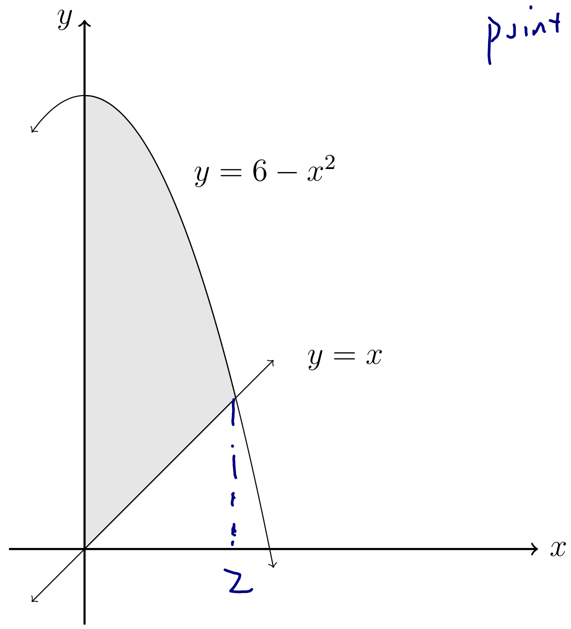
5.1 Problems

Question 1. Find the area of the shaded region:

(a)



point of intersection:

$$6 - x^2 = x$$

$$x^2 + x - 6 = 0$$

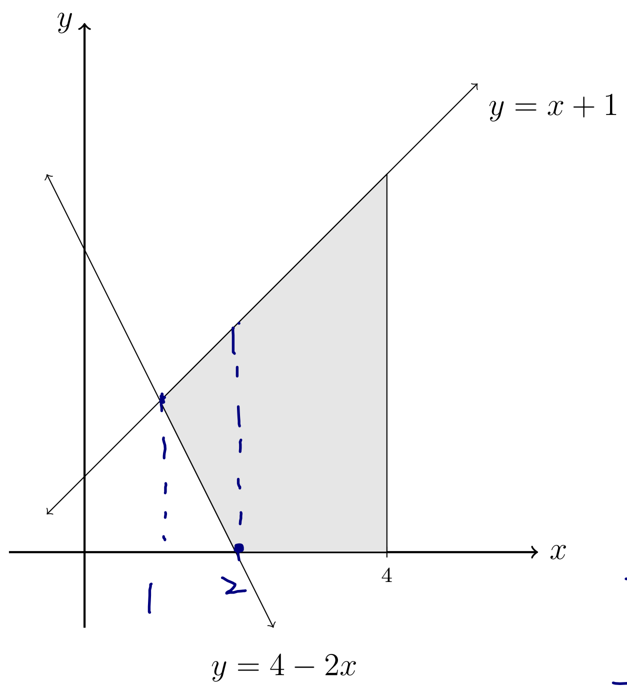
$$(x+3)(x-2) = 0$$

$$x = -3,\ 2$$

$$\int_0^2 (6-x^2) - x \, dx = \left(6x - \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right)\Big|_0^2$$

$$= 10 - 8/3$$

(b)



x intercept

$$4 - 2x = 0 \Rightarrow x = 2$$

point of intersection:

