4.5b Problems

Question 1. Evaluate

(a) $\displaystyle\int_{-1}^{1} \frac{\tan x}{\underbrace{1+x^2+x^4}_{\text{odd}}}\, dx \quad = 0$

(b) $\displaystyle\int_{0}^{\pi/2} \cos x \;\sin(\sin x) dx$

$u = \sin x$

$du = \cos x \, dx$

when $x = 0$, $u = 0$

when $x = \pi/2$, $u = 1$

$= \displaystyle\int_0^1 \sin u \, du$

$= -\cos u \Big|_0^1 = \cos(0) - \cos(1)$

Question 2. Evaluate

(a) $\displaystyle\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$

$u = 1+\sqrt{x} \Rightarrow \sqrt{x} = u-1$

$du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$

$2\sqrt{x}\,du = dx$

$2(u-1)\,du = dx$

when $x=0$, $u=1$

when $x=1$, $u=2$

$= \displaystyle\int_1^2 \frac{2(u-1)\,du}{u^4}$

$= \displaystyle\int_1^2 \left(\frac{2}{u^3} - \frac{2}{u^4}\right) du$

$= \left(-u^{-2} + \frac{2}{3}u^{-3}\right)\Big|_1^2$

$= -\left(\frac{1}{4}-1\right) + \frac{2}{3}\left(\frac{1}{8}-1\right)$

(b) $\displaystyle\int_0^1 x\sqrt{1-x^4}\,dx$

$u = 1-x^4$

$du = -4x^3\,dx$

$\frac{du}{-4x^3} = dx$

$= \displaystyle\int x\sqrt{u}\,\frac{du}{-4x^3}$

$= \displaystyle\int \sqrt{u}\,\frac{du}{-4x^2}$

when $x=0$, $u=1$

when $x=1$, $u=0$

$= \displaystyle\int_1^0 \sqrt{u}\,\frac{du}{-4\sqrt{1-u}}$

$u = \sin^2 t$

$du = 2\sin t\cos t\,dt$

when $u=0$, $t=0$

when $u=1$, $t=\pi/2$

$= \displaystyle\int_{\frac{\pi}{2}}^0 \frac{\sin t\cdot 2\sin t\cancel{\cos t}}{-4\cancel{\cos t}}\,dt$

$= -\frac{1}{2}\displaystyle\int_{\frac{\pi}{2}}^0 \sin^2 t\,dt$

$= -\frac{1}{2}\displaystyle\int_{\frac{\pi}{2}}^0 \frac{1-\cos 2t}{2}\,dt$

$= \left(-\frac{1}{4}t + \frac{1}{2}\sin 2t\right)\Big|_{\pi/2}^0 = \pi/8$

Question 3. Find the average value of the function on the given interval

(a) $g(t) = \dfrac{t}{\sqrt{3+t^2}}$, $[1, 3]$

$u = 3 + t^2$

$du = 2t\,dt$

when $t = 1$ $u = 4$

when $t = 3$ $u = 12$

$$\frac{\int_1^3 g(t)\,dt}{3-1} = \frac{\int_1^3 \frac{t}{\sqrt{3+t^2}}\,dt}{2} = \frac{\int_4^{12} \frac{1}{2}\frac{du}{\sqrt{u}}}{2} = \frac{1}{4}\cdot 2u^{1/2}\Big|_4^{12}$$

$$= \frac{1}{2}\left(\sqrt{12} - \sqrt{4}\right)$$

$$= \sqrt{3} - 1$$

(b) $h(x) = \cos^4 x \sin x$, $[0, \pi]$

$u = \cos x$

$du = -\sin x\,dx$

when $x = 0$, $u = 1$

when $x = \pi$, $u = -1$

$$\frac{\int_0^\pi h(x)\,dx}{\pi - 0} = \frac{\int_0^\pi \cos^4 x \sin x\,dx}{\pi} = \frac{-\int_1^{-1} u^4\,du}{\pi} = \frac{-1}{5\pi} u^5 \Big|_1^{-1}$$

$$= \frac{2}{5\pi}$$

Question 4. If f is continuous on $[0,1]$, prove that $\int_0^1 f(x)\ dx = \int_0^1 f(1-x)\ dx$

Right hand side $= \int_0^1 f(1-x)\,dx$ $\qquad u = 1-x$

$du = -dx$

$= \int_1^0 f(u) \cdot -du$

$= \int_0^1 f(u)\,du$ = Left hand side