2.4a Problems

Fun Trig Limits

Example 1. Find the limits:

(a) $\lim_{x\to 0} \frac{\sin 4x}{\sin 7x}$ $= \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \frac{7x}{\sin 7x} \cdot \frac{4x}{7x}$

$= \lim_{x\to 0} 1 \cdot 1 \cdot \frac{4x}{7x} = \frac{4}{7}$

(b) $\lim_{t\to 0} \frac{\sin 4t}{\cos(-3t)}$ $= \frac{0}{1} = 0$

(c) $\lim_{\theta\to 0} \frac{\sin\theta}{\theta + \tan\theta}$ $= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \frac{\theta}{\theta + \frac{\sin\theta}{\cos\theta}}$

$= \lim_{\theta\to 0} 1 \cdot \frac{\theta}{\theta + \frac{\sin\theta}{\cos\theta}} \cdot \frac{\frac{1}{\theta}}{\frac{1}{\theta}}$

$= \lim_{\theta\to 0} \frac{1}{1 + \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta}}$

$= \lim_{\theta\to 0} \frac{1}{1 + 1 \cdot \frac{1}{\cos\theta}} = \frac{1}{2}$

Example 2. Find the limits:

(a) $\lim\limits_{t\to\pi/4} \dfrac{1-\tan t}{\sin t - \cos t}$ $\tan t = \frac{\sin}{\cos}$ $= \lim\limits_{t\to\frac{\pi}{4}} \dfrac{1-\frac{\sin t}{\cos t}}{\sin t - \cos t}$

(combine the terms in the numerator) $= \lim\limits_{t\to\frac{\pi}{4}} \dfrac{\frac{\cos t}{\cos t} - \frac{\sin t}{\cos t}}{\sin t - \cos t} = \lim\limits_{t\to\frac{\pi}{4}} \dfrac{\frac{\cos t - \sin t}{\cos t}}{\sin t - \cos t}$

(canceling) $= \lim\limits_{t\to\frac{\pi}{4}} \dfrac{\frac{-\cancel{(\sin t - \cos t)}}{\cos t}}{\cancel{\sin t - \cos t}} = \lim\limits_{t\to\frac{\pi}{4}} \dfrac{-1}{\cos t}$

(plugging in values) $= \dfrac{-1}{\cos\frac{\pi}{4}} = -\sqrt{2}$

(b) $\lim\limits_{x\to 0} \dfrac{\sec x}{1-\sin x}$

$\lim\limits_{x\to 0} \dfrac{\sec x}{1-\sin x}$ $\sec x = \frac{1}{\cos x}$ $= \lim\limits_{x\to 0} \dfrac{\frac{1}{\cos x}}{1-\sin x}$

(plugging in values) $= \dfrac{\frac{1}{\cos 0}}{1-\sin 0}$

$= \dfrac{1}{1}$

$= 1$

(c) $\lim\limits_{t\to 0} \dfrac{t^3}{\tan^3 2t}$ $\tan = \frac{\sin}{\cos}$ $= \lim\limits_{t\to 0} \dfrac{t^3}{\sin^3(2t)} \cdot \cos^3(2t)$

(make out $\frac{t}{\sin t}$) $= \lim\limits_{t\to 0} \underbrace{\dfrac{(2t)^3}{\sin^3(2t)}}_{\to 1} \cdot \dfrac{t^3 \cdot \cos^3(2t)}{(2t)^3}$

(simplify) $= \lim\limits_{t\to 0} 1 \cdot \dfrac{\cancel{t^3}\cos^3(2t)}{(2\cancel{t})^3} = \lim\limits_{t\to 0} \dfrac{\cos^3(2t)}{8}$

(plugging in values) $= \dfrac{\cos^3(2\cdot 0)}{8} = \dfrac{1}{8}$

Derivatives and Such

Example 3. Find the equation of the tangent line to the curve

(a) $y = \underbrace{(1+x)\cos x}_{f(x)}$ through the point $(0, 1)$.

