Homework 8

Problem 1 (2pts)**.** *Let*

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 5 & 3 \\ 2 & 3 & 3 \end{pmatrix}.$$

Find a Cholesky decomposition

$$A = LL^T,$$

where L is lower triangular.

Solution. *Write*

$$L = \begin{pmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix}.$$

From

$$A = LL^T,$$

we compare entries.

First,

$$\ell_{11}^2 = 4,$$

so we take

$$\ell_{11} = 2.$$

Then

$$\ell_{11}\ell_{21} = 2, \qquad \ell_{11}\ell_{31} = 2,$$

giving

$$\ell_{21} = 1, \qquad \ell_{31} = 1.$$

Next,

$$\ell_{21}^2 + \ell_{22}^2 = 5,$$

so

$$1 + \ell_{22}^2 = 5,$$

and hence

$$\ell_{22} = 2.$$

From the $(3,2)$ entry,

$$\ell_{31}\ell_{21} + \ell_{32}\ell_{22} = 3,$$

so

$$1 + 2\ell_{32} = 3,$$

giving

$$\ell_{32} = 1.$$

Finally,

$$\ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 = 3,$$

so

$$1 + 1 + \ell_{33}^2 = 3,$$

and therefore

$$\ell_{33} = 1.$$

Thus

$$\boxed{L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.}$$

Indeed,

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 5 & 3 \\ 2 & 3 & 3 \end{pmatrix}.$$

Problem 2 (4pts)**.** *Let*

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

1. *Compute a compact singular value decomposition*

$$A = U_r \Sigma_r V_r^*.$$

2. *Use the SVD to give orthonormal bases for*

$$\mathcal{C}(A), \qquad \mathcal{N}(A), \qquad \mathcal{C}(A^*), \qquad \mathcal{N}(A^*).$$

Solution. *We first compute*

$$A^*A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

The eigenvalues of A^*A are

$$8, \qquad 4, \qquad 0.$$

Therefore the nonzero singular values are

$$\boxed{\sigma_1 = 2\sqrt{2}, \qquad \sigma_2 = 2.}$$

Corresponding orthonormal eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

For the nonzero singular values,

$$u_j = \frac{Av_j}{\sigma_j}.$$

Thus

$$u_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

and

$$u_2 = \frac{1}{2}A\left(\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}\sqrt{2}\\-\sqrt{2}\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

Therefore the compact SVD is

$$\boxed{A = U_r\Sigma_r V_r^*,}$$

where

$$U_r = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\\0 & 0\end{pmatrix},$$

$$\Sigma_r = \begin{pmatrix}2\sqrt{2} & 0\\0 & 2\end{pmatrix},$$

and

$$V_r = \begin{pmatrix}1 & 0\\0 & 1/\sqrt{2}\\0 & 1/\sqrt{2}\end{pmatrix}.$$

Since the columns of U_r form an orthonormal basis for $\mathcal{C}(A)$,

$$\boxed{\mathcal{C}(A) = \operatorname{span}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}\right\}.}$$

The columns of V_r form an orthonormal basis for $\mathcal{C}(A^*)$, so

$$\boxed{\mathcal{C}(A^*) = \operatorname{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}.}$$

The remaining right singular vector spans $\mathcal{N}(A)$:

$$\boxed{\mathcal{N}(A) = \operatorname{span}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}.}$$

Finally,

$$\mathcal{N}(A^*) = \mathcal{C}(A)^{\perp}.$$

Since both vectors in $\mathcal{C}(A)$ have third component zero,

$$\boxed{\mathcal{N}(A^*) = \operatorname{span}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.}$$

Problem 3 (4pts)**.** *In class, we constructed the singular value decomposition of*

$$A \in \mathbb{C}^{m\times n}$$

using the eigenvalues and eigenvectors of A^*A.

This problem shows that the SVD can be constructed in a completely symmetric way using the eigenvalues and eigenvectors of AA^* instead.

Suppose

$$AA^*u = \lambda u, \qquad \lambda > 0, \qquad \|u\|_2 = 1.$$

Define

$$\sigma = \sqrt{\lambda}, \qquad v = \frac{A^*u}{\sigma}.$$

Show that this construction indeed gives a singular vector pair; that is, prove that

$$\boxed{Av = \sigma u,}$$

and that

$$\|v\|_2 = 1.$$

Solution. *Since*

$$v = \frac{A^*u}{\sigma},$$

we have

$$Av = A\left(\frac{A^*u}{\sigma}\right) = \frac{1}{\sigma}AA^*u.$$

Because

$$AA^*u = \lambda u,$$

it follows that

$$Av = \frac{\lambda}{\sigma}u.$$

Since

$$\sigma = \sqrt{\lambda},$$

we have

$$\lambda = \sigma^2.$$

Therefore,

$$Av = \frac{\sigma^2}{\sigma}u = \boxed{\sigma u}.$$

It remains to show that v has unit norm.

We compute

$$\|v\|_2^2 = v^*v = \frac{1}{\sigma^2}(A^*u)^*(A^*u).$$

Since

$$(A^*u)^* = u^*A,$$

we obtain

$$\|v\|_2^2 = \frac{1}{\sigma^2}u^*AA^*u.$$

Using

$$AA^*u = \lambda u,$$

we get

$$\|v\|_2^2 = \frac{\lambda}{\sigma^2}u^*u.$$

Since

$$\|u\|_2 = 1 \qquad \textit{and} \qquad \sigma^2 = \lambda,$$

we conclude that

$$\|v\|_2^2 = 1.$$

Hence

$$\boxed{\|v\|_2 = 1}.$$

Therefore,

$$\boxed{Av = \sigma u, \qquad \|v\|_2 = 1,}$$

so u and v form a left/right singular vector pair associated with the singular value

$$\boxed{\sigma = \sqrt{\lambda}}.$$