

Homework 5 Solutions

Problem 1.

Let $L \subseteq \mathbb{R}^n$ be a linear subspace, and let P_L denote the orthogonal projection onto L .

1. By definition of orthogonal projection,

$$P_L(x) \in L$$

for every $x \in \mathbb{R}^n$.

2. Again by the definition of orthogonal projection,

$$x - P_L(x) \in L^\perp$$

for every $x \in \mathbb{R}^n$.

3. Suppose $x \in L$.

Since $P_L(x) \in L$, we have

$$x - P_L(x) \in L.$$

From part (2),

$$x - P_L(x) \in L^\perp.$$

Therefore,

$$x - P_L(x) \in L \cap L^\perp.$$

Since

$$L \cap L^\perp = \{0\},$$

we obtain

$$x - P_L(x) = 0.$$

Hence,

$$\boxed{P_L(x) = x.}$$

4. We prove that P_L is linear.

Let $x, y \in \mathbb{R}^n$ and let $\alpha, \beta \in \mathbb{R}$.

Write

$$x = P_L(x) + (x - P_L(x)),$$

$$y = P_L(y) + (y - P_L(y)),$$

where

$$P_L(x), P_L(y) \in L$$

and

$$x - P_L(x), y - P_L(y) \in L^\perp.$$

Therefore,

$$\alpha x + \beta y = (\alpha P_L(x) + \beta P_L(y)) + (\alpha(x - P_L(x)) + \beta(y - P_L(y))).$$

Since L and $L^\perp$ are linear subspaces,

$$\alpha P_L(x) + \beta P_L(y) \in L$$

and

$$\alpha(x - P_L(x)) + \beta(y - P_L(y)) \in L^\perp.$$

By uniqueness of the orthogonal decomposition,

$$P_L(\alpha x + \beta y) = \alpha P_L(x) + \beta P_L(y).$$

Hence,

$$\boxed{P_L \text{ is linear.}}$$

5. Since

$$P_L(x) \in L,$$

part (3) gives

$$P_L(P_L(x)) = P_L(x).$$

Therefore,

$$\boxed{P_L^2 = P_L.}$$

6. We have

$$x = P_L(x) + (x - P_L(x)),$$

with

$$P_L(x) \in L$$

and

$$x - P_L(x) \in L^\perp.$$

Hence,

$$P_L(x) \perp x - P_L(x).$$

By the Pythagorean theorem,

$$\|x\|_2^2 = \|P_L(x)\|_2^2 + \|x - P_L(x)\|_2^2.$$

Thus,

$$\boxed{\|x\|_2^2 = \|P_L(x)\|_2^2 + \|x - P_L(x)\|_2^2.}$$

Problem 2.

Let

$$U = \{u_\ell\}_{\ell=1}^r \quad \text{and} \quad V = \{v_\ell\}_{\ell=1}^r$$

be two orthonormal bases of the same subspace

$$L = \text{span}(U) = \text{span}(V).$$

(a)

Since

$$v_1, \dots, v_r$$

form an orthonormal basis of L , the orthogonal projection of x onto L is

$$P_L(x) = \sum_{\ell=1}^r \langle v_\ell, x \rangle v_\ell.$$

Since $u_j \in L$, we have

$$u_j \perp x - P_L(x).$$

Therefore,

$$\langle u_j, x \rangle = \langle u_j, P_L(x) \rangle.$$

Substituting the expression for $P_L(x)$ gives

$$\langle u_j, x \rangle = \left\langle u_j, \sum_{\ell=1}^r \langle v_\ell, x \rangle v_\ell \right\rangle.$$

Using linearity of the inner product,

$$\langle u_j, x \rangle = \sum_{\ell=1}^r \langle v_\ell, x \rangle \langle u_j, v_\ell \rangle.$$

Hence,

$$\boxed{\langle u_j, x \rangle = \sum_{\ell=1}^r \langle v_\ell, x \rangle \langle u_j, v_\ell \rangle.}$$

(b)

By definition,

$$P_U(x) = \sum_{j=1}^r \langle u_j, x \rangle u_j.$$

Using part (a),

$$P_U(x) = \sum_{j=1}^r \left(\sum_{\ell=1}^r \langle v_\ell, x \rangle \langle u_j, v_\ell \rangle \right) u_j.$$

Rearranging the sums,

$$P_U(x) = \sum_{\ell=1}^r \langle v_\ell, x \rangle \left(\sum_{j=1}^r \langle u_j, v_\ell \rangle u_j \right).$$

Since

$$u_1, \dots, u_r$$

form an orthonormal basis of L , and

$$v_\ell \in L,$$

we have

$$v_\ell = \sum_{j=1}^r \langle u_j, v_\ell \rangle u_j.$$

Therefore,

$$P_U(x) = \sum_{\ell=1}^r \langle v_\ell, x \rangle v_\ell.$$

But the right-hand side is exactly

$$P_V(x).$$

Hence,

$$\boxed{P_U(x) = P_V(x)}$$

for every $x \in \mathbb{R}^n$.

Thus, the orthogonal projection onto L is independent of the choice of orthonormal basis.