

Homework 6

Problem 1 (6pts). Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Define

$$f(x) = \|Ax - b\|_2^2, \quad x \in \mathbb{R}^n.$$

(a) Show that

$$\boxed{\nabla f(x) = 2A^T(Ax - b).}$$

(b) Using part (a) to show that a minimizer of $f(x)$ satisfies the normal equation

$$\boxed{A^T A \hat{x} = A^T b.}$$

(c) Find all solutions $x \in \mathbb{R}^2$ to the following least-squares problem:

$$x \in \arg \min_{y \in \mathbb{R}^2} \left\| \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} y - \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\|_2^2.$$

Your answer should be a one-dimensional affine subspace.

Problem 2 (4pts). Prove that

$$\text{rank}(A^* A) = \text{rank}(A) = \text{rank}(A A^*).$$