$f'(x) = (1+x)'\cos x + (1+x)\cos' x$

$= \cos x + (1+x)(-\sin x)$

Hence $f'(0) = \cos 0 + (1+0)(-\sin 0) = 1$

The tangent line at $(0,1)$ is

$y - 1 = 1\cdot(x-0) \iff y = x+1$

(b) $y = \underbrace{\cos x - \sin x}_{f(x)}$ through the point $(\pi, -1)$.

$f'(x) = (\cos x - \sin x)' = \cos' x - \sin' x = -\sin x - \cos x$

Hence $f'(\pi) = -\sin \pi - \cos \pi = -0 - (-1) = 1$

The tangent line at $(\pi, -1)$ is

$y - (-1) = 1\cdot(x - \pi) \iff y = x - 1 - \pi$

(c) $y = \underbrace{2x \sin x}_{f(x)}$ through the point $(\pi/2, \pi)$.

$f'(x) = (2x\sin x)' = (2x)'\sin x + 2x(\sin x)' = 2\sin x + 2x\cos x$

Hence $f'(\frac{\pi}{2}) = 2\sin\frac{\pi}{2} + 2\frac{\pi}{2}\cos\frac{\pi}{2} = 2$

The tangent line at $(\pi/2, \pi)$ is

$y - \pi = 2\cdot(x - \frac{\pi}{2}) \iff y = 2x$

Example 4. Suppose that $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let $g(x) = f(x)\sin x$ and $h(x) = \dfrac{\cos x}{f(x)}$. Find

(a) $g'(\pi/3)$

(b) $h'(\pi/3)$

(a) $g(x) = f(x)\sin x$, so $g'(x) = (f(x)\sin x)' = f'(x)\sin x + f(x)(\sin x)'$

$= f'(x)\sin x + f(x)\cos x$

Hence $g'(\pi/3) = f'(\frac{\pi}{3})\sin\frac{\pi}{3} + f(\frac{\pi}{3})\cos\frac{\pi}{3}$

$= (-2)\cdot\frac{\sqrt{3}}{2} + 4\cdot\frac{1}{2}$

$= 2 - \sqrt{3}$

(b) $h(x) = \dfrac{\cos x}{f(x)}$, so $h'(x) = \dfrac{(\cos x)' f(x) - \cos x f'(x)}{f^2(x)} = \dfrac{-\sin x f(x) - \cos x f'(x)}{f^2(x)}$

Hence $h'(\pi/3) = \dfrac{-\sin\frac{\pi}{3} f(\frac{\pi}{3}) - \cos\frac{\pi}{3} f'(\frac{\pi}{3})}{f^2(\frac{\pi}{3})} = \dfrac{-\frac{\sqrt{3}}{2}4 - \frac{1}{2}(-2)}{4^2} = \dfrac{1-2\sqrt{3}}{16}$

Example 5. Find the points on the curve $y = \underbrace{\dfrac{\cos x}{2 + \sin x}}_{f(x)}$ at which the tangent is horizontal.

$f'(x) = \dfrac{(\cos x)'(2+\sin x) - \cos x(2+\sin x)'}{(2+\sin x)^2}$

$= \dfrac{(-\sin x)(2+\sin x) - \cos x(\cos x)}{(2+\sin x)^2}$

(simplify) $= \dfrac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2}$

apply $\boxed{\sin^2 x + \cos^2 x = 1}$ $= \dfrac{-2\sin x - 1}{(2+\sin x)^2}$

The tangent is horizontal means $f'(x) = 0$

$\Leftrightarrow \dfrac{-2\sin x - 1}{(2+\sin x)^2} = 0$

$\Leftrightarrow -2\sin x - 1 = 0$

$\Leftrightarrow \sin x = -\frac{1}{2}$

$\Leftrightarrow x = -\frac{\pi}{6} + 2k\pi$ or $\frac{7\pi}{6} + 2k\pi \quad \forall k \in \mathbb{Z}$