$$4 - 2x = x + 1 \Rightarrow x = 1$$

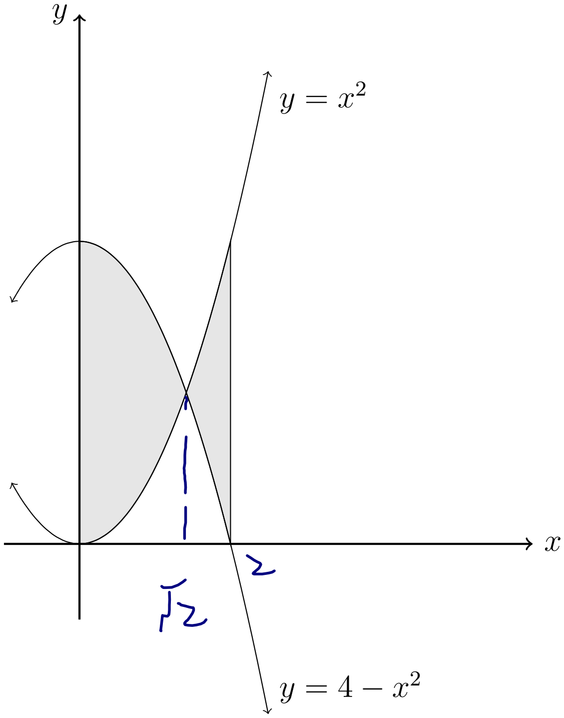
$$\int_1^2 (x+1) - (4-2x) \, dx + \int_2^4 (x+1) \, dx$$

$$= \int_1^2 (3x-3) \, dx + \int_2^4 (x+1) \, dx$$

$$= \left(\tfrac{3}{2}x^2 - 3x\right)\Big|_1^2 + \left(\tfrac{1}{2}x^2 + x\right)\Big|_2^4$$

$$= 7$$

(c)



point of intersection

$$x^2 = 4 - x^2 \Rightarrow x = \pm\sqrt{2}$$

x-intercept

$$4 - x^2 = 0 \Rightarrow x = \pm 2$$

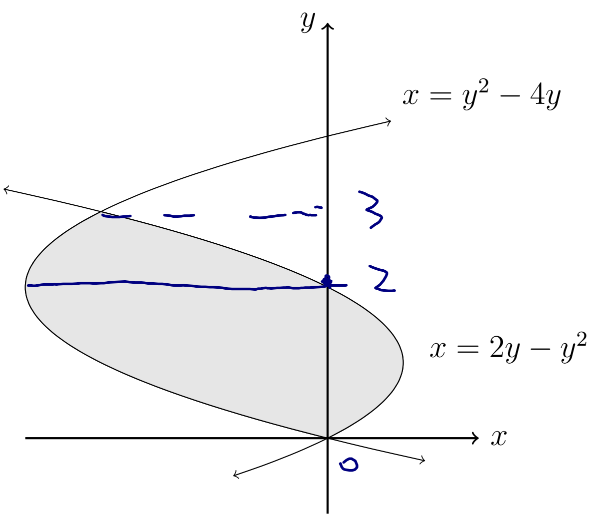
$$\int_0^{\sqrt{2}} (4 - x^2 - x^2)\,dx + \int_{\sqrt{2}}^{2} (x^2 - (4 - x^2))\,dx$$

$$= \int_0^{\sqrt{2}} (4 - 2x^2)\,dx + \int_{\sqrt{2}}^{2} (2x^2 - 4)\,dx$$

$$= 4x - \frac{2}{3}x^3 \Big|_0^{\sqrt{2}} + \left(\frac{2}{3}x^3 - 4x\right)\Big|_{\sqrt{2}}^{2}$$

$$= \frac{16\sqrt{2}}{3} + \frac{8}{3}$$

(d)



point of intersection

$$y^2 - 4y = 2y - y^2$$

$$\Rightarrow 2y^2 = 6y \Rightarrow y = 0, 3$$

$$\int_0^3 -(y^2 - 4y) + (2y - y^2)\,dy$$

$$= \int_0^3 -(2y^2 - 6y)\,dy$$

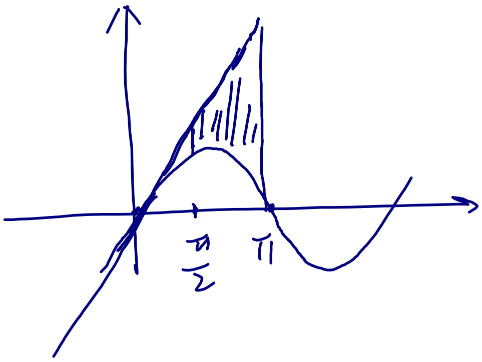
$$= \left(-\frac{2}{3}y^3 + 3y^2\right)\Big|_0^3$$

$$= -\frac{2}{3}\cdot 27 + 3\cdot 9$$

$$= 9$$

Question 2. Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or to y then find the area of the region.

