Homework 7 Solutions

Solution. ***Problem 1.***

Let

$$A = \begin{pmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

1. *Since*

$$A^* = \begin{pmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = A,$$

the matrix A is Hermitian.

2. *The characteristic polynomial is*

$$\det(A - \lambda I) = (-\lambda) \det \begin{pmatrix} 2-\lambda & i \\ -i & 2-\lambda \end{pmatrix}.$$

Hence

$$\det(A - \lambda I) = -\lambda \left((2-\lambda)^2 - 1\right) = -\lambda(\lambda - 1)(\lambda - 3).$$

Therefore the eigenvalues are

$$\boxed{0, 1, 3}.$$

For $\lambda = 3$,

$$(A - 3I)x = 0$$

gives

$$-x_1 + ix_2 = 0,$$

so

$$E_3 = \operatorname{span}\left\{ \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \right\}.$$

For $\lambda = 1$,

$$(A - I)x = 0$$

gives

$$x_1 + ix_2 = 0,$$

so

$$E_1 = \operatorname{span}\left\{ \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \right\}.$$

For $\lambda = 0$,

$$E_0 = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

3. *Normalized eigenvectors are*

$$v_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}, \qquad v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}, \qquad v_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For example,

$$v_3^* v_1 = \frac{1}{2}\begin{pmatrix} -i & 1 & 0 \end{pmatrix}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\left((-i)^2 + 1\right) = 0.$$

Also,

$$v_3^* v_0 = 0, \qquad v_1^* v_0 = 0.$$

Thus eigenvectors corresponding to distinct eigenvalues are orthogonal.

4. *Since the nullspace is the eigenspace corresponding to eigenvalue* 0*,*

$$\boxed{\mathcal{N}(A) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}.}$$

Hence

$$\operatorname{nullity}(A) = 1,$$

and by the rank-nullity theorem,

$$\boxed{\operatorname{rank}(A) = 2}.$$

Solution. ***Problem 2.***

Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and satisfy

$$A^2 = A.$$

1. *Suppose*

$$Av = \lambda v$$

for some nonzero eigenvector v. Then

$$A^2 v = A(\lambda v) = \lambda A v = \lambda^2 v.$$

Since $A^2 = A$,

$$A^2 v = Av = \lambda v.$$

Thus

$$\lambda^2 = \lambda,$$

so

$$\lambda(\lambda - 1) = 0.$$

Therefore

$$\boxed{\lambda = 0 \text{ or } \lambda = 1}.$$

2. *By definition,*

$$\boxed{\mathcal{N}(A) = \{x : Ax = 0\}.}$$

Now suppose $y \in \mathcal{C}(A)$*. Then*

$$y = Ax$$

for some x*, and therefore*

$$Ay = A^2x = Ax = y.$$

Hence

$$\mathcal{C}(A) \subseteq \{y : Ay = y\}.$$

Conversely, if

$$Ay = y,$$

then $y = Ay$*, so* $y \in \mathcal{C}(A)$*. Therefore*

$$\boxed{\mathcal{C}(A) = \{x : Ax = x\}.}$$

3. *Let* $z \in \mathcal{N}(A)$ *and* $y \in \mathcal{C}(A)$*. Write*

$$y = Ax.$$

Since A *is Hermitian,*

$$\langle z, y\rangle = z^*Ax = (Az)^*x = 0.$$

Therefore

$$\mathcal{N}(A) \subseteq \mathcal{C}(A)^\perp.$$

Conversely, suppose

$$z \in \mathcal{C}(A)^\perp.$$

Since $Ax \in \mathcal{C}(A)$ *for every* x*,*

$$z^*Ax = 0$$

for every x*. Since* $A = A^*$*,*

$$(Az)^*x = 0$$

for every x*, which implies*

$$Az = 0.$$

Thus

$$z \in \mathcal{N}(A).$$

Hence

$$\boxed{\mathcal{N}(A) = \mathcal{C}(A)^\perp.}$$

4. *For any* x*,*

$$Ax \in \mathcal{C}(A).$$

Also,

$$A(x - Ax) = Ax - A^2x = 0.$$

Therefore

$$x - Ax \in \mathcal{N}(A) = \mathcal{C}(A)^\perp.$$

Thus

$$x = Ax + (x - Ax),$$

where

$$Ax \in \mathcal{C}(A)$$

and

$$x - Ax \in \mathcal{C}(A)^{\perp}.$$

Therefore

$$\boxed{A \text{ is the orthogonal projection onto } \mathcal{C}(A).}$$