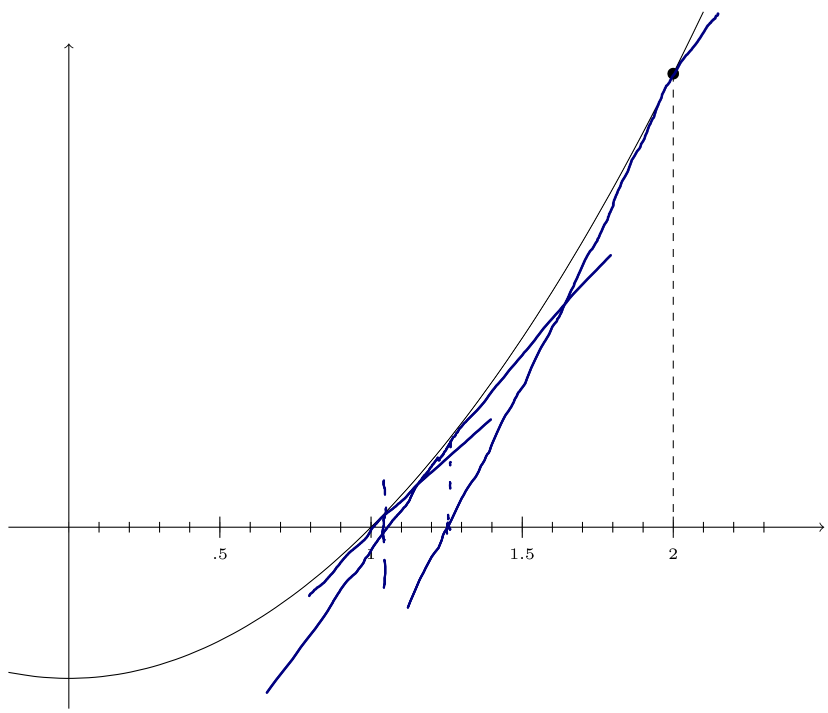
3.8 Problems

Newton's Method

Example 1. Understanding Newton's method

(a) Using the graph of $f(x) = x^2 - 1$ and a starting value of $x_1 = 2$, draw the first 3 iterations of Newton's Method.



(b) Using a picture and the definition of slope, derive the recursive formula for Newton's Method.

$$x^k = x^{k-1} - \frac{f(x^{k-1})}{f'(x^{k-1})}$$

Example 2. For each initial approximation, determine graphically what happens if Newton's method is used for the function whose graph is shown.