(a) $y = \sin x,\ y = x,\ x = \pi/2,\ x = \pi.$



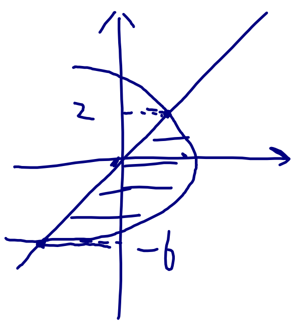
$$\int_{\frac{\pi}{2}}^{\pi} (x - \sin x)\, dx$$

$$= \left(\tfrac{1}{2}x^2 + \cos x\right)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{3}{8}\pi^2 - 1$$

$x = 3 - y^2/4$

(b) $4x + y^2 = 12,\ x = y$



points of intersection:

$$3 - y^2/4 = y$$

$$y^2 + 4y - 12 = 0$$

$$(y+6)(y-2) = 0$$

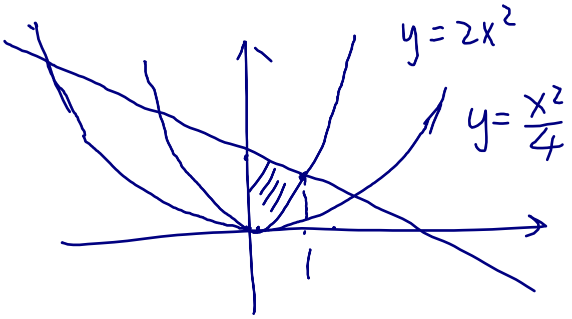
$$y = 2, -6$$

$$\int_{-6}^{2} \left(3 - y^2/4 - y\right) dy$$

$$= \left(3y - \tfrac{1}{12}y^3 - \tfrac{1}{2}y^2\right)\Big|_{-6}^{2}$$

$$= 40 - \frac{56}{3}$$

Question 3. Sketch the region enclosed by the given curves: $y = \frac{x^2}{4}$, $y = 2x^2$, $x + y = 3$, $x \geq 0$. Find the area of the region.



$y = 3 - x$

point of intersection: $2x^2 = 3 - x \Rightarrow 2x^2 + x - 3 = 0$

$(2x+3)(x-1) = 0$

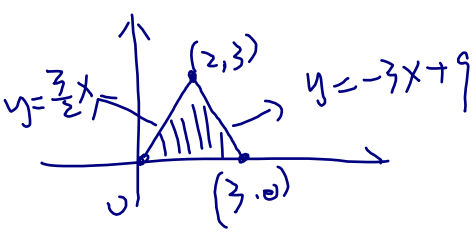
$x = -3/2,\ 1$

$$\int_0^1 3 - x - 2x^2\,dx$$

$$= \left(3x - \frac{1}{2}x^2 - \frac{2}{3}x^3\right)\Big|_0^1$$

$$= \frac{11}{6}$$

Question 4. Use calculus to find the area of the triangle with vertices $(0,0)$, $(3,0)$, and $(2,3)$.



$$y - 0 = \frac{0-3}{3-2}(x-3)$$

$$y = -3x + 9$$

$$y - 0 = \frac{3-0}{2-0}(x-0)$$

$$y = \frac{3}{2}x$$

$$\text{Area} = \int_0^2 \frac{3}{2}x\,dx + \int_2^3 (-3x+9)\,dx$$

$$= \frac{3}{4}x^2\Big|_0^2 + \left(-\frac{3}{2}x^2 + 9x\right)\Big|_2^3$$

$$= 3 + 3/2$$