Midterm Exam Solutions

MTH 415 — Fall 2026

Problem 1: True or False?

(a) **True.**

By the fundamental subspace relation,

$$\mathcal{N}(A) = \mathcal{C}(A^T)^\perp.$$

Therefore every

$$x \in \mathcal{N}(A)$$

is orthogonal to every vector in $\mathcal{C}(A^T)$.

(b) **True.**

If the columns of A are linearly independent, then

$$Ax = 0$$

has only the trivial solution. Hence

$$\mathcal{N}(A) = \{0\}.$$

(c) **False.**

The matrix

$$QQ^T$$

is the orthogonal projection onto $\mathcal{C}(Q)$, so in general it does not preserve the norm of every vector in $\mathbb{R}^m$.

For example, let

$$Q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$QQ^T x = 0,$$

so

$$\|QQ^T x\|_2 = 0 \neq 1 = \|x\|_2.$$

(d) **True.**

Since

$$Av = 0$$

for some nonzero v, we have

$$v \in \mathcal{N}(A),$$

so

$$\dim \mathcal{N}(A) \geq 1.$$

By the rank-nullity theorem,

$$\operatorname{rank}(A) = n - \dim \mathcal{N}(A) < n.$$

(e) **True.**

Suppose

$$x \in K^{\perp}.$$

Then x is orthogonal to every vector in K. Since

$$L \subseteq K,$$

x is also orthogonal to every vector in L. Hence

$$x \in L^{\perp}.$$

Therefore,

$$K^{\perp} \subseteq L^{\perp}.$$

(f) **True.**

If

$$b \in \mathcal{C}(A),$$

then there exists some x_0 such that

$$Ax_0 = b.$$

Thus

$$\|Ax_0 - b\|_2 = 0.$$

Since a norm cannot be negative,

$$\boxed{\min_x \|Ax - b\|_2 = 0.}$$

Problem 2

Let

$$f(x_1, x_2) = \max\{|x_1|, |x_2|\}.$$

1. Recall that

$$\mathrm{ReLU}(x) = \begin{cases} x, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

If $x \geq 0$, then

$$\mathrm{ReLU}(x) + \mathrm{ReLU}(-x) = x = |x|.$$

If $x < 0$, then

$$\mathrm{ReLU}(x) + \mathrm{ReLU}(-x) = 0 + (-x) = |x|.$$

Therefore,

$$\boxed{|x| = \mathrm{ReLU}(x) + \mathrm{ReLU}(-x).}$$

2. If $a \geq b$, then

$$\mathrm{ReLU}(a - b) + b = (a - b) + b = a = \max\{a, b\}.$$

If $a < b$, then

$$\mathrm{ReLU}(a - b) + b = 0 + b = b = \max\{a, b\}.$$

Therefore,

$$\boxed{\max\{a, b\} = \mathrm{ReLU}(a - b) + b.}$$

3. First compute

$$|x_1| = \mathrm{ReLU}(x_1) + \mathrm{ReLU}(-x_1),$$

and

$$|x_2| = \mathrm{ReLU}(x_2) + \mathrm{ReLU}(-x_2).$$

Define the first hidden layer by

$$h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = \mathrm{ReLU}\left(\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right).$$

Thus

$$h_1 = \mathrm{ReLU}(x_1), \qquad h_2 = \mathrm{ReLU}(-x_1),$$
$$h_3 = \mathrm{ReLU}(x_2), \qquad h_4 = \mathrm{ReLU}(-x_2),$$

and hence

$$|x_1| = h_1 + h_2, \qquad |x_2| = h_3 + h_4.$$

For the second hidden layer, define

$$g = \mathrm{ReLU}\left(\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} h \right).$$

Then

$$g_1 = \mathrm{ReLU}(|x_1| - |x_2|)$$

and, since $|x_2| \geq 0$,

$$g_2 = \mathrm{ReLU}(|x_2|) = |x_2|.$$

Finally, use the linear output layer

$$f(x_1, x_2) = \begin{pmatrix} 1 & 1 \end{pmatrix} g.$$

Therefore,

$$f(x_1, x_2) = \mathrm{ReLU}(|x_1| - |x_2|) + |x_2| = \max\{|x_1|, |x_2|\}.$$

Thus one possible network has

$$W_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, \qquad b_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$W_2 = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad b_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$W_3 = \begin{pmatrix} 1 & 1 \end{pmatrix}, \qquad b_3 = 0.$$

Problem 3

Let

$$S = \left\{ x \in \mathbb{R}^4 : x_1 + x_2 = x_3 + x_4 \right\}.$$

We verify that S is a linear subspace of $\mathbb{R}^4$.

