin 3x \sin 5x}{x^2} = \lim_{x\to 0} \underbrace{\frac{\sin 3x}{3x}}_{\to 1} \cdot \underbrace{\frac{\sin 5x}{5x}}_{\to 1} \cdot \frac{3x \cdot 5x}{x^2}$

$= 15$

Continuity and differentiability

6. Explain why the following function is discontinuous at $x = 4$. Could you change the function (minimally) to make it continuous?

$$f(x) = \begin{cases} \dfrac{x^2-16}{x-4} & x \ne 4 \\ 7 & x = 4 \end{cases}$$

$\lim_{x\to 4} f(x) = \lim_{x\to 4} \frac{x^2-16}{x-4} = \lim_{x\to 4} \frac{(x+4)\cancel{(x-4)}}{\cancel{x-4}} = 8$

Since $f(4) \ne 8 = \lim_{x\to 4} f(x)$, $f(x)$ is not continuous at $x = 4$. Changing $f(4) = 7$ to $f(4) = 8$ makes it continuous

$\sin x = x \iff \underbrace{\sin x - x}_{f(x)} = 0$

$f(x)$ is continuous in $(-\infty, \infty)$

$f(\pi) = \sin \pi - \pi < 0$

$f(-\pi) = \sin(-\pi) + \pi > 0$,

By IVT, $\exists\, c \in (-\pi, \pi)$, s.t. $f(c) = 0$ which means c is a root of $\sin x = x$.

7. Find a value of a such that the following function is continuous.

$$f(x) = \begin{cases} x^2 - 3x + 7 & x \le 4 \\ ax + 8 & x > 4 \end{cases}$$

$\lim_{x \to 4^-} f(x) = 16 - 12 + 7 = 11$ $\quad \lim_{x \to 4^+} f(x) = 4a + 8$

For f to be cont. at 4, we need $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x)$, use above

$\iff 11 = 4a + 8$

$\iff a = 3/4$

8. Prove that $\sin x = x$ has a solution.

Derivatives and Derivative functions

9. Calculate the derivatives of the following expressions.

a) $3x^2 + \cos x$

$6x - \sin x$

b) $\cot x + \tan x$

$-\csc^2 x + \sec^2 x$

c) $x^2 \sin x$

$2x \sin x + x^2 \cos x$

d) $\left(5x^5 + \sqrt{x}\right) \csc x$

$\left(25x^4 + \frac{1}{2}x^{-\frac{1}{2}}\right)\csc - (5x^5 + \sqrt{x}) \cdot \csc x \cot x$

e)* $\underbrace{\sqrt[3]{x^5}}_{x^{\frac{5}{3}}}\ \underline{\sin x \cos x} = \frac{1}{2}\sin 2x$

$\frac{5}{3}x^{2/3} \cdot \frac{1}{2}\sin 2x + x^{5/3} \cdot \cos 2x$

f) $\dfrac{\sin x}{x} = \sin x \cdot x^{-1}$

$\cos x \cdot x^{-1} - \sin x \cdot x^{-2}$

g) $\dfrac{x^2+\tan x}{\sin x}$

$$\frac{(2x+\sec^2 x)\sin x - (x^2+\tan x)\cos x}{\sin^2 x}$$

h) $\dfrac{x^2+2x\sin x}{\cot x}$

$$\frac{(2x+2\sin x+2x\cos x)\cot x + (x^2+2x\sin x)\csc^2 x}{\cot^2 x}$$

i) $\sin^2 x$

$2\sin x\cos x$

j) $\sin x^2$

$\cos x^2 \cdot 2x$

k) $\sqrt{\cos x}$

$\frac{1}{2}\cos^{-\frac{1}{2}} x \cdot (-\sin x)$

l) $\tan\sqrt{x}$

$\sec^2\sqrt{x} \cdot \frac{1}{2}x^{-\frac{1}{2}}$

m) $\dfrac{\csc x}{4x-7}$

$$\frac{-\csc x\cot x(4x-7) - 4\csc x}{(4x-7)^2}$$

n)* $\sin(\sec x\tan x)$

$\cos(\sec x\tan x)(\sec x\tan^2 x+\sec^3 x)$

o) $\sqrt{x^2\sin x+\tan x}$ $= \frac{1}{2}(x^2\sin x+\tan x)^{-\frac{1}{2}}\cdot(2x\sin x+x^2\cos x+\sec^2 x)$