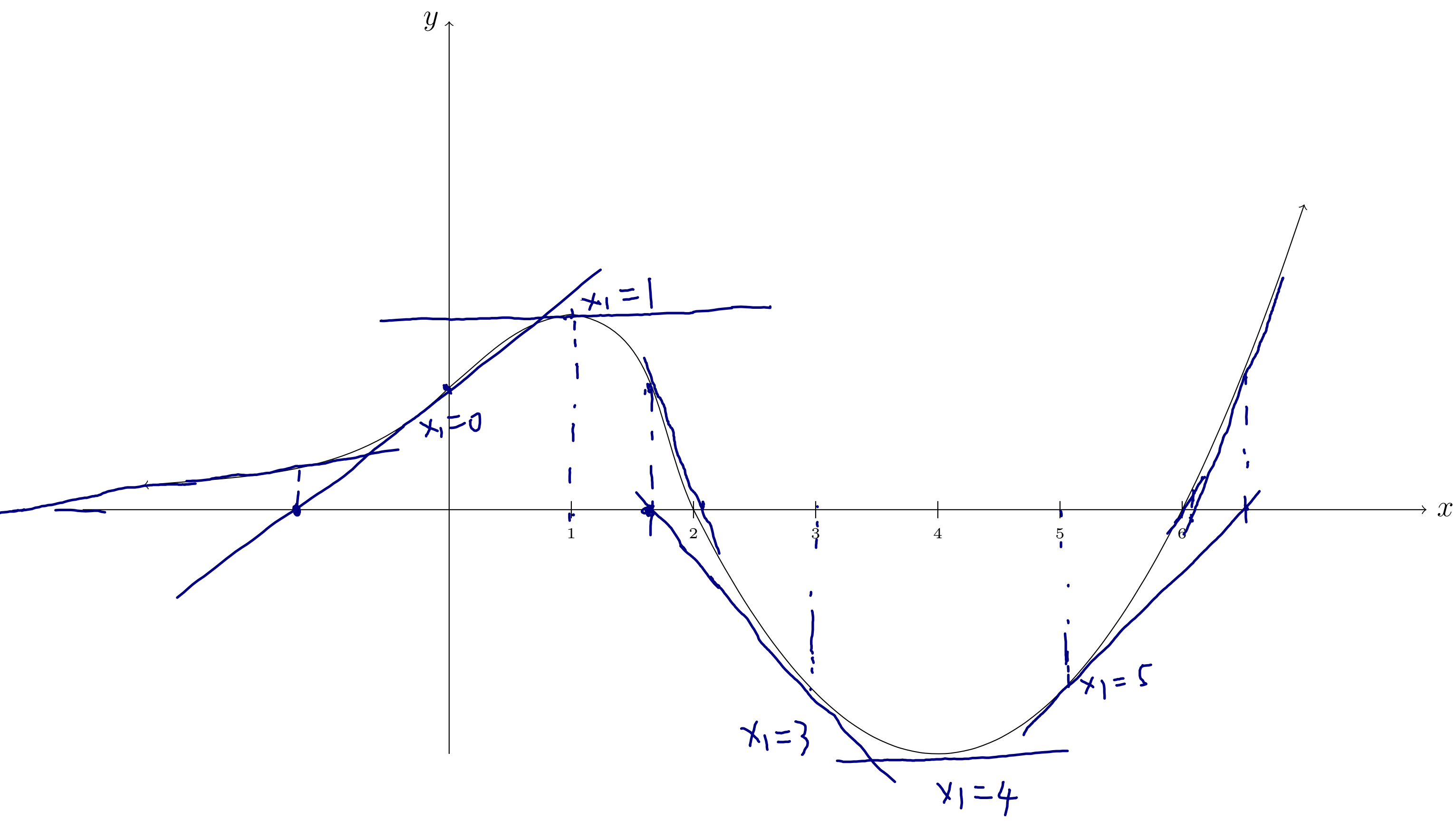
(a) $x_1 = 0$ divarge

(b) $x_1 = 1$ tangent line is horizontal, cannot find x_2

(c) $x_1 = 3$ converge to the root $x=2$

(d) $x_1 = 4$ tangent line is horizontal, cannot find x_2

(e) $x_1 = 5$ converge to the root $x=6$



Example 3. Use Newton's method with $x_1 = 1$ to find x_3 for the equation $\underbrace{x^5 - x - 1}_{f(x)} = 0$

$f'(x) = 5x^4 - 1$

$x_1 = 1$

$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{4} = \frac{5}{4}$

$x_3 = \frac{5}{4} - \frac{f(\frac{5}{4})}{f'(\frac{5}{4})} \approx 1.178$

Example 4. Use Newton's method to approximate a solution to $3\cos x = x - 1$ as follows: Let $x_1 = 1$ be the initial approximation. Find the next two approximations, x_2 and x_3, to four decimal places each.

$3\cos x = x - 1$

$\Leftrightarrow \underbrace{3\cos x - x + 1}_{f(x)} = 0$

Use Newton's method to find a root of this $f(x)$

$f'(x) = -3\sin x - 1$

$x_1 = 1$

$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{3\cos 1}{-3\sin 1 - 1} \approx 3.85$

$x_3 = 3.85 - \frac{f(3.85)}{f'(3.85)} \approx 4$

Example 5. Use Linearization to approximate $\sqrt[4]{17}$.

let $f(x) = \sqrt[4]{x}$ then $\sqrt[4]{17} = f(17)$

$x_0 = 17$ $a = 16$, $f'(x) = \frac{1}{4} x^{-3/4}$ $f(a) = \sqrt[4]{16} = 2$. $f'(a) = \frac{1}{32}$

$$f(17) \approx L(17)$$
$$= f(a) + f'(a)(x_0 - a)$$
$$= 2 + \frac{1}{32}(17-16) = \frac{65}{32}$$

Example 6. Newton's Method to approximate $\sqrt[4]{17}$. Make good choice for x_1 then calculate x_2 and x_3.

To approximate a number $\sqrt[4]{17}$, set

$$x = \sqrt[4]{17} \quad (1)$$

Then $\sqrt[4]{17}$ is the solution to (1). Simplify (1) by raising both hand side to the 4th power

$$x^4 = 17$$
$$\underbrace{x^4 - 17}_{f(x)} = 0$$
$$\Rightarrow f'(x) = 4x^3$$

choose $x_1 = \sqrt[4]{16} = 2$, then run Newton's iteration

$$x_2 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-1}{32} = \frac{65}{32}$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{65}{32} - \frac{(65/32)^4 - 17}{4(65/32)^3} \approx 2.03$$