First,

$$0 + 0 = 0 + 0,$$

so

$$0 \in S.$$

Now let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in S, \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \in S.$$

Then

$$x_1 + x_2 = x_3 + x_4$$

and

$$y_1 + y_2 = y_3 + y_4.$$

For any $\alpha, \beta \in \mathbb{R}$,

$$(\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2)$$

equals

$$\alpha(x_1 + x_2) + \beta(y_1 + y_2).$$

Using the defining conditions for x and y,

$$\alpha(x_1 + x_2) + \beta(y_1 + y_2) = \alpha(x_3 + x_4) + \beta(y_3 + y_4),$$

which equals

$$(\alpha x_3 + \beta y_3) + (\alpha x_4 + \beta y_4).$$

Therefore,

$$\alpha x + \beta y \in S.$$

Hence,

$$\boxed{S \text{ is a linear subspace of } \mathbb{R}^4.}$$

Problem 4

Let

$$A = \begin{pmatrix} 1 & i \\ i & 0 \\ 0 & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Let the columns of A be

$$a_1 = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}.$$

We first construct an orthonormal basis for $\mathcal{C}(A)$.

Since

$$\|a_1\|_2^2 = a_1^* a_1 = 2,$$

we have

$$q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}.$$

Next,

$$q_1^* a_2 = \frac{i}{\sqrt{2}}.$$

Thus

$$u_2 = a_2 - (q_1^* a_2) q_1.$$

Therefore,

$$u_2 = \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} - \frac{i}{2} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} i/2 \\ 1/2 \\ 1 \end{pmatrix}.$$

Since

$$\|u_2\|_2^2 = \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2},$$

we obtain

$$q_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} i \\ 1 \\ 2 \end{pmatrix}.$$

Hence

$$\{q_1, q_2\}$$

is an orthonormal basis for $\mathcal{C}(A)$.

The orthogonal projection of x onto $\mathcal{C}(A)$ is

$$P_{\mathcal{C}(A)} x = (q_1^* x) q_1 + (q_2^* x) q_2.$$

We have

$$q_1^* x = \frac{1}{\sqrt{2}},$$

and

$$q_2^* x = \frac{2-i}{\sqrt{6}}.$$

Therefore,

$$P_{\mathcal{C}(A)} x = \frac{1}{2}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} + \frac{2-i}{6}\begin{pmatrix} i \\ 1 \\ 2 \end{pmatrix}.$$

Thus,

$$\boxed{P_{\mathcal{C}(A)} x = \begin{pmatrix} \dfrac{2+i}{3} \\ \dfrac{1+i}{3} \\ \dfrac{2-i}{3} \end{pmatrix}.}$$

Now consider the least-squares problem

$$\min_{z \in \mathbb{C}^2} \|Az - x\|_2^2.$$

The vector Az always belongs to $\mathcal{C}(A)$. Therefore, the least-squares problem asks for the vector in $\mathcal{C}(A)$ that is closest to x.

Hence, at the least-squares solution $\widehat{z}$,

$$\boxed{A\widehat{z} = P_{\mathcal{C}(A)} x.}$$

Thus we solve

$$A\widehat{z} = \begin{pmatrix} \dfrac{2+i}{3} \\ \dfrac{1+i}{3} \\ \dfrac{2-i}{3} \end{pmatrix}.$$

Write

$$\widehat{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} z_1 + iz_2 \\ iz_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \dfrac{2+i}{3} \\ \dfrac{1+i}{3} \\ \dfrac{2-i}{3} \end{pmatrix}.$$

From the third equation,

$$z_2 = \frac{2-i}{3}.$$

From the second equation,

$$iz_1 = \frac{1+i}{3},$$

so

$$z_1 = \frac{1-i}{3}.$$

Therefore,

$$\boxed{\widehat{z} = \begin{pmatrix} \dfrac{1-i}{3} \\ \dfrac{2-i}{3} \end{pmatrix}.}$$

The relationship is therefore

$$\boxed{A\widehat{z} = P_{\mathcal{C}(A)}x,}$$

and the residual

$$x - A\widehat{z}$$

is orthogonal to $\mathcal{C}(A)$.